Bull. Korean Math. Soc. **0** (0), No. 0, pp. 1–0
https://doi.org/10.4134/BKMS.b200816
pISSN: 1015-8634 / eISSN: 2234-3016

On ϕ-w-flat modules and their homological dimensions

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Abstract. In this paper, we introduce and study the class of ϕ-w-flat modules which are generalizations of both ϕ-flat modules and w-flat modules. The ϕ-w-weak global dimension ϕ-w-w.gl.dim(R) of a commutative ring R is also introduced and studied. We show that, for a ϕ-ring R, ϕ-w-w.gl.dim$(R) = 0$ if and only if w-dim$(R) = 0$ if and only if R is a ϕ-von Neumann ring. It is also proved that, for a strongly ϕ-ring R, ϕ-w-w.gl.dim$(R) \leq 1$ if and only if each nonnil ideal of R is ϕ-w-flat, if and only if R is a ϕ-PvMR, if and only if R is a PvMR.

Throughout this paper, R denotes a commutative ring with $1 \neq 0$ and all modules are unitary. We denote by $\mathrm{Nil}(R)$ the nilpotent radical of R, $\mathrm{Z}(R)$ the set of all zero-divisors of R and $\mathrm{T}(R)$ the localization of R at the set of all regular elements. The R-submodules I of $\mathrm{T}(R)$ such that $sI \subseteq R$ for some regular element s are said to be *fractional ideals*. Recall from [3] that a ring R is an NP-*ring* if $\mathrm{Nil}(R)$ is a prime ideal, and a ZN-*ring* if $\mathrm{Z}(R) = \mathrm{Nil}(R)$. A prime ideal P is said to be *divided prime* if $P \subsetneq (x)$ for every $x \in R - P$. Set $\mathcal{H} = \{R \,|\, R$ is a commutative ring and Nil(R) is a divided prime ideal of $R\}$. A ring R is a ϕ-*ring* if $R \in \mathcal{H}$. Moreover, a ZN ϕ-ring is said to be a *strongly ϕ-ring*. For a ϕ-ring R, there is a ring homomorphism $\phi : \mathrm{T}(R) \to R_{\mathrm{Nil}(R)}$ such that $\phi(a/b) = a/b$ where $a \in R$ and b is a regular element. Denote by the ring $\phi(R)$ the image of ϕ restricted to R. In 2001, Badawi [4] investigated ϕ-chain rings (ϕ-CRs for short) and ϕ-pseudo-valuation rings as a ϕ-version of chain rings and pseudo-valuation rings. In 2004, Anderson and Badawi [1] introduced the concept of ϕ-Prüfer rings and showed that a ϕ-ring R is ϕ-Prüfer if and only if $R_{\mathfrak{m}}$ is a ϕ-chain ring for any maximal ideal $\mathfrak{m}$ of R if and only if $R/\mathrm{Nil}(R)$ is a Prüfer domain if and only if $\phi(R)$ is Prüfer. Later, the authors in [2, 5] generalized the concepts of Dedekind domains, Krull domains and Mori domains to the context of rings that are in the class $\mathcal{H}$. In 2013, Zhao et al. [19] introduced and studied the conceptions of ϕ-flat modules and ϕ-von Neumann rings and obtained that a ϕ-ring is ϕ-von Neumann if and only if its Krull dimension is 0. Recently, Zhao [18] gave a homological characterization

Received September 23, 2020; Revised January 16, 2021; Accepted January 29, 2021.
2010 *Mathematics Subject Classification.* Primary 13A15; Secondary 13F05.
Key words and phrases. ϕ-w-flat module, ϕ-w-weak global dimension, ϕ-von Neumann ring, ϕ-PvMR.

of ϕ-Prüfer rings as follows: a strongly ϕ-ring R is ϕ-Prüfer, if and only if each submodule of a ϕ-flat module is ϕ-flat, if and only if each nonnil ideal of R is ϕ-flat.

Some other important generalizations of classical notions are their w-versions. In 1997, Wang and McCasland [15] introduced the w-modules over strong Mori domains (SM domains for short) which can be seen as a w-version of Noetherian domains. In 2011, Yin *et al.* [17] extended w-theories to commutative rings containing zero divisors. The notion of w-flat modules appeared first in [10] for integral domains and was extended to arbitrary commutative rings in [13]. In 2012, Kim and Wang [7] introduced ϕ-SM rings which can be seen as both a ϕ-version and a w-version of Noetherian domains and obtained that a ϕ-ring R is ϕ-SM if and only if $R/\mathrm{Nil}(R)$ is an SM domain if and only if $\phi(R)$ is an SM ring. In 2014, Wang and Kim [12] introduced w-w.gl.dim(R) as a generalization of the classical weak global dimension and obtained that a ring R is a von Neumann ring if and only if each R-module is w-flat, i.e., w-w.gl.dim$(R) = 0$. In 2015, Wang and Qiao [16] studied several properties of the w-weak global dimension, and proved that an integral domain R is a Prüfer v-multiplication domain (PvMD for short) if and only if w-w.gl.dim$(R) \leq 1$ if and only if $R_{\mathfrak{m}}$ is a valuation domain for any maximal w-ideal $\mathfrak{m}$ of R. As ϕ-rings are natural extensions of integral domains, we introduce and study the ϕ-versions of w-flat modules, von Neumann rings and PvMDs in this article. As our work involves w-theories, we give a review as below.

Let R be a commutative ring and J a finitely generated ideal of R. Then J is called a GV-*ideal* if the natural homomorphism $R \to \mathrm{Hom}_R(J, R)$ is an isomorphism. The set of all GV-ideals is denoted by $\mathrm{GV}(R)$. An R-module M is said to be GV-*torsion* if for any $x \in M$ there is a GV-ideal J such that $Jx = 0$; an R-module M is said to be GV-*torsion free* if $Jx = 0$, then $x = 0$ for any $J \in \mathrm{GV}(R)$ and $x \in M$. A GV-torsion free module M is said to be a w-*module* if for any $x \in E(M)$ there is a GV-ideal J such that $Jx \subseteq M$ where $E(M)$ is the injective envelope of M. The w-*envelope* M_w of a GV-torsion free module M is defined by the minimal w-module that contains M. Therefore, a GV-torsion free module M is a w-module if and only if $M_w = M$. A *maximal* w-*ideal* for which is maximal among the w-submodules of R is proved to be prime (see [17, Proposition 3.8]). The set of all maximal w-ideals is denoted by w-Max(R). The w-*dimension* w-dim(R) of a ring R is defined to be the supremum of the heights of all maximal w-ideals.

