Bull. Korean Math. Soc. **00** (20XX), No. 0, pp. 000–000
https://doi.org/10.4134/BKMS.b210101
pISSN: 1015-8634 / eISSN: 2234-3016

A DECOMPOSITION THEOREM FOR UTUMI AND DUAL-UTUMI MODULES

Yasser Ibrahim and Mohamed Yousif

Abstract. We show that if M is a Utumi module, in particular if M is quasi-continuous, then $M = Q \oplus K$, where Q is quasi-injective that is both a square-full as well as a dual-square-full module, K is a square-free module, and Q & K are orthogonal. Dually, we also show that if M is a dual-Utumi module whose local summands are summands, in particular if M is quasi-discrete, then $M = P \oplus K$ where P is quasi-projective that is both a square-full as well as a dual-square-full module, K is a dual-square-free module, and P & K are factor-orthogonal.

1. Preliminaries

A module Y is called a square if $Y \cong X \oplus X$ for some module X. A module M is called square-free if it does not contain a non-zero square. A submodule X of a module M is called a square-root in M if $X \oplus X$ embeds in M. The module M is called square-full if every non-zero submodule of M contains a non-zero square-root. A well-known result of Mohamed and Müller, [8, Theorem 2.37], asserts that every quasi-continuous module M has a decomposition $M = M_1 \oplus M_2$, unique up to superspectivity, such that:

(1) M_1 is square-free;
(2) M_2 is square-full and quasi-injective;
(3) M_1 and M_2 are orthogonal.

The notion of square-free was dualized in [1] as follows: a right R-module M is called dual-square-free if M has no proper submodules A and B with $M = A + B$ and $M/A \cong M/B$. Equivalently, [7], if L is a factor module of M such that $L \cong N \oplus N$ for some module N, then $N = 0$. Subsequently, a thorough investigation of dual-square-free modules was carried out in [2].

In [6], the notion of factor-square-full modules was introduced and a dualization of the aforementioned result of Mohamed and Müller was established.

Received February 1, 2021; Accepted July 12, 2021.

2010 *Mathematics Subject Classification.* Primary 16D40, 16D50, 16D60; Secondary 16L30, 16P20, 16P60.

Key words and phrases. Utumi and dual Utumi modules, quasi-injective and quasi-projective modules, square-free and dual-square-free modules, discrete and quasi-discrete modules, $D3$-modules and $D4$-modules.

According to [6], a submodule $Y \subseteq M$ is called dual-square-root if there is an epimorphism $f : M \to (M/Y)^2$, where $(M/Y)^2 := (M/Y) \oplus (M/Y)$. A module M is called factor-square-full if, every proper submodule X of M is contained in a proper dual-square-root Y of M. It was shown in [6, Proposition 3.4 and Theorem 3.7] that every quasi-discrete module M is a direct sum $M_1 \oplus M_2$ of a factor-square-full module M_1 and a dual-square-free module M_2, which are factor orthogonal. Moreover, such a decomposition is unique up to isomorphism and the module M_1 is quasi-projective.

In this paper we show that if M is a Utumi module (U-module, for short), then $M = Q \oplus K$ where Q is quasi-injective that is both a square-full as well as a dual-square-full module, K is a square-free module, and Q and K are orthogonal. In particular, such a decomposition holds for quasi-continuous modules. Dually, we also show that if M is a Dual-Utumi module (DU-module, for short) whose local summands are summands, then $M = P \oplus K$, where P is quasi-projective that is both a square-full as well as a dual-square-full module, K is a dual-square-free module, and P & K are factor-orthogonal. In particular, such a decomposition holds for quasi-discrete modules. Our results may be considered as an improvement of the work on quasi-discrete modules in [6].

Let's recall first some definitions. According to [3], the notion of a U-module was introduced as a non-trivial and simultaneous generalization of quasi-continuous, square-free and automorphism-invariant modules, where a right R-module M is called a U-module if, whenever A and B are submodules of M with $A \cong B$ and $A \cap B = 0$, there exist two summands K and T of M such that $A \subseteq^{ess} K$, $B \subseteq^{ess} T$ and $K \oplus T \subseteq^{\oplus} M$. Dually, in [4], the notion of DU-modules was introduced as a strict and simultaneous generalization of the quasi-discrete, pseudo-discrete and dual-square-free modules. As defined in [4], a right R-module M is called a DU-module if, for any two proper submodules A and B of M with $M/A \cong M/B$ and $A + B = M$, there exist two summands K and L of M such that A lies over K, B lies over L and $K \cap L \subseteq^{\oplus} M$. For the definitions of quasi-continuous, quasi-discrete, discrete, quasi-injective, and quasi-projective, we refer the reader to the textbooks [8] and [9].

