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ON THE GEOMETRY OF THE CROSSED PRODUCT OF GROUPS

Firat Ateş, Ahmet Sinan Çevik, and Eylem Güzel Karpuz

Abstract. In this paper, firstly, we work on the presentation of the crossed product of groups of general types. After that we find the generating pictures (Second Homotopy Group) of this product by looking the relations from a geometric viewpoint. Finally, we give some applications.

1. Introduction

Let H and G be two groups. A crossed system of these groups is a quadruple (H, G, α, f), where $\alpha : G \longrightarrow \mathrm{Aut}(H)$, $g \mapsto \alpha_g(h)$ and $f : G \times G \longrightarrow H$ are two maps which satisfy

$$\alpha_{g_1}(\alpha_{g_2}(h)) = f(g_1, g_2)\alpha_{g_1g_2}(h)f(g_1, g_2)^{-1}, \tag{1}$$

$$f(g_1, g_2)f(g_1g_2, g_3) = \alpha_{g_1}(f(g_2, g_3))f(g_1, g_2g_3) \tag{2}$$

for all $g, g_1, g_2 \in G$ and $h \in H$, where $\mathrm{Aut}(H)$ denotes the group automorphism of H. The crossed system (H, G, α, f) is called normalized if $f(1, 1) = 1$. Also α is called a weak action and f is called an α-cocycle. The crossed product of H and G associated to the crossed system, denoted by $H\#^f_\alpha G$, is the set $H \times G$ with the multiplication

$$(h_1, g_1)(h_2, g_2) = (h_1\alpha_{g_1}(h_2)f(g_1, g_2), g_1g_2).$$

Assume that (H, G, α, f) is a normalized crossed system. Then $H\#^f_\alpha G$ is a group with the identity $(1, 1)$ and we have $f(g_1, 1) = f(1, g_2) = 1$ for all $g_1, g_2 \in G$. Here we recall that we have $(h, g)^{-1} = (f(g^{-1}, g)^{-1}\alpha_{g^{-1}}(h^{-1}), g^{-1})$ for $(h, g) \in H\#^f_\alpha G$.

Consider the crossed system (H, G, α, f) such that f is a trivial map. In this case, it is easily seen that $H\#^f_\alpha G$ is isomorphic to semidirect product of H and G if α is a homomorphism. Let us again consider the crossed system (H, G, α, f) such that α is trivial. Then one can show that $H\#^f_\alpha G$ is isomorphic to twisted product of H and G if we have $\mathrm{Im}f \subseteq Z(H)$, where $Z(H)$ is the center of H.

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Now we give the following theorem as the main application of the crossed product construction. The proof of this theorem can be found in [1, 3, 19].

Theorem 1.1. *Let E be a group, H be a normal subgroup of E and G be the quotient of E by H. Then there exist two maps $\alpha : G \longrightarrow \mathrm{Aut}(H)$ and $f : G \times G \longrightarrow H$ such that (H, G, α, f) is a normalized crossed system and $E \cong H \#_{\alpha}^{f} G$.*

Some other results and details relative to the crossed product of two groups can be found in [2, 3, 6–8, 15].

Note 1.1. Let $(E, \cdot)$ be such a group structure and let H and G be groups. Then the problem arises that what are the all group structures E which containing H as a normal subgroup such that $E/H \cong G$. This problem is called extension problem which was first stated by Holder [14]. In [2, 3], authors give some results on the crossed products about this problem. Also they say that the set of these $(E, \cdot)$ group structures is one to one correspondence with the set of all normalised crossed systems (H, G, α, f).

In this paper, we work on the presentation of the crossed product for given groups in Section 2. After that we will define the generating pictures (see Section 3) of this product. Finally, we will give some applications in Section 4. Thus we look the extension problem from a geometric viewpoint under this product (see Corollary 3.2 and Corollary 4.3).

2. Presentation

In [1–3], the authors give some results on the presentation of the crossed product of cyclic groups. As far as we know there is no more work about the presentation of this product for arbitrary groups. In this section, we aim to work on the presentation of this product for given groups. In order to do that, let us consider the groups H and G presented by

$$\mathcal{P}_H = \langle \mathbf{x} \, ; \, \mathbf{r} \rangle \text{ and } \mathcal{P}_G = \langle \mathbf{y} \, ; \, \mathbf{s} \rangle \,, \tag{3}$$

respectively, and two maps α, f as in (1) and (2). Also let us think that (H, G, α, f) is a normalized crossed system. Let $\mathbf{x}^{-1}$ be a set in one-to-one correspondence $x \longleftrightarrow x^{-1}$ with $\mathbf{x}$ for $x \in \mathbf{x}$ and also let $\mathbf{y}^{-1}$ be a set in one-to-one correspondence $y \longleftrightarrow y^{-1}$ with $\mathbf{y}$ for $y \in \mathbf{y}$.

Let us denote $f(y, y^{-1}) \in H$ by $W_{y,y^{-1}}$ and let W_S be a word on the set $\mathbf{x} \cup \mathbf{x}^{-1}$ for $S \in \mathbf{s}$. Then we can give the following theorem as the one of the main results of this paper.

Theorem 2.1. *Let us consider the groups H and G given by the presentations in* (3)*. Then the crossed product of them is defined by generators*

$$\mathbf{x} \cup \mathbf{y}$$

and the relations

$$R = 1 \ (R \in \mathbf{r}), \tag{4}$$

(5) $$S = W_S \ (S \in \mathbf{s}),$$

(6) $$yx^{\delta} = \alpha_y(x^{\delta})y,$$

(7) $$y^{-1}x^{\delta} = \alpha_{y^{-1}}(W^{-1}_{y,y^{-1}}x^{\delta}W_{y,y^{-1}})y^{-1},$$

where $\delta = \pm 1$, $y \in \mathbf{y}$, $y^{-1} \in \mathbf{y}^{-1}$, $x \in \mathbf{x}$ *and* $x^{-1} \in \mathbf{x}^{-1}$.