An R-homomorphism $f : M \to N$ is said to be a w-*monomorphism* (resp., w-*epimorphism*, w-*isomorphism*) if for any $p \in w$-Max(R), $f_p : M_p \to N_p$ is a monomorphism (resp., an epimorphism, an isomorphism). Note that f is a w-monomorphism (resp., w-epimorphism) if and only if $\mathrm{Ker}(f)$ (resp., $\mathrm{Coker}(f)$) is GV-torsion. A sequence $A \to B \to C$ is said to be w-*exact* if for any $p \in w$-Max(R), $A_p \to B_p \to C_p$ is exact. A class $\mathcal{C}$ of R-modules is said to be *closed under* w-*isomorphisms* provided that for any w-isomorphism $f : M \to N$, if one of the modules M and N is in $\mathcal{C}$, so is the other. An R-module M is said to be of *finite type* if there exist a finitely generated free module F and a w-epimorphism $g : F \to M$, or equivalently, if there exists a finitely generated R-submodule N of M such that $N_w = M_w$. Certainly, the class of finite type modules is closed under w-isomorphisms. Now we proceed to introduce the notion of ϕ-w-flat modules.

1. ϕ-w-flat modules

We say an ideal I of R is nonnil provided that there is a non-nilpotent element in I. Denote by $\mathrm{NN}(R)$ the set of all nonnil ideals of R. Certainly, GV-ideals are nonnil. Let R be an NP-ring. It is easy to verify that $\mathrm{NN}(R)$ is a multiplicative system of ideals. That is $R \in \mathrm{NN}(R)$ and for any $I \in \mathrm{NN}(R)$, $J \in \mathrm{NN}(R)$, we have $IJ \in \mathrm{NN}(R)$. Let M be an R-module. Define

$$\phi\text{-}tor(M) = \{x \in M \mid Ix = 0 \text{ for some } I \in \mathrm{NN}(R)\}.$$

An R-module M is said to be *ϕ-torsion* (resp., *ϕ-torsion free*) provided that ϕ-$tor(M) = M$ (resp., ϕ-$tor(M) = 0$). Clearly, if R is an NP-ring, the class of ϕ-torsion modules is closed under submodules, quotients, direct sums and direct limits. Thus an NP-ring R is ϕ-torsion free if and only if every flat module is ϕ-torsion free if and only if R is a ZN-ring (see [18, Proposition 2.2]). The classes of ϕ-torsion modules and ϕ-torsion free modules constitute a hereditary torsion theory of finite type. For more details, refer to [9].

Lemma 1.1. *Let R be an* NP*-ring, $\mathfrak{m}$ a maximal w-ideal of R and I an ideal of R. Then $I \in \mathrm{NN}(R)$ if and only if $I_\mathfrak{m} \in \mathrm{NN}(R_\mathfrak{m})$.*

Proof. Let $I \in \mathrm{NN}(R)$ and x a non-nilpotent element in I. We will show the element $x/1$ in $I_\mathfrak{m}$ is a non-nilpotent element of $R_\mathfrak{m}$. If $(x/1)^n = x^n/1 = 0$ in $R_\mathfrak{m}$ for some positive integer n, there is an $s \in R - \mathfrak{m}$ such that $sx^n = 0$ in R. Since R is an NP-ring, $\mathrm{Nil}(R)$ is the minimal prime w-ideal of R. In the integral domain $R/\mathrm{Nil}(R)$, we have $\overline{s}\overline{x^n} = \overline{0}$, thus $\overline{x^n} = \overline{0}$ since $s \notin \mathrm{Nil}(R)$. So $x \in \mathrm{Nil}(R)$, a contradiction.

Let x/s be a non-nilpotent element in $I_\mathfrak{m}$ where $x \in I$ and $s \in R - \mathfrak{m}$. Clearly, x is non-nilpotent and thus $I \in \mathrm{NN}(R)$. □

Proposition 1.2. *Let R be an* NP*-ring, $\mathfrak{m}$ a maximal w-ideal of R and M an R-module. Then M is ϕ-torsion over R if and only $M_\mathfrak{m}$ is ϕ-torsion over $R_\mathfrak{m}$.*

Proof. Let M be an R-module and $x \in M$. If $M_\mathfrak{m}$ is ϕ-torsion over $R_\mathfrak{m}$, there is an ideal $I_\mathfrak{m} \in \mathrm{NN}(R_\mathfrak{m})$ such that $I_\mathfrak{m} x/1 = 0$ in $R_\mathfrak{m}$. Let I be the preimage of $I_\mathfrak{m}$ in R. Then I is nonnil by Lemma 1.1. Thus there is a non-nilpotent element $t \in I$ such that $tkx = 0$ for some $k \notin m$. Let $s = tk$. Then we have $(s) \in \mathrm{NN}(R)$ and $(s)x = 0$. Thus M is ϕ-torsion. Suppose M is ϕ-torsion over R. Let x/s be an element in $M_\mathfrak{m}$. Then there is an ideal $I \in \mathrm{NN}(R)$ such that $Ix = 0$, and thus $I_\mathfrak{m} x/s = 0$ with $I_\mathfrak{m} \in \mathrm{Nil}(R_\mathfrak{m})$ by Lemma 1.1. It follows that $M_\mathfrak{m}$ is ϕ-torsion over $R_\mathfrak{m}$. □

Recall from [13] that an R-module M is said to be *w-flat* if for any w-monomorphism $f : A \to B$, the induced sequence $f \otimes_R 1 : A \otimes_R M \to B \otimes_R M$ is also a w-monomorphism. Obviously, GV-torsion modules and flat modules are all w-flat. It was proved that the class of w-flat modules is closed under w-isomorphisms (see [14, Corollary 6.7.4]). Following [19, Definition 3.1], an R-module M is said to be *ϕ-flat* if for every monomorphism $f : A \to B$ with $\mathrm{Coker}(f)$ ϕ-torsion, $f \otimes_R 1 : A \otimes_R M \to B \otimes_R M$ is a monomorphism. Obviously flat modules are both ϕ-flat and w-flat. Now we give a generalization of both ϕ-flat modules and w-flat modules.

Definition 1.3. Let R be a ring. An R-module M is said to be *ϕ-w-flat* if for every monomorphism $f : A \to B$ with $\mathrm{Coker}(f)$ ϕ-torsion, $f \otimes_R 1 : A \otimes_R M \to B \otimes_R M$ is a w-monomorphism; equivalently, if $0 \to A \to B \to C \to 0$ is an exact exact sequence with C ϕ-torsion, then $0 \to A \otimes_R M \to B \otimes_R M \to C \otimes_R M \to 0$ is w-exact.