Throughout, all rings R are associative with unity and all modules are unitary R-modules. For a module M, we use $rad(M)$, $E(M)$ and $End(M_R)$ to denote the Jacobson radical, the injective hull and the endomorphism ring of M, respectively. If $M = R$, we write $J(R) = rad(R)$. We write $N \subseteq M$ if N is a submodule of M, $N \subseteq^{ess} M$ if N is an essential submodule of M, $N \subseteq^{\oplus} M$ if N is a direct summand of M, and $N \ll M$ if N is a small submodule of M. A submodule N of M is called proper if $N \subsetneqq M$. A submodule N of a right R-module M is said to lie over a direct summand of M if there is a decomposition $M = M_1 \oplus M_2$ with $M_1 \subseteq N$ and $N \cap M_2 \ll M$. Furthermore, two right R-modules M and N are called orthogonal, if they do not contain non-zero isomorphic submodules. Dually, M and N are called factor orthogonal if no non-zero factor of M is isomorphic to a factor of N.

2. Results

Lemma 2.1 ([3, Theorem 3.13])**.** *If M is a U-module, then $M = Q \oplus T$, where*

(1) *Q is a quasi-injective module*;
(2) *$Q = A \oplus B \oplus D$, where $A \cong B$ and D is isomorphic to a direct summand of $A \oplus B$*;
(3) *T is a square-free module*;
(4) *T is Q-injective, and*
(5) *Q and T are orthogonal.*

Recall that a local summand of a module M is a direct sum $L := \oplus_{i \in I} N_i$ of submodules of M such that $\oplus_{i \in F} N_i$ is a summand of M for any finite subset F of I.

Lemma 2.2 ([4, Theorem 4.4])**.** *Let M be a DU-module whose local summands are summands. Then $M = Q \oplus P$, where*

(1) *Q is a DSF-module*;
(2) *$Q = \oplus_{\lambda \in \Lambda} Q_\lambda$, a direct sum of pairwise non-isomorphic indecomposable modules*;
(3) *$P = C \oplus A \oplus B$ is a quasi-projective and discrete module with $A \cong B$, and C is isomorphic to a direct summand of $A \oplus B$*;
(4) *Q is P-projective*;
(5) *P and Q are factor-orthogonal.*

Lemma 2.3. *If $M = A \oplus B \oplus C$ with $A \overset{f}{\cong} B$, and C is isomorphic to a direct summand of $A \oplus B$, then M is both a square-full as well as a dual square-full module.*

Proof. First we show that M is square-full. Let $0 \neq X \subseteq M = (A \oplus B) \oplus C$ and suppose that $Q =: X \cap A \neq 0$. Therefore, $Q \cong f(Q)$ with $Q \cap f(Q) = 0$. This means that Q is a non-zero square root embedded in M. Similarly, if $S = X \cap B \neq 0$, then S is a non-zero square root embedded in M. Now, suppose that $E =: X \cap C \neq 0$, and let $\sigma : C \longrightarrow A \oplus B$ be an embedding. Clearly, $E \cong \sigma(E)$ with $E \cap \sigma(E) = 0$, and so E is a non-zero square root embedded in M. Therefore, it remains to consider the case when $X \cap A = X \cap B = X \cap C = 0$. By [8, Lemma 1.31], X and one of A, B or C have non-zero isomorphic submodules. Without loss of generality, let $X' \subseteq X$ and $A' \subseteq A$ be such that $X' \cong A'$. Inasmuch as $X' \cap A' = 0$, we infer that X' is a square-root in M. This shows that M is a square-full module. Next, we show that M is dual-square-full. Let X be a proper submodule of M. Clearly, we have the following epimorphism:

$$M \to A \oplus B \cong B \oplus B \cong M/(A \oplus C) \oplus M/(A \oplus C) \to M/(A+X+C) \oplus M/(A+X+C).$$

Now, if $Y := A+X+C \neq M$, then Y is a proper factor-square-full submodule containing X. Otherwise, suppose that $Y := A + X + C = M$. In this case

$M/(X+C) \cong A/\left(A \cap (X+C)\right)$, and we have the following epimorphism:

$$\begin{aligned} M \to A \oplus B \cong A \oplus A \to A/\left(A \cap (X+C)\right) \oplus A/\left(A \cap (X+C)\right) \\ \cong M/(X+C) \oplus M/(X+C). \end{aligned}$$

Now, if $X + C \neq M$, then $X + C$ is a proper factor-square-full submodule containing X. If $M = X + C$, then by the hypothesis, $C \cong D \subseteq^{\oplus} A \oplus B$ for a submodule $D \subseteq M$, and we have the following epimorphism:

$$M = A \oplus B \oplus C \to D \oplus C \cong C \oplus C \to C/(X \cap C) \oplus C/(X \cap C) \cong M/X \oplus M/X.$$

In this case X is a proper factor-square-full submodule. This shows that M is dual-square-full, completing the proof. □

Now, the next two results are immediate consequences of Lemma 2.1, Lemma 2.2 and Lemma 2.3. Recall first that a module M is said to satisfy the $C1$-condition if every submodule of M is essential in a direct summand. M is said to satisfy the $C3$-condition if the sum of any two summands of M with zero intersection is a summand of M. A module is called *quasi-continuous* if it satisfies both the $C1$- and $C3$-conditions. Moreover, a module M is called *automorphism-invariant* (*auto-invariant*) if it is invariant under any automorphism of its injective hull.