Proof. Let us denote the set of all words in A by A^{*} and consider $x \in \mathbf{x}$, $y \in \mathbf{y}$ and $\delta = \pm 1$. Also for $\mathbf{z} = \mathbf{x} \cup \mathbf{y}$, let

$$\psi : \mathbf{z}^* \longrightarrow H\#^f_{\alpha}G$$

be a homomorphism defined by $\psi(x) = (x, 1)$ and $\psi(y) = (1, y)$. Here we realise that

$$(x, 1)(x^{-1}, 1) = (x\alpha_1(x^{-1})f(1, 1), 1) = (xx^{-1}, 1) = (1, 1),$$
$$(x^{-1}, 1)(x, 1) = (x^{-1}\alpha_1(x)f(1, 1), 1) = (x^{-1}x, 1) = (1, 1).$$

Thus we have $(x, 1)^{-1} = (x^{-1}, 1)$. So we have that $\psi(x^{-1}) = (x^{-1}, 1)$. Also since we have $(1, y)^{-1} = (f(y^{-1}, y)^{-1}, y^{-1})$, then we see that

$$\psi(y^{-1}) = (f(y^{-1}, y)^{-1}, y^{-1}).$$

Now let us think that $(x^{\delta}, 1)(1, y) = (x^{\delta}, y)$ and $(x^{\delta}, 1)(1, y^{-1}) = (x^{\delta}, y^{-1})$. Then we say that ψ is onto. Now let us check whether $H\#^f_{\alpha}G$ satisfies relations (4)-(7).

Let $R = x_1^{\delta_1}x_2^{\delta_2}\cdots x_s^{\delta_s} \in \mathbf{r}$, where $x_1, x_2, \ldots, x_s \in \mathbf{x}$ and $\delta_i = \pm 1$ $(1 \le i \le s)$. Thus the relation (4) follows from $(x_1^{\delta_1}, 1)(x_2^{\delta_2}, 1)\cdots(x_s^{\delta_s}, 1) = (R, 1) = (1, 1)$. Also for $S \in \mathbf{s}$, let $S = y_1^{\epsilon_1}y_2^{\epsilon_2}\cdots y_k^{\epsilon_k}$, where $y_1, y_2, \ldots, y_k \in \mathbf{y}$ and $\epsilon_i = \pm 1$ $(1 \le i \le k)$. Then the relation (5) follows from

$$(1, y_1)^{\epsilon_1}(1, y_2)^{\epsilon_2}\cdots(1, y_k)^{\epsilon_k} = (W_S, S) = (W_S, 1),$$

where $W_S \in (\mathbf{x} \cup \mathbf{x}^{-1})^*$.

We can get the relation (6) by

$$(1, y)(x, 1)^{\delta} = (1, y)(x^{\delta}, 1) = (\alpha_y(x^{\delta}), y) = (\alpha_y(x^{\delta}), 1)(1, y).$$

Now we show that the relation (7) holds. By using (2), we get

$$f(y^{-1}, y)f(y^{-1}y, y^{-1}) = \alpha_{y^{-1}}(f(y, y^{-1}))f(y^{-1}, yy^{-1})$$
$$\Longrightarrow f(y^{-1}, y)f(1, y^{-1}) = \alpha_{y^{-1}}(f(y, y^{-1}))f(y^{-1}, 1)$$

(8) $$\Longrightarrow f(y^{-1}, y) = \alpha_{y^{-1}}(f(y, y^{-1})).$$

Then we have

$$\begin{aligned}
&(1, y)^{-1}(x^{\delta}, 1)\\
&= (f(y^{-1}, y)^{-1}, y^{-1})(x^{\delta}, 1)\\
&= (f(y^{-1}, y)^{-1}\alpha_{y^{-1}}(x^{\delta}), y^{-1})\\
&= (f(y^{-1}, y)^{-1}\alpha_{y^{-1}}(x^{\delta})f(y^{-1}, y)f(y^{-1}, y)^{-1}, y^{-1})
\end{aligned}$$

$$\begin{aligned}
&= (f(y^{-1},y)^{-1}\alpha_{y^{-1}}(x^{\delta})f(y^{-1},y),1)(f(y^{-1},y)^{-1},y^{-1})\\
&= (\alpha_{y^{-1}}(f(y,y^{-1})^{-1})\alpha_{y^{-1}}(x^{\delta})\alpha_{y^{-1}}(f(y,y^{-1})),1)(f(y^{-1},y)^{-1},y^{-1})\\
&= (\alpha_{y^{-1}}(W_{y,y^{-1}}^{-1}x^{\delta}W_{y,y^{-1}}),1)(1,y)^{-1},
\end{aligned}$$

where $W_{y,y^{-1}} = f(y,y^{-1})$. Therefore ψ induces an epimorphism $\overline{\psi}$ from the group C defined by (4)-(7) onto $H\#_{\alpha}^{f}G$.