Clearly ϕ-flat modules and w-flat modules are ϕ-w-flat. It is well known that an R-module M is flat if and only if the induced homomorphism $1 \otimes_R f : M \otimes_R I \to M \otimes_R R$ is exact for any (finitely generated) ideal I, if and only if the multiplication homomorphism $i : I \otimes_R M \to IM$ is an isomorphism for any (finitely generated) ideal I, if and only if $\mathrm{Tor}_1^R(R/I, M) = 0$ for any (finitely generated) ideal I of R. Some similar characterizations of w-flat modules and ϕ-flat modules are given in [12, Proposition 1.1] and [19, Theorem 3.2], respectively. We can also obtain some similar characterizations of ϕ-w-flat modules.

Theorem 1.4. *Let R be an* NP*-ring. The following statements are equivalent for an R-module M:*

1. *M is ϕ-w-flat;*
2. *$M_{\mathfrak{m}}$ is ϕ-flat over $R_{\mathfrak{m}}$ for all $\mathfrak{m} \in w$-$Max(R)$;*
3. *$\mathrm{Tor}_1^R(T, M)$ is* GV*-torsion for all (finite type) ϕ-torsion R-modules T;*
4. *$\mathrm{Tor}_1^R(R/I, M)$ is* GV*-torsion for all (finite type) nonnil ideals I of R;*
5. *$f \otimes_R 1 : I \otimes_R M \to R \otimes_R M$ is w-exact for all (finite type) nonnil ideals I of R;*
6. *the multiplication homomorphism $i : I \otimes_R M \to IM$ is a w-isomorphism for all (finite type) ideals I;*
7. *let $0 \to K \to F \to M \to 0$ be an exact sequence of R-modules, where F is free. Then $(K \cap FI)_w = (IK)_w$ for all (finite type) nonnil ideals I of R.*

Proof. (1) $\Rightarrow$ (2): Let $\mathfrak{m}$ be a maximal w-ideal of R, $f : A_{\mathfrak{m}} \to B_{\mathfrak{m}}$ an $R_{\mathfrak{m}}$-homomorphism with $\mathrm{Coker}(f)$ ϕ-torsion over $R_{\mathfrak{m}}$. Then $\mathrm{Coker}(f)$ is ϕ-torsion over R by Proposition 1.2. It follows that $f \otimes_R M : A_{\mathfrak{m}} \otimes_R M \to B_{\mathfrak{m}} \otimes_R M$ is a w-monomorphism over R. Localizing at $\mathfrak{m}$, we have $f \otimes_{R_{\mathfrak{m}}} M_{\mathfrak{m}} : A_{\mathfrak{m}} \otimes_{R_{\mathfrak{m}}} M_{\mathfrak{m}} \to B_{\mathfrak{m}} \otimes_{R_{\mathfrak{m}}} M_{\mathfrak{m}}$ is a monomorphism over $R_{\mathfrak{m}}$ since $N_{\mathfrak{m}} \otimes_R M_{\mathfrak{m}} \cong N_{\mathfrak{m}} \otimes_{R_{\mathfrak{m}}} M_{\mathfrak{m}}$ for any R-module N. It follows that $M_{\mathfrak{m}}$ is ϕ-flat over $R_{\mathfrak{m}}$.

(2) $\Rightarrow$ (1): Let $f: A \to B$ be a monomorphism with $\mathrm{Coker}(f)$ ϕ-torsion. For any $\mathfrak{m} \in w\text{-Max}(R)$, we have $f_{\mathfrak{m}}: A_{\mathfrak{m}} \to B_{\mathfrak{m}}$ is a monomorphism with $\mathrm{coker}(f_{\mathfrak{m}})$ ϕ-torsion over $R_{\mathfrak{m}}$ by Proposition 1.2. Since $M_{\mathfrak{m}}$ is ϕ-flat over $R_{\mathfrak{m}}$, $f_{\mathfrak{m}} \otimes_{R_{\mathfrak{m}}} M_{\mathfrak{m}} : A_{\mathfrak{m}} \otimes_{R_{\mathfrak{m}}} M_{\mathfrak{m}} \to B_{\mathfrak{m}} \otimes_{R_{\mathfrak{m}}} M_{\mathfrak{m}}$ is a monomorphism. Thus $f \otimes_R M : A \otimes_R M \to B \otimes_R M$ is a w-monomorphism. Consequently, M is ϕ-w-flat.

The equivalences of (2)-(7) hold from [19, Theorem 3.2] by localizing at all maximal w-ideals. □

Corollary 1.5. *Let R be an* NP*-ring. The class of ϕ-w-flat modules is closed under w-isomorphisms.*

Proof. Let $f: M \to N$ be a w-isomorphism and T a ϕ-torsion module. There exist two exact sequences $0 \to T_1 \to M \to L \to 0$ and $0 \to L \to N \to T_2 \to 0$ with T_1 and T_2 GV-torsion. Considering the induced two long exact sequences $\mathrm{Tor}_1^R(T, T_1) \to \mathrm{Tor}_1^R(T, M) \to \mathrm{Tor}_1^R(T, L) \to T \otimes T_1$ and $\mathrm{Tor}_2^R(T, T_2) \to \mathrm{Tor}_1^R(T, L) \to \mathrm{Tor}_1^R(T, N) \to \mathrm{Tor}_1^R(T, T_2)$, we have M is ϕ-w-flat if and only if N is ϕ-w-flat by Theorem 1.4. □

Lemma 1.6. *Let R be a ϕ-ring and I a nonnil ideal of R. Then* $\mathrm{Nil}(R) = I\mathrm{Nil}(R)$.

Proof. Let I be a nonnil ideal of R with a non-nilpotent element $s \in I$. Then $\mathrm{Nil}(R) \subseteq (s)$. Thus for any $a \in \mathrm{Nil}(R)$, there exists $b \in R$ such that $a = sb$. Thus $\overline{a} = \overline{s}\overline{b}$ in the integral domain $R/\mathrm{Nil}(R)$. Since $\overline{a} = 0$ and $\overline{s} \neq 0$, we have $\overline{b} = 0$. So $b \in \mathrm{Nil}(R)$ and then $\mathrm{Nil}(R) \subseteq s\mathrm{Nil}(R) \subseteq I\mathrm{Nil}(R) \subseteq \mathrm{Nil}(R)$. It follows that $\mathrm{Nil}(R) = I\mathrm{Nil}(R)$. □

Proposition 1.7. *Let R be a ϕ-ring and M an R-module. Then $M/\mathrm{Nil}(R)M$ is ϕ-flat over R if and only if $M/\mathrm{Nil}(R)M$ is flat over $R/\mathrm{Nil}(R)$. Consequently, $R/\mathrm{Nil}(R)$ is always ϕ-flat over R.*