Theorem 2.4. *If M is a U-module, then $M = Q \oplus K$, where Q is quasi-injective that is both a square-full as well as a dual-square-full module, K is a square-free module, and Q and K are orthogonal. In particular, such a decomposition holds for both quasi-continuous and auto-invariant modules.*

A module M is said to satisfy the $D1$-condition if every submodule N of M lies over a direct summand of M. The module M is said to satisfy the $D3$-condition if M_1 and M_2 are direct summands of M, and $M = M_1 + M_2$, then $M_1 \cap M_2$ is a direct summand of M. A module is called *quasi-discrete* if it satisfies both the $D1$- and $D3$-conditions.

Theorem 2.5. *Let M be a DU-module whose local summands are summands. Then $M = P \oplus K$, where P is quasi-projective and discrete that is both a square-full as well as a dual-square-full module, K is a dual-square-free module, and P and K are factor-orthogonal. In particular, such a decomposition holds for quasi-discrete modules.*

A module M is called H-supplemented [6] if, for any submodule $X \subseteq M$, there exist a submodule $Y \subseteq M$ and a decomposition $M = A \oplus B$ such that $X \subseteq Y$, $A \subseteq Y$, $Y/X \ll M/X$ and $Y/A \ll M/A$. If A and B are modules, then A is called radical-B-projective [6] if, for every homomorphism $f : A \to X$ and every epimorphism $g : B \to X$ there exists a homomorphism $h : A \to B$ such that $\operatorname{Im}(f - gh) \ll X$. A module M is called quasi-radical-projective if M is radical-M-projective.

Theorem 2.6. *Let M be an H-supplemented module that satisfies the D3-condition, then $M = Q \oplus P$, where Q is a dual-square-free module, P is a quasi-radical-projective module that is both a square-full as well as a dual-square-full module, and P and Q are factor-orthogonal.*

Proof. It follows from Lemma 2.3 and the proof of Proposition 2.16 in [5]. □

References

[1] N. Ding, Y. Ibrahim, M. Yousif, and Y. Zhou, *D4-modules*, J. Algebra Appl. **16** (2017), no. 9, 1750166, 25 pp. https://doi.org/10.1142/S0219498817501663

[2] Y. Ibrahim and M. Yousif, *Dual-square-free modules*, Comm. Algebra **47** (2019), no. 7, 2954–2966. https://doi.org/10.1080/00927872.2018.1543429

[3] ———, *Utumi modules*, Comm. Algebra **46** (2018), no. 2, 870–886. https://doi.org/10.1080/00927872.2017.1339064

[4] ———, *Dual Utumi modules*, Comm. Algebra **47** (2019), no. 9, 3889–3904. https://doi.org/10.1080/00927872.2019.1572166

[5] I. Kikumasa and Y. Kuratomi, *On H-supplemented modules over a right perfect ring*, Comm. Algebra **46** (2018), no. 5, 2063–2072. https://doi.org/10.1080/00927872.2017.1372451

[6] I. Kikumasa, Y. Kuratomi, and Y. Shibata, *Factor square full modules*, Comm. Algebra **49** (2021), no. 6, 2326–2336. https://doi.org/10.1080/00927872.2020.1870997

[7] D. Keskin Tütüncü, I. Kikumasa, Y. Kuratomi, and Y. Shibata, *On dual of square free modules*, Comm. Algebra **46** (2018), no. 8, 3365–3376. https://doi.org/10.1080/00927872.2017.1412449

[8] S. H. Mohamed and B. J. Müller, *Continuous and discrete modules*, London Mathematical Society Lecture Note Series, 147, Cambridge University Press, Cambridge, 1990. https://doi.org/10.1017/CBO9780511600692

[9] R. Wisbauer, *Foundations of module and ring theory*, revised and translated from the 1988 German edition, Algebra, Logic and Applications, 3, Gordon and Breach Science Publishers, Philadelphia, PA, 1991.

Yasser Ibrahim
Department of Mathematics
Faculty of Science
Cairo University
Giza, Egypt
and
Department of Mathematics
Faculty of Science
Taibah University
Madina, Saudi Arabia
Email address: yfibrahim@sci.cu.edu.eg, yabdelwahab@taibahu.edu.sa

Mohamed Yousif
Department of Mathematics
The Ohio State University
Lima, Ohio 45804, USA
Email address: yousif.1@osu.edu