Let $w \in \mathbf{z}^*$ be any non-empty word. By using the relations (6) and (7), there exist words $w_x \in (\mathbf{x}\cup\mathbf{x}^{-1})^*$ and $w_y \in (\mathbf{y}\cup\mathbf{y}^{-1})^*$ such that $w = w_x w_y$ in C. Therefore, for any word $w \in \mathbf{z}^*$, we have

$$\begin{aligned}
\overline{\psi} = \psi(w) = \psi(w_x w_y) &= \psi(w_x)\psi(w_y)\\
&= (w_x,1)(w_{x'},w_y)\\
&= (w_x w_{x'},w_y),
\end{aligned}$$

where $w_{x'} \in (\mathbf{x}\cup\mathbf{x}^{-1})^*$. Now, if $\psi(w') = \psi(w'')$ for some $w',w'' \in \mathbf{z}^*$, then we deduce that $w'_x w'_{x'} = w''_x w''_{x'}$ in H and $w'_y = w''_y$ in G, where $w' = w'_x w'_y$ and $w'' = w''_x w''_y$. Here since $w'_y = w''_y$, we get $w'_{x'} = w''_{x'}$. So we have $w'_x = w''_x$ in H. This implies that $w'_x = w''_x$ and $w'_y = w''_y$ hold in C. So that $w' = w''$ holds. We get that $\overline{\psi}$ is injective. This says that there is an isomorphism between the group C defined by (4)-(7) and $H\#_{\alpha}^{f}G$. Hence the result. □

Let us consider the presentations $\mathcal{P}_H$ and $\mathcal{P}_G$ given in (3) for the groups H and G, respectively. Then by Theorem 2.1, we get the presentation of the group $H\#_{\alpha}^{f}G$ as follows,

$$\begin{aligned}
\mathcal{P}_{H\#_{\alpha}^{f}G} = \langle \mathbf{x},\, \mathbf{y}\,;\, &R = 1\ (R\in\mathbf{r}),\ \ S = W_S\ (S\in\mathbf{s}),\\
&yx^{\delta} = \alpha_y(x^{\delta})y,\ \ y^{-1}x^{\delta} = \alpha_{y^{-1}}(W_{y,y^{-1}}^{-1}x^{\delta}W_{y,y^{-1}})y^{-1}\rangle. \qquad (9)
\end{aligned}$$

Let us consider the presentation of the group $H\#_{\alpha}^{f}G$ given in (9). Here let us think that f is a trivial map. In this case, it is known that $H\#_{\alpha}^{f}G$ is isomorphic to semidirect product of H and G if α is a homomorphism. Then we get the following well known corollary.

Corollary 2.2. *The presentation of the semidirect product of the groups H and G is*

$$\langle \mathbf{x},\, \mathbf{y}\,; R = 1\ (R\in\mathbf{r}),\ \ S = 1\ (S\in\mathbf{s}),\ \ yx^{\delta} = \alpha_y(x^{\delta})y,\ \ y^{-1}x^{\delta} = \alpha_{y^{-1}}(x^{\delta})y^{-1}\rangle.$$

Also let us think that α is trivial in (9). Then we know that $H\#_{\alpha}^{f}G$ is isomorphic to twisted product of H and G if we have $\mathrm{Im}f \subseteq Z(H)$, where $Z(H)$ is the center of H. Then we get the following corollary.

Corollary 2.3. *The presentation of the twisted product of the groups H and G is*

$$\langle \mathbf{x},\, \mathbf{y}\,; R = 1\ (R\in\mathbf{r}),\ \ S = W_S\ (S\in\mathbf{s}),\ \ yx^{\delta} = x^{\delta}y,\ \ y^{-1}x^{\delta} = x^{\delta}y^{-1}\rangle.$$

Proof. Let us think the relation $y^{-1}x^{\delta} = \alpha_{y^{-1}}(W^{-1}_{y,y^{-1}}x^{\delta}W_{y,y^{-1}})y^{-1}$. Since α is trivial, then we have $y^{-1}x^{\delta} = W^{-1}_{y,y^{-1}}x^{\delta}W_{y,y^{-1}}y^{-1}$. Also we have $\mathrm{Im}f \subseteq Z(H)$, then we get $W^{-1}_{y,y^{-1}}x^{\delta}W_{y,y^{-1}} = W^{-1}_{y,y^{-1}}W_{y,y^{-1}}x^{\delta} = x^{\delta}$. □

3. Generating pictures

We may say that not every presentation of the form in (9) defines an extension of H by G since the natural map $H \longrightarrow H\#^f_\alpha G$ may not be injective. Let us think that $(E, \cdot)$ is a such a group structure and H, G are subgroups of E such that $E = H \cdot G$. By [5, 12], if we know generating pictures (Second Homotopy Group) of E, then we can determine whether E is an extension of H by G or not. So if we define generating pictures of $H\#^f_\alpha G$, then we can give necessary and sufficient conditions for the natural map $H \longrightarrow H\#^f_\alpha G$ to be injective. Thus, in this section we will give the generating pictures of $H\#^f_\alpha G$ to define an extension of H by G. In order to do that let us give some explanation about generating pictures by followings.

Let K be a finitely presented group with a presentation $\mathcal{P}_K = \langle \mathbf{z}\ ;\ \mathbf{t}\rangle$. If we regard $\mathcal{P}_K$ as a 2-complex in the standard way with a single 0-cell, with 1-cells in bijective correspondence with the elements of $\mathbf{z}$, and with 2-cells attached by the boundary paths determined by the spelling of the corresponding elements of $\mathbf{t}$, then K is just the fundamental group of $\mathcal{P}_K$. Therefore we can talk about the second homotopy group $\pi_2(\mathcal{P}_K)$ of $\mathcal{P}_K$, which is a left $\mathbb{Z}K$-module. The elements of $\pi_2(\mathcal{P}_K)$ can be represented by geometric configurations called *spherical pictures.* Spherical (and non-spherical) pictures are described in detail in [11] and [16].

Suppose $\mathbf{Y}$ is a collection of spherical pictures over $\mathcal{P}_K$. Then one can define certain additional operations on spherical pictures [16]. Allowing this additional operation leads to the notion of *equivalence* (*rel* $\mathbf{Y}$) *of spherical pictures.* Then, again in [16], Pride proved that *the elements* $\langle\mathbb{P}\rangle$ $(\mathbb{P} \in \mathbf{Y})$ *generate* $\pi_2(\mathcal{P}_K)$ *as a module if and only if every spherical picture is equivalent* (*rel* $\mathbf{Y}$) *to the empty picture.* Therefore one can easily say that if the elements $\langle\mathbb{P}\rangle$ $(\mathbb{P} \in \mathbf{Y})$ generate $\pi_2(\mathcal{P}_K)$, then $\mathbf{Y}$ *generates* $\pi_2(\mathcal{P}_K)$.