Proof. For the "only if" part, let $\overline{I} = I/\mathrm{Nil}(R)$ be an ideal of $\overline{R} = R/\mathrm{Nil}(R)$. If $\overline{I}$ is zero, certainly $\mathrm{Tor}_1^{\overline{R}}(\overline{R}/\overline{I}, M/\mathrm{Nil}(R)M) = 0$. Let $\overline{I}$ be a non-zero ideal of $\overline{R}$ with $I \in \mathrm{NN}(R)$. Since $M/\mathrm{Nil}(R)M$ is ϕ-flat over R,

$$\mathrm{Tor}_1^R(R/I, M/\mathrm{Nil}(R)M) = 0.$$

By Lemma 1.6,

$$\mathrm{Tor}_1^R(R/\mathrm{Nil}(R), R/I) \cong I \cap \mathrm{Nil}(R)/I\mathrm{Nil}(R) = \mathrm{Nil}(R)/I\mathrm{Nil}(R) = 0.$$

We have $\mathrm{Tor}_1^{\overline{R}}(\overline{R}/\overline{I}, M/\mathrm{Nil}(R)M) \cong \mathrm{Tor}_1^R(R/I, M/\mathrm{Nil}(R)M) = 0$ by change of rings.

For the "if" part, let I be a nonnil ideal of R. Similarly to the proof of "only if" part, since $\mathrm{Tor}_1^R(R/\mathrm{Nil}(R), R/I) = 0$, we have $\mathrm{Tor}_1^R(R/I, M/\mathrm{Nil}(R)M) \cong \mathrm{Tor}_1^{\overline{R}}(\overline{R}/\overline{I}, M/\mathrm{Nil}(R)M) = 0$. It follows that $M/\mathrm{Nil}(R)M$ is ϕ-flat over R. □

By localizing at all maximal w-ideals, we obtain the following corollary.

Corollary 1.8. *Let R be a ϕ-ring and M an R-module. Then $M/\mathrm{Nil}(R)M$ is ϕ-w-flat over R if and only if $M/\mathrm{Nil}(R)M$ is w-flat over $R/\mathrm{Nil}(R)$.*

Proof. See Proposition 1.7, Theorem 1.4 and [8, Theorem 3.3]. □

Certainly if R is an integral domain, every ϕ-w-flat module is w-flat. Conversely, this property characterizes integral domains.

Theorem 1.9. *The following statements are equivalent for a ϕ-ring R:*

(1) *R is an integral domain;*
(2) *every ϕ-w-flat module is w-flat;*
(3) *every ϕ-flat module is w-flat.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3): Trivial.

(3) $\Rightarrow$ (1): Let s be a nilpotent element of R. Then

$$\mathrm{Tor}_1^R(R/(s), R/\mathrm{Nil}(R)) \cong (s) \cap \mathrm{Nil}(R)/s\mathrm{Nil}(R) = (s)/s\mathrm{Nil}(R)$$

is GV-torsion since $R/\mathrm{Nil}(R)$ is w-flat by (3) and Proposition 1.7. Thus there is a GV-ideal J such that $sJ \subseteq s\mathrm{Nil}(R)$. Since J is a nonnil ideal, $\mathrm{Nil}(R) = J\mathrm{Nil}(R)$ by Lemma 1.6. Thus $sJ \subseteq s\mathrm{Nil}(R) = sJ\mathrm{Nil}(R) \subseteq sJ$. That is, $sJ = sJ\mathrm{Nil}(R)$. Since sJ is finitely generated, $sJ = 0$ by Nakayama's lemma. Since $J \in \mathrm{GV}(R)$, $s \in R$ is GV-torsion free, then $s = 0$. Consequently, $\mathrm{Nil}(R) = 0$ and R is an integral domain. □

Recall from [11] that a ring R is said to be a DW *ring* if every ideal of R is a w-ideal. Then a ring R is a DW ring if and only if every R-module is a w-module, if and only if $\mathrm{GV}(R) = \{R\}$ (see [11, Theorem 3.8]). Certainly if R is a DW ring, every ϕ-w-flat module is ϕ-flat. Conversely, this property characterizes DW rings.

Theorem 1.10. *The following statements are equivalent for an* NP*-ring R:*

(1) *R is a* DW *ring;*
(2) *every ϕ-w-flat module is ϕ-flat;*
(3) *every w-flat module is ϕ-flat.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3): Trivial.

(3) $\Rightarrow$ (1): For any $J \in \mathrm{GV}(R)$, R/J is GV-torsion, and thus w-flat. By (3), R/J is ϕ-flat. Since every GV-ideal J is a nonnil ideal of R, we have $\mathrm{Tor}_1^R(R/J, R/J) \cong J/J^2 = 0$. It follows that J is a finitely generated idempotent ideal, and thus J is projective. So $J = J_w = R$ by [14, Exercise 6.10(1)] and thus R is a DW ring by [14, Theorem 6.3.12]. □

Some non-integral domain examples are provided by the idealization construction $R(+)M$ where M is an R-module (see [6]). We recall this construction. Let $R(+)M = R \oplus M$ as an R-module, and define

(1) $(r, m)+(s, n)=(r + s, m + n)$.
(2) $(r, m)(s, n)=(rs, sm + rn)$.

Under these definitions, $R(+)M$ becomes a commutative ring with identity. Denote by $(0:_R M)$ the set $\{r \in R \,|\, rM = 0\}$. Now we compute some examples of GV-ideals of $R(+)M$.

Proposition 1.11. *Let T be a commutative ring and E a w-module over T such that $(0:_T E) = 0$. Set $R = T(+)E$. Then $J(+)E$ is a* GV*-ideal of R for any $J \in \mathrm{GV}(T)$.*

Proof. Let J be a GV-ideal of T. Then we claim that $J(+)E \in \mathrm{GV}(R)$. Indeed, since $T(+)E/J(+)E \cong T/J$, for any $i = 0, 1$, we have

$$\mathrm{Ext}^i_R(T(+)E/J(+)E, R) \cong \mathrm{Ext}^i_T(T/J, \mathrm{Hom}_R(T, R)).$$

Note that

$$\mathrm{Hom}_R(T, R) = \mathrm{Hom}_R(R/0(+)E, R) \cong 0(+)E \cong E$$

since $(0:_T E) = 0$. Thus $\mathrm{Ext}^i_R(T(+)E/J(+)E, R) \cong \mathrm{Ext}^i_T(T/J, E)$ for any $i = 0, 1$. If $J \in \mathrm{GV}(T)$ then $J(+)E \in \mathrm{GV}(R)$ since E is a w-module over T. □

Now we give an example to show the notion of ϕ-w-flat modules is a strict generalization of ϕ-flat modules and w-flat modules.

Example 1.12. Let D be a non-DW integral domain and K its quotient field. Then $R = D(+)K$ is a ϕ-ring (see [2, Remark 1]). However, by Proposition 1.11, R is neither an integral domain nor a DW ring. Consequently, there is a ϕ-w-flat module over R which is neither ϕ-flat nor w-flat by Theorem 1.9 and Theorem 1.10.

2. Homological properties of ϕ-w-flat modules

Let R be a ring. It is well known that the flat dimension of an R-module M is defined as the shortest flat resolution of M and the weak global dimension of R is the supremum of the flat dimensions of all R-modules. The w-flat dimension w-$\mathrm{fd}_R(M)$ of an R-module M and w-weak global dimension w-w.gl.dim(R) of a ring R were introduced and studied in [16]. We now introduce the notion of ϕ-w-flat dimension of an R-module as follows.

Definition 2.1. Let R be a ring and M an R-module. We write ϕ-w-$\mathrm{fd}_R(M) \leq n$ (ϕ-w-fd abbreviates *ϕ-w-flat dimension*) if there is a w-exact sequence of R-modules

$$(\diamondsuit) \qquad 0 \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0,$$

where each F_i is w-flat for $i = 0, \ldots, n-1$ and F_n is ϕ-w-flat. The w-exact sequence $(\diamondsuit)$ is said to be a ϕ-w-flat w-resolution of length n of M. If such finite w-resolution does not exist, then we say ϕ-w-$\mathrm{fd}_R(M) = \infty$; otherwise, define ϕ-w-$\mathrm{fd}_R(M) = n$ if n is the length of the shortest ϕ-w-flat w-resolution of M.

It is obvious that an R-module M is ϕ-w-flat if and only if ϕ-w-$\mathrm{fd}_R(M) = 0$. Certainly, ϕ-w-$\mathrm{fd}_R(M) \leq w$-$\mathrm{fd}_R(M)$. If R is an integral domain, then ϕ-w-$\mathrm{fd}_R(M) = w$-$\mathrm{fd}_R(M)$.

Lemma 2.2 ([16, Lemma 2.2]). *Let N be an R-module and $0 \to A \to F \to C \to 0$ a w-exact sequence of R-modules with F a w-flat module. Then for any $n > 0$, the induced map $\mathrm{Tor}^R_{n+1}(C, N) \to \mathrm{Tor}^R_n(A, N)$ is a w-isomorphism. Hence, $\mathrm{Tor}^R_{n+1}(C, N)$ is* GV*-torsion if and only if so is $\mathrm{Tor}^R_n(A, N)$.*

Proposition 2.3. *Let R be an* NP*-ring. The following statements are equivalent for an R-module M:*

(1) *ϕ-w-$fd_R(M) \leq n$;*
(2) *$\mathrm{Tor}^R_{n+k}(M, N)$ is* GV*-torsion for all ϕ-torsion R-modules N and all $k > 0$;*
(3) *$\mathrm{Tor}^R_{n+1}(M, N)$ is* GV*-torsion for all ϕ-torsion R-modules N;*
(4) *$\mathrm{Tor}^R_{n+1}(M, R/I)$ is* GV*-torsion for all nonnil ideals I;*
(5) *$\mathrm{Tor}^R_{n+1}(M, R/I)$ is* GV*-torsion for all finite type nonnil ideals I;*
(6) *if $0 \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0$ is an exact sequence, where $F_0, F_1, \ldots, F_{n-1}$ are flat R-modules, then F_n is ϕ-w-flat;*
(7) *if $0 \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0$ is an w-exact sequence, where $F_0, F_1, \ldots, F_{n-1}$ are w-flat R-modules, then F_n is ϕ-w-flat;*
(8) *if $0 \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0$ is an exact sequence, where $F_0, F_1, \ldots, F_{n-1}$ are w-flat R-modules, then F_n is ϕ-w-flat;*
(9) *if $0 \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0$ is an w-exact sequence, where $F_0, F_1, \ldots, F_{n-1}$ are flat R-modules, then F_n is ϕ-w-flat.*

Proof. (1) $\Rightarrow$ (2): We prove (2) by induction on n. For the case $n = 0$, (2) holds by Theorem 1.4 as M is ϕ-w-flat. If $n > 0$, then there is a w-exact sequence $0 \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0$, where each F_i is w-flat for $i = 0, \ldots, n-1$ and F_n is ϕ-w-flat. Set $K_0 = \ker(F_0 \to M)$. Then both $0 \to K_0 \to F_0 \to M \to 0$ and $0 \to F_n \to F_{n-1} \to \cdots \to F_1 \to K_0 \to 0$ are w-exact, and ϕ-w-$\mathrm{fd}_R(K_0) \leq n-1$. By induction, $\mathrm{Tor}^R_{n-1+k}(K_0, N)$ is GV-torsion for all ϕ-torsion R-modules N and all $k > 0$. Thus, it follows from Lemma 2.2 that $\mathrm{Tor}^R_{n+k}(M, N)$ is GV-torsion.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5): Trivial.

(5) $\Rightarrow$ (6): Let $K_0 = \ker(F_0 \to M)$ and $K_i = \ker(F_i \to F_{i-1})$, where $i = 1, \ldots, n-1$. Then $K_{n-1} = F_n$. Since all $F_0, F_1, \ldots, F_{n-1}$ are flat, $\mathrm{Tor}^R_1(F_n, R/I) \cong \mathrm{Tor}^R_{n+1}(M, R/I)$ is GV-torsion for all finite type nonnil ideal I. Hence F_n is a ϕ-w-flat module by Theorem 1.4.

(6) $\Rightarrow$ (1): Obvious.

(3) $\Rightarrow$ (7): Set $L_n = F_n$ and $L_i = \mathrm{Im}(F_i \to F_{i-1})$, where $i = 1, \ldots, n-1$. Then both $0 \to L_{i+1} \to F_i \to L_i \to 0$ and $0 \to L_1 \to F_0 \to M \to 0$ are w-exact sequences. By using Lemma 2.2 repeatedly, we can obtain that $\mathrm{Tor}^R_1(F_n, N)$ is GV-torsion for all ϕ-torsion R-modules N. Thus F_n is ϕ-w-flat.

(7) $\Rightarrow$ (8) $\Rightarrow$ (6), (7) $\Rightarrow$ (9) and (9) $\Rightarrow$ (6): Trivial. □

Definition 2.4. The *ϕ-w-weak global dimension* of a ring R is defined by

$$\phi\text{-}w\text{-w}.gl.\dim(R) = \sup\{w_\phi\text{-}fd_R(M) \mid M \text{ is an } R\text{-module}\}.$$

Obviously, by definition, ϕ-w-w.gl.dim$(R) \le w$-w.gl.dim(R). Notice that if R is an integral domain, then ϕ-w-w.gl.dim$(R) = w$-w.gl.dim(R).