The reader can find some examples, and more details about second homotopy group $\pi_2(\mathcal{P}_K)$, in [4, 5, 16, 17].

Let us consider the groups H and G given by the presentations in (3). Also let us think the second homotopy groups $\pi_2(\mathcal{P}_H)$ and $\pi_2(\mathcal{P}_G)$ which are the collections of the spherical pictures over the groups H and G, respectively. Let us think the words W_S on the set $\mathbf{x} \cup \mathbf{x}^{-1}$ for $S \in \mathbf{s}$ and maps f, α which satisfies the conditions (1) and (2). Now we can give the following theorem as the another main result of this paper.

Theorem 3.1. *Let us consider the group* $H\#^f_\alpha G$ *presented by* (9)*. For each* $y \in \mathbf{y}$ *and* $S \in \mathbf{s}$*, let us suppose that* $\alpha_y(W_S) = W_S$*. Then second homotopy* *group* $\pi_2(\mathcal{P}_{H\#_\alpha^f G})$ *is generated by the pictures given in Figures* 1-9 *and the pictures in* $\pi_2(\mathcal{P}_H)$ *and* $\pi_2(\mathcal{P}_G)$.

Proof. Let us consider the group $H\#_\alpha^f G$. Since we have the following relations $R = 1$, $S = W_S$, $yx^\delta = \alpha_y(x^\delta)y$ and $y^{-1}x^\delta = \alpha_{y^{-1}}(W_{y,y^{-1}}^{-1}x^\delta W_{y,y^{-1}})y^{-1}$, we have to think about the following overlapping word pairs

$$x^\delta x^{-\delta}x^\delta,\ y^\epsilon y^{-\epsilon}y^\epsilon,\ yx^\delta x^{-\delta},\ y^{-1}x^\delta x^{-\delta},$$
$$yy^{-1}x^\delta,\ y^{-1}yx^\delta,\ Sx^\delta,\ yW_S,\ y^{-1}W_S,\ yR,\ y^{-1}R$$

for defining the elements of $\pi_2(\mathcal{P}_{H\#_\alpha^f G})$, where $\delta = \mp 1$ and $\epsilon = \mp 1$. It is known that spherical pictures which are obtained from the resolutions of these pairs give the elements of $\pi_2(\mathcal{P}_{H\#_\alpha^f G})$ by [5].

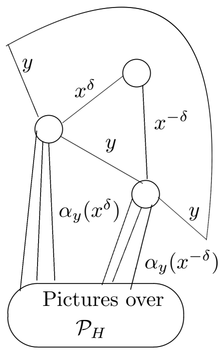

Figure 1

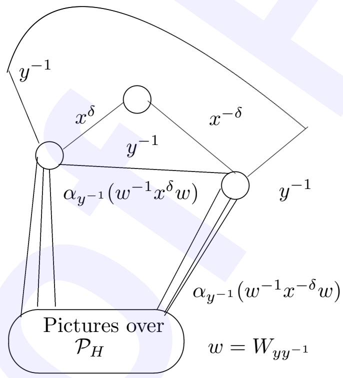

Figure 2

Let us consider the pairs $x^\delta x^{-\delta}x^\delta$ and $y^\epsilon y^{-\epsilon}y^\epsilon$. Here the pictures which are obtained from these pairs are already spherical. Now let us think the pairs $yx^\delta x^{-\delta}$ and $y^{-1}x^\delta x^{-\delta}$. Since we have $yx^\delta = \alpha_y(x^\delta)y$, $yx^{-\delta} = \alpha_y(x^{-\delta})y$ and $\alpha_y(x^\delta)\alpha_y(x^{-\delta}) = \alpha_y(x^\delta x^{-\delta}) = 1$, we get the spherical pictures in Figure 1. Also we have

$$y^{-1}x^\delta = \alpha_{y^{-1}}(W_{y,y^{-1}}^{-1}x^\delta W_{y,y^{-1}})y^{-1},\ y^{-1}x^{-\delta} = \alpha_{y^{-1}}(W_{y,y^{-1}}^{-1}x^{-\delta}W_{y,y^{-1}})y^{-1}$$

and

$$\alpha_{y^{-1}}(W_{y,y^{-1}}^{-1}x^\delta W_{y,y^{-1}})\alpha_{y^{-1}}(W_{y,y^{-1}}^{-1}x^{-\delta}W_{y,y^{-1}})$$
$$= \alpha_{y^{-1}}(W_{y,y^{-1}}^{-1}x^\delta W_{y,y^{-1}}W_{y,y^{-1}}^{-1}x^{-\delta}W_{y,y^{-1}}) = 1.$$

Then we get the spherical pictures in Figure 2.