Proposition 2.5. *Let R be an* NP*-ring. The following statements are equivalent for R.*

1. (1) *ϕ-w-fd$_R(M) \le n$ for all R-modules M.*
2. (2) $\mathrm{Tor}^R_{n+k}(M,N)$ *is* GV*-torsion for all R-modules M and ϕ-torsion N and all $k > 0$.*
3. (3) $\mathrm{Tor}^R_{n+1}(M,N)$ *is* GV*-torsion for all R-modules M and ϕ-torsion N.*
4. (4) $\mathrm{Tor}^R_{n+1}(M,R/I)$ *is* GV*-torsion for all R-modules M and nonnil ideals I of R.*
5. (5) $\mathrm{Tor}^R_{n+1}(M,R/I)$ *is* GV*-torsion for all R-modules M and finite type nonnil ideals I of R.*
6. (6) *ϕ-w-fd$_R(R/I) \le n$ for all nonnil ideals I of R.*
7. (7) *ϕ-w-fd$_R(R/I) \le n$ for all finite type nonnil ideals I of R.*
8. (8) *ϕ-w-w.gl.*$\dim(R) \le n$.

Consequently, the ϕ-w-weak global dimension of R is determined by the formulas:

$$\begin{aligned}\phi\text{-}w\text{-}w.gl.\dim(R) &= \sup\{\phi\text{-}w\text{-}fd_R(R/I) \mid I \textit{ is a nonnil ideal of } R\}\\ &= \sup\{\phi\text{-}w\text{-}fd_R(R/I) \mid I \textit{ is a finite type nonnil ideal of } R\}.\end{aligned}$$

Proof. (1) $\Leftrightarrow$ (8) and (1) $\Rightarrow$ (6) $\Rightarrow$ (7) $\Rightarrow$ (8): Trivial.

(1) $\Rightarrow$ (2) and (5) $\Rightarrow$ (1): Follows from Proposition 2.3.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5): Trivial.

(8) $\Rightarrow$ (1): Let M be an R-module and $0 \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0$ an exact sequence, where $F_0, F_1, \dots, F_{n-1}$ are flat R-modules. To complete the proof, it suffices, by Proposition 2.3, to prove that F_n is ϕ-w-flat. Let I be a finite type nonnil ideal of R. Thus ϕ-w-fd$_R(R/I) \le n$ by (8). It follows from Lemma 2.2 that $\mathrm{Tor}^R_1(R/I, F_n) \cong \mathrm{Tor}^R_{n+1}(R/I, M)$ is GV-torsion. □

3. Rings with ϕ-w-weak global dimension at most one

It is well known that a commutative ring R with weak global dimension 0 is exactly a *von Neumann regular ring*, equivalently $a \in (a^2)$ for any $a \in R$. It was proved in [12, Theorem 4.4] that a commutative ring R has w-weak global dimension 0, if and only if $a \in (a^2)_w$ for any $a \in R$, if and only if $R_{\mathfrak{m}}$ is a field for any maximal w-ideal $\mathfrak{m}$ of R, if and only if R is a von Neumann regular ring. Recall from [19] that a ϕ-ring R is said to be *ϕ-von Neumann regular* provided that every R-module is ϕ-flat. A ϕ-ring R is ϕ-von Neumann regular, if and only if there is an element $x \in R$ such that $a = xa^2$ for any non-nilpotent element $a \in R$, if and only if $R/\mathrm{Nil}(R)$ is a von Neumann regular ring, if and only if R is zero-dimensional (see [19, Theorem 4.1]). Now, we give some more characterizations of ϕ-von Neumann regular rings.

Theorem 3.1. *Let R be a ϕ-ring. The following statements are equivalent for R:*

(1) ϕ-w-$w.gl.\dim(R) = 0$;
(2) *every R-module is ϕ-w-flat*;
(3) $a \in (a^2)_w$ *for any non-nilpotent element* $a \in R$;
(4) w-$\dim(R) = 0$;
(5) $\dim(R) = 0$;
(6) *R is ϕ-von Neumann regular.*

Proof. (1) $\Leftrightarrow$ (2) By definition.

(2) $\Rightarrow$ (3): Let a be a non-nilpotent element in R. Then Ra is a nonnil ideal of R. It follows that $\mathrm{Tor}_1^R(R/Ra, R/Ra)$ is GV-torsion since R/Ra is ϕ-torsion and ϕ-w-flat. That is, Ra/Ra^2 is GV-torsion, and thus $a \in Ra \subseteq (Ra)_w = (Ra^2)_w$.

(3) $\Rightarrow$ (4): Since R is a ϕ-ring, $\mathrm{Nil}(R)$ is the minimal prime w-ideal R. We claim that the ring $\overline{R}_{\overline{\mathfrak{m}}} := (R/\mathrm{Nil}(R))_{\mathfrak{m}/\mathrm{Nil}(R)}$ is a field for any $\mathfrak{m} \in w$-$\mathrm{Max}(R)$. Indeed, let a be a non-nilpotent element in R. By (3), $(a)_w = (a^2)_w$. Thus $(a)_\mathfrak{m} = (a^2)_\mathfrak{m}$. We have $(\overline{a})_{\overline{\mathfrak{m}}} = (\overline{a^2})_{\overline{\mathfrak{m}}}$ as an ideal of $\overline{R}_{\overline{\mathfrak{m}}}$. So $\overline{R}_{\overline{\mathfrak{m}}}$ is a local von Neumann regular ring, and thus a field. Note that $\overline{R}_{\overline{\mathfrak{m}}} = R_\mathfrak{m}/\mathrm{Nil}(R_\mathfrak{m})$. It follows that $R_\mathfrak{m}$ is 0-dimensional (see [6, Theorem 3.1]). Thus w-$\dim(R) = 0$.

(4) $\Rightarrow$ (1): By Theorem 1.4, we just need to show $\mathrm{Tor}_1^R(R/I, R/J)$ is GV-torsion for all nonnil ideals I and all ideals J of R. Since R is a ϕ-ring with w-$\dim(R) = 0$, $\mathrm{Nil}(R)$ is the unique maximal w-ideal of R. We just need to show $\mathrm{Tor}_1^R(R/I, R/J)_{Nil(R)} = 0$. That is, $(I \cap J/IJ)_{Nil(R)} = 0$.