Let us think the pairs $yy^{-1}x^\delta$ and $y^{-1}yx^\delta$. Also, by using (1), let us consider the followings:

$$\alpha_y(\alpha_{y^{-1}}(W_{y,y^{-1}}^{-1}x^\delta W_{y,y^{-1}})) = W_{y,y^{-1}}\alpha_{yy^{-1}}(W_{y,y^{-1}}^{-1}x^\delta W_{y,y^{-1}})W_{y,y^{-1}}^{-1}$$

$$
\begin{aligned}
&= W_{y,y^{-1}}(W_{y,y^{-1}}^{-1}x^{\delta}W_{y,y^{-1}})W_{y,y^{-1}}^{-1} = x^{\delta},\\
\alpha_{y^{-1}}(W_{y,y^{-1}}^{-1}\alpha_y(x^{\delta})W_{y,y^{-1}}) &= \alpha_{y^{-1}}(\alpha_y(W_{y^{-1},y}^{-1})\alpha_y(x^{\delta})\alpha_y(W_{y^{-1},y}))\\
&= \alpha_{y^{-1}}(\alpha_y(W_{y^{-1},y}^{-1}x^{\delta}W_{y^{-1},y}))\\
&= W_{y^{-1},y}\alpha_{y^{-1},y}(W_{y^{-1},y}^{-1}x^{\delta}W_{y^{-1},y})W_{y^{-1},y}^{-1}\\
&= W_{y^{-1},y}(W_{y^{-1},y}^{-1}x^{\delta}W_{y^{-1},y})W_{y^{-1},y}^{-1} = x^{\delta}.
\end{aligned}
$$

Since we have

$$
\begin{aligned}
&y^{-1}x^{\delta} = \alpha_{y^{-1}}(W_{y,y^{-1}}^{-1}x^{\delta}W_{y,y^{-1}})y^{-1},\\
&y\alpha_{y^{-1}}(W_{y,y^{-1}}^{-1}x^{\delta}W_{y,y^{-1}}) = \alpha_y(\alpha_{y^{-1}}(W_{y,y^{-1}}^{-1}x^{\delta}W_{y,y^{-1}}))y,
\end{aligned}
$$

$yy^{-1} = 1$ and $\alpha_y(\alpha_{y^{-1}}(W_{y,y^{-1}}^{-1}x^{\delta}W_{y,y^{-1}})) = x^{\delta}$, then we get the spherical pictures in Figure 3. Similarly, by considering $yx^{\delta} = \alpha_y(x^{\delta})y$, $y^{-1}\alpha_y(x^{\delta}) = \alpha_{y^{-1}}(W_{y,y^{-1}}^{-1}\alpha_y(x^{\delta})W_{y,y^{-1}})y^{-1}$, $y^{-1}y = 1$ and $\alpha_{y^{-1}}(W_{y,y^{-1}}^{-1}\alpha_y(x^{\delta})W_{y,y^{-1}}) = x^{\delta}$, we have the spherical pictures in Figure 4.

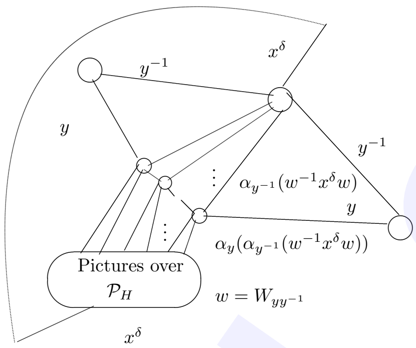

Figure 3

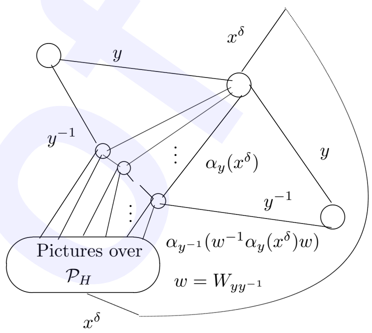

Figure 4

Now, let us think the pairs Sx^{δ}. Here, let us take $S = y_1^{\epsilon_1}y_2^{\epsilon_2}\cdots y_k^{\epsilon_k}$, where $y_i \in \mathbf{y}$ and $\epsilon_i = \pm 1$ for $1 \le i \le k$ for $S \in \mathbf{s}$. Also let us denote

$$
w_i = \begin{cases} W_{y_i,y_i^{-1}}, & \text{if } \epsilon_i = -1,\\ 1, & \text{if } \epsilon_i = 1. \end{cases}
$$

Then we have

$$
Sx^{\delta} = \alpha_{y_1^{\epsilon_1}}(w_1^{-1}\alpha_{y_2^{\epsilon_2}}(w_2^{-1}\cdots\alpha_{y_{k-1}^{\epsilon_{k-1}}}(w_{k-1}^{-1}\alpha_{y_k^{\epsilon_k}}(w_k^{-1}x^{\delta}w_k)w_{k-1})\cdots w_2)w_1)S.
$$

Let us denote

$$
T = \alpha_{y_1^{\epsilon_1}}(w_1^{-1}\alpha_{y_2^{\epsilon_2}}(w_2^{-1}\cdots\alpha_{y_{k-1}^{\epsilon_{k-1}}}(w_{k-1}^{-1}\alpha_{y_k^{\epsilon_k}}(w_k^{-1}x^{\delta}w_k)w_{k-1})\cdots w_2)w_1).
$$

Since we have $W_S\alpha_S(x^{\delta}) = Sx^{\delta} = TS = TW_S$ and $\alpha_S(x^{\delta}) = x^{\delta}$, then we get $TW_S = W_S\alpha_S(x^{\delta}) = W_Sx^{\delta}$. Thus we get the spherical pictures in Figure 5.

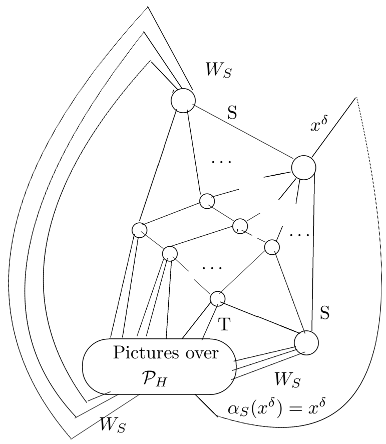

Figure 5

Let us consider the pairs yW_S and $y^{-1}W_S$. For each $y \in \mathbf{y}$ and $S \in \mathbf{s}$, let us suppose that $\alpha_y(W_S) = W_S$. Therefore, since we have $\alpha_{y^{-1}}(\alpha_y(W_S)) = W_{y^{-1},y}\alpha_{y^{-1},y}(W_S)W^{-1}_{y^{-1},y}$ by (1), we get $\alpha_{y^{-1}}(W_S) = W_{y^{-1},y}W_S W^{-1}_{y^{-1},y}$. From this we have

$$W_S = W^{-1}_{y^{-1},y}\alpha_{y^{-1}}(W_S)W_{y^{-1},y} = \alpha_{y^{-1}}(W^{-1}_{y,y^{-1}})\alpha_{y^{-1}}(W_S)\alpha_{y^{-1}}(W_{y,y^{-1}})$$
$$= \alpha_{y^{-1}}(W^{-1}_{y,y^{-1}}W_S W_{y,y^{-1}})$$

by (8).

Therefore we find

$$y\alpha_{y^{-1}}(W^{-1}_{y,y^{-1}}W_S W_{y,y^{-1}}) = \alpha_y(\alpha_{y^{-1}}(W^{-1}_{y,y^{-1}}W_S W_{y,y^{-1}}))y = W_S y$$

and $y^{-1}\alpha_y(W_S) = \alpha_{y^{-1}}(W^{-1}_{y,y^{-1}}\alpha_y(W_S)W_{y,y^{-1}})y^{-1} = W_S y^{-1}$. Thus since we have $S = W_S$, $yW_S = \alpha_y(W_S)y$ and $W_S y = y\alpha_{y^{-1}}(W^{-1}_{y,y^{-1}}W_S W_{y,y^{-1}}) = yW_S$, then we get the spherical pictures in Figure 6. Similarly, by considering $y^{-1}W_S = \alpha_{y^{-1}}(W^{-1}_{y,y^{-1}}W_S W_{y,y^{-1}})y^{-1} = W_S y^{-1}$, thus we get the spherical pictures in Figure 7.

Finally, let us think the pairs yR and $y^{-1}R$. Here we have $yR = \alpha_y(R)y$, $y^{-1}R = \alpha_{y^{-1}}(W^{-1}_{y,y^{-1}}RW_{y,y^{-1}})y^{-1}$, $\alpha_y(R) = 1$ and $\alpha_{y^{-1}}(W^{-1}_{y,y^{-1}}RW_{y,y^{-1}}) = 1$. Then we get the spherical pictures in Figure 8 and Figure 9.

We recall that the second homotopy groups $\pi_2(\mathcal{P}_H)$ and $\pi_2(\mathcal{P}_G)$ are the collections of the spherical pictures over the groups H and G. So the second homotopy group $\pi_2(\mathcal{P}_{H\#^f_\alpha G})$ consists of the spherical pictures given in Figures 1-9 and the pictures in $\pi_2(\mathcal{P}_H)$ and $\pi_2(\mathcal{P}_G)$. □

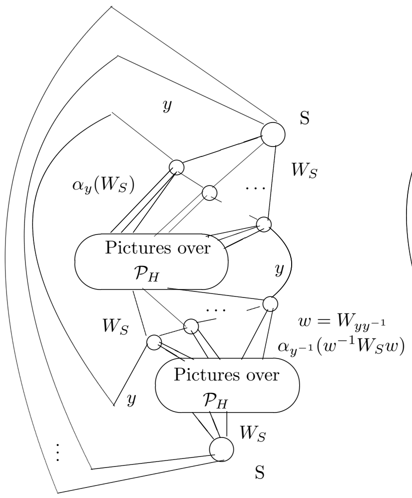

Figure 6

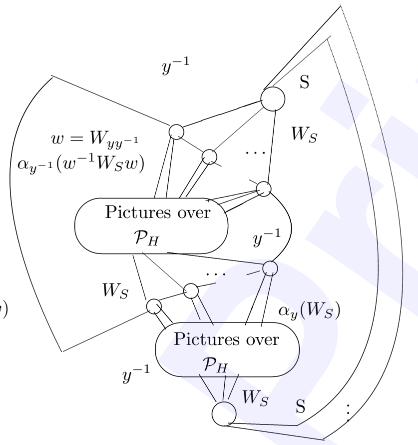

Figure 7

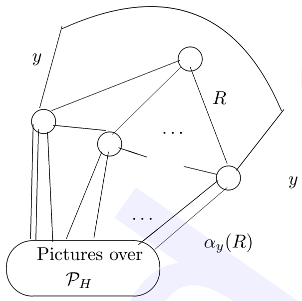

Figure 8

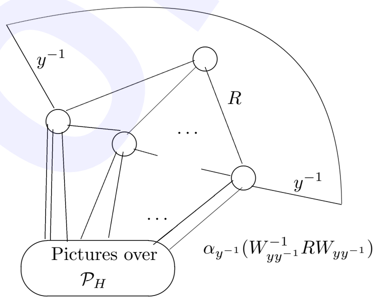

Figure 9

For each $y \in \mathbf{y}$ and $S \in \mathbf{s}$, let us have $\alpha_y(W_S) = W_S$. Therefore we can give the following corollary.

Corollary 3.2. *A presentation of the form $\mathcal{P}_{H\#_\alpha^f G}$ represents a crossed product if and only if the diagrams given in Figures* 1-9 *are spherical.*

Proof. Let $\mathcal{P}_{H\#_\alpha^f G}$ represents a crossed product. Since we have $\alpha_y(W_S) = W_S$, the generating pictures of $H\#_\alpha^f G$ consist of the pictures in Figures 1-9. All these pictures must be spherical. This implies that the diagrams given in Figures 1-9 must be spherical.