If J is a nonnil ideal of R, there are non-nilpotent elements $s \in I$ and $t \in J$ such that $st \in IJ$. Since $st \notin \mathrm{Nil}(R)$, $(I \cap J/IJ)_{Nil(R)} = 0$. If J is a nilpotent ideal of R, $I \cap J = J$. Thus $\mathrm{Tor}_1^R(R/I, R/J)_{\mathrm{Nil}(R)} = (I \cap J/IJ)_{\mathrm{Nil}(R)} = (J/IJ)_{\mathrm{Nil}(R)}$. Let s be a non-nilpotent element in I. We have $s(j + IJ) = 0 + (IJ)$ in J/IJ for any $j \in J$. Thus $(I \cap J/IJ)_{Nil(R)} = 0$.

(4) $\Rightarrow$ (5): By (4), $\mathrm{Nil}(R)$ is the unique w-maximal ideal of R. If $\mathrm{Nil}(R)$ is a maximal ideal of R, (6) holds obviously. Otherwise, there is a non-unit element a which is not nilpotent. Since (a) is not a GV-ideal, there is maximal w-ideal $\mathfrak{m}$ such that $\mathrm{Nil}(R) \subsetneq (a) \subseteq (a)_w \subseteq \mathfrak{m}$, Thus w-$\dim(R) \geq 1$, which is a contradiction.

(5) $\Rightarrow$ (4): Trivial.

(5) $\Leftrightarrow$ (6): See [19, Theorem 4.1]. □

Recall from [6] that a ring R is said to be a *Prüfer ring* provided that every finitely generated regular ideal I is invertible, i.e., $II^{-1} = R$ where $I^{-1} = \{x \in T(R) \mid Ix \subseteq R\}$, or equivalently, there is a fractional ideal J of R such that $IJ = R$. It is well known that an integral domain is a Prüfer domain if and only if the weak global dimension of $R \leq 1$. Recall that a ring R is said to be a PvMR if every finitely generated regular ideal I is w-invertible, i.e., $(II^{-1})_w = R$, or equivalently, there is a fractional ideal J of R such that $(IJ)_w = R$. PvMDs are exactly integral domains which are PvMRs. It is known that an integral domain R is a PvMD if and only if $R_{\mathfrak{m}}$ is a valuation domain for each $\mathfrak{m} \in w\text{-Max}(R)$ if and only if w-w.gl.dim$(R) \leq 1$ (see [12,16]).

Following [4], a ϕ-ring R is said to be a *ϕ-chain ring* (ϕ-CR for short) if for any $a, b \in R - \mathrm{Nil}(R)$, either $a \mid b$ or $b \mid a$ in R. A ϕ-ring R is said to be a *ϕ-Prüfer ring* if every finitely generated nonnil ideal I is ϕ-invertible, i.e., $\phi(I)\phi(I^{-1}) = \phi(R)$. It follows from [1, Corollary 2.10] that a ϕ-ring R is ϕ-Prüfer, if and only if $R_{\mathfrak{m}}$ is a ϕ-CR for any maximal ideal $\mathfrak{m}$ of R, if and only if $R/\mathrm{Nil}(R)$ is a Prüfer domain, if and only if $\phi(R)$ is Prüfer. For a strongly ϕ-ring R, Zhao [18, Theorem 4.3] showed that R is a ϕ-Prüfer ring if and only if all ϕ-torsion free R-modules are ϕ-flat, if and only if each submodule of a ϕ-flat R-module is ϕ-flat, if and only if each nonnil ideal of R is ϕ-flat.

Let R be a ϕ-ring. Recall from [7] that a nonnil ideal J of R is said to be a *ϕ-GV-ideal* (resp., *ϕ-w-ideal*) of R if $\phi(J)$ is a GV-ideal (resp., w-ideal) of $\phi(R)$. A ϕ-ring R is called a ϕ-SM *ring* if it satisfies the ACC on ϕ-w-ideals. An ideal I of R is *ϕ-w-invertible* if $(\phi(I)\phi(I)^{-1})_W = \phi(R)$ where W is the w-operation of $\phi(R)$. A ϕ-ring is ϕ-Krull provided that any nonnil ideal is ϕ-w-invertible (see [7, Theorem 2.23]). By extending ϕ-Krull rings and PvMDs, we give the definition of ϕ-Prüfer v-multiplication rings.

Definition 3.2. Let R be a ϕ-ring. R is said to be a *ϕ-Prüfer v-multiplication ring* (ϕ-PvMR for short) provided that any finitely generated nonnil ideal is ϕ-w-invertible.

Now we characterize ϕ-Prüfer multiplication rings in terms of ϕ-w-flat modules.

Theorem 3.3. *Let R be a ϕ-ring. The following statements are equivalent for R:*

(1) *R is a* ϕ-PvMR;
(2) *$R_{\mathfrak{m}}$ is a ϕ-CR for any* $\mathfrak{m} \in w\text{-Max}(R)$;
(3) $R/\mathrm{Nil}(R)$ *is a* PvMD;
(4) $\phi(R)$ *is a* PvMR.

Moreover, if R is a strongly ϕ-ring, all above are equivalent to

(5) *R is a ϕ-w-w.gl.*$\dim(R) \leq 1$;
(6) *every submodule of a w-flat module is ϕ-w-flat*;
(7) *every submodule of a flat module is ϕ-w-flat*;
(8) *every ideal of R is ϕ-w-flat*;
(9) *every nonnil ideal of R is ϕ-w-flat*;
(10) *every finite type nonnil ideal of R is ϕ-w-flat.*

Proof. Let R be a ϕ-ring. Denote by W, w and $\overline{w}$ the w-operations of $\phi(R)$, R and $R/\mathrm{Nil}(R)$ respectively. We will prove the equivalences of (1)-(4) and (5)-(10).

(1) $\Rightarrow$ (4): Let K be a finitely generated regular ideal of $\phi(R)$. Then $K = \phi(I)$ for some finitely generated nonnil ideal I of R by [1, Lemma 2.1]. Since R is a ϕ-PvMR, $(KK^{-1})_W = (\phi(I)\phi(I)^{-1})_W = \phi(R)$. Thus $\phi(R)$ is a PvMR.

(4) $\Rightarrow$ (1): Let I be a finitely generated nonnil ideal of R. We will show I is ϕ-w-invertible. By [1, Lemma 2.1], $\phi(I)$ is a finitely generated regular ideal of $\phi(R)$. Thus $(\phi(I)\phi(I)^{-1})_W = \phi(R)$ since $\phi(R)$ is a PvMR.

(2) $\Leftrightarrow$ (3): By [1, Theorem 3.7, Corollary 2.10], $R_{\mathfrak{m}}$ is a ϕ-CR for any $\mathfrak{m} \in w$-Max(R) if and only if $R_{\mathfrak{m}}/Nil(R_{\mathfrak{m}}) = (R/Nil(R))_{\mathfrak{m}}$ is a valuation domain for any $\mathfrak{m} \in w$-Max(R) if and only if $R/Nil(R)$ is a PvMD (see [12, Theorem 4.9]).