Conversely, since the diagrams given in Figures 1-9 are spherical and the generating pictures of $H\#_{\alpha}^{f}G$ consist of them with the spherical pictures in $\pi_2(\mathcal{P}_H)$ and $\pi_2(\mathcal{P}_G)$, then the natural map $H \longrightarrow H\#_{\alpha}^{f}G$ becomes injective, by [5,12]. This says that presentation $\mathcal{P}_{H\#_{\alpha}^{f}G}$ defines an extension of H by G. Therefore, we get our result. □

Let us think that f is a trivial map and α is a homomorphism in $H\#_{\alpha}^{f}G$. Then we recall that $H\#_{\alpha}^{f}G$ is isomorphic to semidirect product of H and G. Since we have $W_S = 1$, then we get $\alpha_y(W_S) = W_S$. So we get the following corollary. This corollary can be found in [9,10,18].

Theorem 3.3. *The second homotopy group of semidirect product of the group H by G is generated by the pictures given in Figure* 5-a *and Figure* 8-a *and the pictures in $\pi_2(\mathcal{P}_H)$, $\pi_2(\mathcal{P}_G)$.*

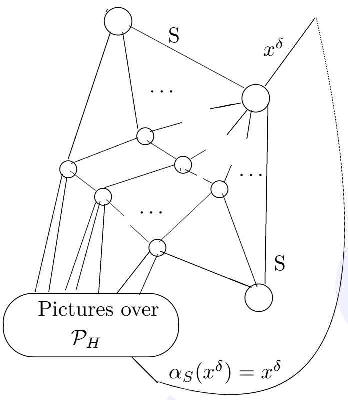

Figure 5-a

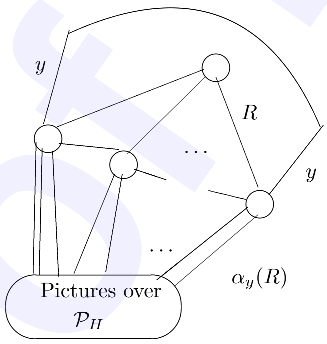

Figure 8-a

4. Some applications

In this section, we will present some applications of the main theorems. Throughout this section some notations will be used as in the previous ones.

Let H be a cyclic group with the presentation

$$\mathcal{P}_H = \langle x \; ; \; x^n = 1 \rangle$$

and let G be an abelian group with the presentation

$$\mathcal{P}_G = \left\langle y_1, y_2, \ldots, y_k \; ; \; y_i^{m_i} = 1, \, y_j y_r y_j^{-1} y_r^{-1} = 1 \right\rangle,$$

where $1 \leq i \leq k$,$1 \leq j < r \leq k$ and n, m_i is a non-negative integer. When $n = m_i = 0$ the order of x and y_i are infinite.

Then by Theorem 2.1, we have

$$\mathcal{P}_{H\#_{\alpha}^{f}G} = \langle \, x, y_1, y_2, \ldots, y_k; \, x^n = 1,$$

$$y_i^{m_i} = x^{a_i}, \;\; y_j y_r y_j^{-1} y_r^{-1} = x^{b_{jr}},$$
$$y_i x = x^{c_i} y_i, \;\; y_i^{-1} x = x^{d_i} y_i^{-1},$$
$$y_i x^{-1} = x^{-c_i} y_i, \;\; y_i^{-1} x^{-1} = x^{-d_i} y_i^{-1} \,\rangle, \tag{10}$$

where $a_i, b_{jr}, c_i, d_i \in \{0, 1, \ldots, n-1\}$.

Thus we have the following theorem.

Theorem 4.1. *The presentation of the form $\mathcal{P}_{H\#_\alpha^f G}$ in* (10) *represents a crossed product $H\#_\alpha^f G$ if and only if we have $c_i^{m_i} \equiv 1 (mod\ n)$, $d_i^{m_i} \equiv 1 (mod\ n)$, $a_i(c_t - 1) \equiv 0 : (mod\ n)$, $a_i(d_t - 1) \equiv 0\ (mod\ n)$, $b_{jr}(c_t - 1) \equiv 0\ (mod\ n)$, $b_{jr}(d_t - 1) \equiv 0\,(mod\ n)$ and $c_t d_t \equiv 1\,(mod\ n)$, where $1 \leq i, t \leq k$, $1 \leq j < r \leq k$ and $a_i, b_{jr}, c_i, d_i, c_t, d_t \in \{0, 1, \ldots, n-1\}$.*

Proof. Assume that $\mathcal{P}_{H\#_\alpha^f G}$ represents a crossed product. Thus the diagrams Figures 1-9 must be spherical by Corollary 3.2. We can define these diagrams as follows. Let us take $1 \leq i, t \leq k$, $1 \leq j < r \leq k$ and $a_i, b_{jr}, c_i, d_i, c_t, d_t \in \{0, 1, \ldots, n-1\}$.

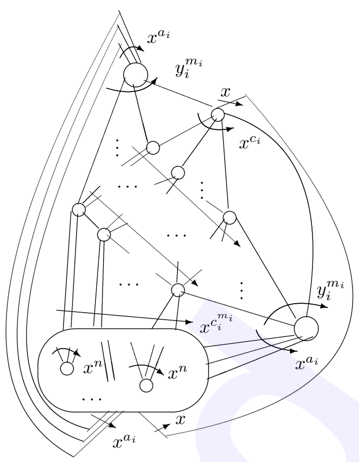

Figure 10

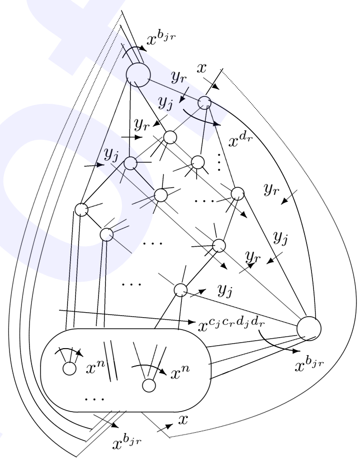

Figure 11

Since we have the relations $y_i^{m_i} = x^{a_i}$ and $y_j y_r y_j^{-1} y_r^{-1} = x^{b_{jr}}$, we get the Figure 10 and Figure 11. Thus in order to be spherical these pictures, we get $c_i^{m_i} \equiv 1$ (mod n) and $c_j c_r d_j d_r \equiv 1$ (mod n). We have also the relations $y_i^{-m_i} = x^{-a_i}$ and $y_r y_j y_r^{-1} y_j^{-1} = x^{-b_{jr}}$. By using them, we get similar pictures given in Figure 10 and Figure 11. So we have $d_i^{m_i} \equiv 1$ (mod n) and $c_r c_j d_r d_j \equiv 1$ (mod n). Also in Figure 10 and Figure 11, when we take x^{-1} instead of x, we get same results.

Let us see the Figure 12 and Figure 13. Here we get $a_i(c_t - 1) \equiv 0$ (mod n), $a_i(d_t - 1) \equiv 0$ (mod n), $b_{jr}(c_t - 1) \equiv 0$ (mod n) and $b_{jr}(d_t - 1) \equiv 0$ (mod n) for doing spherical. For the relations $y_i^{-m_i} = x^{-a_i}$ and $y_r y_j y_r^{-1} y_j^{-1} = x^{-b_{jr}}$, we get also similar pictures given in Figure 12 and Figure 13. Moreover when we take y_t^{-1} instead of y_t in these pictures, we also reach same results.

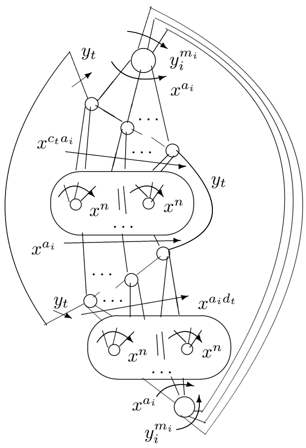

Figure 12

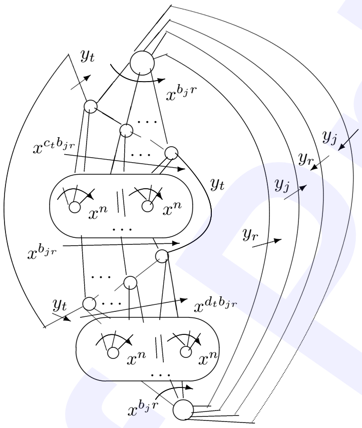

Figure 13

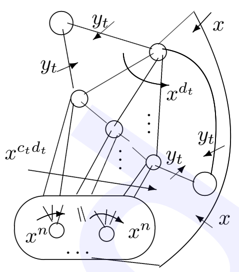

Figure 14

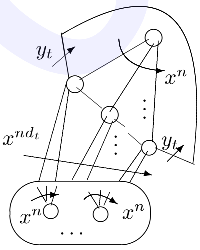

Figure 15

Now let us think the Figure 14 and Figure 15. Here we get $c_t d_t \equiv 1 (\text{mod } n)$. Also in these figures, when we take x^{-1} instead of x and y_t^{-1} instead of y_t, we get same results. In Figure 11, we get the condition $c_r c_j d_r d_j \equiv 1\ (\text{mod } n)$. Here, since we have $c_t d_t \equiv 1 (\text{mod } n)$, then we already have $c_r c_j d_r d_j \equiv 1 (\text{mod } n)$. □

Example 4.2. Let us take $n = 26$, $k = 2$, $m_1 = 3$, $m_2 = 0$, $c_1 = 3$, $c_2 = -1$, $d_1 = 9$, $d_2 = -1$, $a_1 = 13$, $a_2 = 0$ and $b_{12} = 13$. Then

$$\mathcal{P}_{H\#_\alpha^f G} = \langle x, y_1, y_2;\ x^{26} = 1,\ \ y_1^3 = x^{13},\ \ y_1 y_2 y_1^{-1} y_2^{-1} = x^{13},$$

$$y_1 x = x^3 y_1, \;\; y_1^{-1} x = x^9 y_1^{-1}, \;\; y_2 x = x^{-1} y_2\rangle$$

defines the group presentation of the crossed product of the cyclic group generated by x with the order 26 by the direct product of cyclic group generated by y_1 with the order 3 and infinite cyclic group generated by y_2 by above theorem.

We stated about the extension problem in Note 1.1. The following corollary is the first notable result regarding the extension problem which was given by O. L. Holder [13].

Corollary 4.3 (Holder)**.** *Let H and G be finite cyclic groups with the orders n and m, respectively. A finite group E is isomorphic to a crossed product $H\#_{\alpha}^{f}G$ if and only if E is the group generated by two generators x and y subject to the relations*

$$x^n = 1, \;\; y^m = x^a, \;\; y^{-1}xy = x^d,$$

where $1 \leq a, d \leq n-1$ such that $a(d-1) \equiv 0 \, (mod\ n)$ and $d^m \equiv 1 \, (mod\ n)$.

Proof. If we take $k = 1$, $m = m_i$, $a = a_i$, $d = d_i$ and $y = y_i$ in Theorem 4.1, then we get the result by Corollary 3.2. □

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Firat Ateş
Department of Mathematics
Balikesir University
Balikesir 10145, Turkey
Email address: firat@balikesir.edu.tr

Ahmet Sinan Çevik
Department of Mathematics
Selçuk University
Konya 42075, Turkey
Email address: sinan.cevik@selcuk.edu.tr

Eylem Güzel Karpuz
Department of Mathematics
Karamanoğlu Mehmetbey University
Karaman 70100, Turkey
Email address: eylemguzel45@hotmail.com, eylem.guzel@kmu.edu.tr