(3) $\Rightarrow$ (4): Note that $\phi(R)/\mathrm{Nil}(\phi(R)) \cong R/\mathrm{Nil}(R)$ is a PvMD (see [1, Lemma 2.4]). Let $\phi(I)$ be a finitely generated regular ideal of $\phi(R)$. Then, by [1, Lemma 2.1], I is a nonnil ideal of R. Then $\overline{I} = I/Nil(R)$ is w-invertible over $\overline{R} = R/\mathrm{Nil}(R)$ by (3). That is, $(\overline{I}\overline{I^{-1}})_{\overline{w}} = \overline{R}$. There is a GV ideal $\overline{J}$ of $\overline{R}$ such $\overline{J} \subseteq \overline{I}\overline{I^{-1}}$ (see [14, Exercise 6.10(2)]). So $J \subseteq II^{-1}$ where J is a ϕ-GV ideal of R by [7, Lemma 2.3]. Thus $\phi(J) \subseteq \phi(I)\phi(I)^{-1}$. Since $\phi(J) \in \mathrm{GV}(\phi(R))$, $(\phi(I)\phi(I)^{-1})_W = \phi(R)$.

(4) $\Rightarrow$ (3): Suppose $\phi(R)$ is a PvMR. Let $\overline{I}$ is a finitely generated nonzero ideal of $\overline{R}$. Then I is a nonnil ideal of R. Thus $\phi(I)$ is a finitely generated regular ideal of $\phi(R)$ by [1, Lemma 2.1]. So $(\phi(I)\phi(I)^{-1})_W = \phi(R)$ by (4). Hence $J \subseteq II^{-1}$ in R for some ϕ-GV ideal J of R and thus $\overline{J} \subseteq \overline{I}\overline{I^{-1}}$ in $\overline{R}$. By [7, Lemma 2.3], $\overline{J} \in \mathrm{GV}(\overline{R})$, and thus $(\overline{I}\overline{I^{-1}})_{\overline{w}} = \overline{R}$. So $R/\mathrm{Nil}(R)$ is a PvMD.

(5) $\Rightarrow$ (6): Let K be a submodule of a w-flat module F. Then ϕ-w-$\mathrm{fd}_R(F/K) \leq 1$ by (5). Thus K is ϕ-w-flat by Proposition 2.3.

(6) $\Rightarrow$ (7) $\Rightarrow$ (8) $\Rightarrow$ (9) $\Rightarrow$ (10): Trivial.

(10) $\Rightarrow$ (5): Let I be a finite type nonnil ideal of R. Then ϕ-w-$\mathrm{fd}_R(R/I) \leq 1$ by Proposition 2.3. It follows from Proposition 2.5 that ϕ-w-w.gl.dim$(R) \leq 1$.

Now, let R be a strongly ϕ-ring.

(2) $\Rightarrow$ (9): Let $\mathfrak{m}$ be a maximal w-ideal of R and I a nonnil ideal of R. Then $I_{\mathfrak{m}}$ is a nonnil ideal of $R_{\mathfrak{m}}$ by Lemma 1.1 and thus is ϕ-flat by [18, Theorem 4.3]. So I is ϕ-w-flat by Theorem 1.4.

(9) $\Rightarrow$ (2): Let $\mathfrak{m}$ be a maximal w-ideal of R , $I_{\mathfrak{m}}$ a nonnil ideal of $R_{\mathfrak{m}}$. Then I is a nonnil ideal of R by Lemma 1.1. By (9), I is ϕ-w-flat and so $I_{\mathfrak{m}}$ is ϕ-flat by Theorem 1.4. Thus $R_{\mathfrak{m}}$ is a ϕ-CR by [18, Theorem 4.3]. □

Corollary 3.4. *Suppose R is a ϕ-ring. Then R is a ϕ-*Krull *ring if and only if R is both a ϕ-*PvMR *and a ϕ-*SM *ring.*

Proof. By [7, Theorem 2.4] a ϕ-ring R is a ϕ-SM ring if and only if $R/\mathrm{Nil}(R)$ is an SM domain. A ϕ-ring R is a ϕ-Krull ring if and only if $R/\mathrm{Nil}(R)$ is a Krull domain (see [2, Theorem 3.1]). Since R is a Krull domain if and only if R is an SM PvMD (see [8, Theorem 7.9.3]), the equivalence holds by Theorem 3.3. □

Corollary 3.5. *Suppose R is a strongly ϕ-ring. Then R is a ϕ-*PvMR *if and only if R is a* PvMR.

Proof. Suppose R is a ϕ-PvMR and let I be a finitely generated regular ideal of R. Then $\overline{I}$ is a finitely generated regular ideal of $\overline{R}$. By Theorem 3.3, $\overline{R}$ is a PvMD. Then $(\overline{I}\overline{I^{-1}})_{\overline{w}} = \overline{R}$. Thus there is a GV-ideal $\overline{J}$ of $\overline{R}$ with J finitely generated over R such that $\overline{J} \subseteq \overline{I}\overline{I^{-1}}$. Since R is a strongly ϕ-ring, J is a GV-ideal of R by [7, Lemma 2.11]. Since $J \subseteq II^{-1}$ in R, $(II^{-1})_w = R$. Assume R is a PvMR. Since R is a strongly ϕ-ring, $\phi(R) = R$ is a PvMR. Thus R is a ϕ-PvMR by Theorem 3.3. □

The condition that R is a strongly ϕ-ring in Corollary 3.5 can't be removed by the following example.

Example 3.6. Let D be an integral domain which is not a PvMD and K its quotient field. Since K/D is a divisible D-module, the ring $R = D(+)K/D$ is a ϕ-ring but not a strongly ϕ-ring (see [2, Remark 1]). Since $\mathrm{Nil}(R) = 0(+)K/D$, we have $R/\mathrm{Nil}(R) \cong D$ is not a PvMD. Thus R is not a ϕ-PvMR by Theorem 3.3. Denote by $U(R)$ and $U(D)$ the sets of unit elements of R and D respectively. Since $\mathrm{Z}(R) = \{(r, m) \,|\, r \in \mathrm{Z}(D) \cup \mathrm{Z}(K/D)\} = R - \mathrm{U}(D)(+)K/D = R - \mathrm{U}(R)$, R is a PvMR obviously.

Acknowledgement. The first author was supported by the Natural Science Foundation of Chengdu Aeronautic Polytechnic (No. 062026). The second author was supported by the National Natural Science Foundation of China (No. 12061001).

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