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ON FINITE GROUPS WITH THE SAME ORDER TYPE AS SIMPLE GROUPS $F_4(q)$ WITH q EVEN

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Abstract. The main aim of this article is to study quantitative structure of finite simple exceptional groups $F_4(2^n)$ with $n > 1$. Here, we prove that the finite simple exceptional groups $F_4(2^n)$, where $2^{4n}+1$ is a prime number with $n > 1$ a power of 2, can be uniquely determined by their orders and the set of the number of elements with the same order. In conclusion, we give a positive answer to J. G. Thompson's problem for finite simple exceptional groups $F_4(2^n)$.

1. Introduction

For a finite group G, the set $\mathsf{nse}(G)$ of the number of elements in G with the same order links to a well-known problem posed by J. G. Thompson (1987) which is related to algebraic number fields [8, Problem 12.37]:

For a finite group G and a natural number n, set $G(n) = \{g \in G \mid g^n = 1\}$ and define the type of G to be the function whose value at n is the size of $G(n)$. Is it true that a group is solvable if its type is the same as that of a solvable one?

It immediately turns out that if two groups G and H are of the same type, then $|G| = |H|$ and $\mathsf{nse}(G) = \mathsf{nse}(H)$. Therefore, if a group G has been uniquely determined by its order and $\mathsf{nse}(G)$, then Thompson's problem is true for G. One may ask this problem for non-solvable groups, in particular, finite simple groups. In this direction, Shao and et al. [9] studied finite simple groups with at most four prime divisors of their orders and nse. Following this investigation, such problem has been studied for some families of simple groups [1,2] including Suzuki groups $\mathrm{Sz}(q)$ and Small Ree groups $\mathrm{R}(q)$. In this paper, we prove that:

Theorem 1.1. *Let G be a group with $\mathsf{nse}(G) = \mathsf{nse}(F_4(2^n))$ and $|G| = |F_4(2^n)|$, where $2^{4n} + 1$ is a prime number and $n > 1$ is a power of 2. Then G is isomorphic to $F_4(2^n)$.*

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In order to prove Theorem 1.1, we determine the number of elements in $\mathrm{F}_4(2^n)$ with the same order in Proposition 3.1. Then we prove that the prime graph of the group G satisfying hypothesis of Theorem 1.1 has at least two components, see Proposition 3.2, and then we show that a section of G is isomorphic to $\mathrm{F}_4(2^n)$. Finally, we prove that G is isomorphic to $\mathrm{F}_4(2^n)$.

1.1. Definitions and notation

All sets and groups in this paper are finite. The symmetric and alternating groups on n letters are denoted by S_n and A_n, respectively. A Frobenius group G with kernel K and complement H is a semidirect product $G = K \rtimes H$ such that K is a normal subgroup in G, and $\mathbf{C}_K(x) = 1$ for every non-identity element x of H. A group G is a 2-Frobenius group if there exists a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that G/H and K are Frobenius groups with kernels K/H and H, respectively.

For finite simple groups of Lie type, we adopt the standard notation as in [5], and in particular, we use the notation recorded in Table 1 to denote the finite simple classical groups.

TABLE 1. Finite simple classical groups

X	d	$\lvert X\rvert$	$\lvert\mathbf{Out}(X)\rvert$
$\mathrm{PSL}_n(q)$, $n \geqslant 3$	$\gcd(n, q-1)$	$d^{-1}q^{\frac{n(n-1)}{2}}\mathrm{p}_2^n(q)$	$2ad$
$\mathrm{PSL}_2(q)$, $q \neq 2, 3$	$\gcd(2, q-1)$	$d^{-1}q(q^2-1)$	ad
$\mathrm{PSU}_n(q)$, $n \geqslant 3$, $(n, q) \neq (3, 2)$	$\gcd(n, q+1)$	$d^{-1}q^{\frac{n(n-1)}{2}}\mathrm{u}_2^n(q)$	$2ad$
$\mathrm{PSp}_{2m}(q)$, $m \geqslant 3$	$\gcd(2, q-1)$	$d^{-1}q^{m^2}\mathrm{p}_1^m(q^2)$	ad
$\mathrm{PSp}_4(q)$, $q \neq 2$	$\gcd(2, q-1)$	$d^{-1}q^4(q^2-1)(q^4-1)$	$2a$
$\mathrm{P\Omega}_{2m+1}(q)$, q odd and $m \geqslant 3$	2	$2^{-1}q^{m^2}\mathrm{p}_1^m(q^2)$	$2a$
$\mathrm{P\Omega}_{2m}^+(q)$, $m \geqslant 5$	$\gcd(4, q^m-1)$	$d^{-1}q^{m(m-1)}(q^m-1)\mathrm{p}_1^{m-1}(q^2)$	$2ad$
$\mathrm{P\Omega}_8^+(q)$	$\gcd(4, q^4-1)$	$d^{-1}q^{12}(q^4-1)\prod_{i=1}^{3}(q^{2i}-1)$	$6ad$
$\mathrm{P\Omega}_{2m}^-(q)$, $m \geqslant 4$	$\gcd(4, q^m+1)$	$d^{-1}q^{m(m-1)}(q^m+1)\mathrm{p}_1^{m-1}(q^2)$	$2ad$

Note: $\mathrm{p}_t^n(q) = \prod_{i=t}^{n}(q^i-1)$ and $\mathrm{u}_t^n(q) = \prod_{i=t}^{n}(q^i-(-1)^i)$, where $q = p^a$ with p prime.

In this manner, the only repetitions are

$$\mathrm{PSL}_2(4) \cong \mathrm{PSL}_2(5) \cong \mathrm{A}_5, \quad \mathrm{PSL}_2(7) \cong \mathrm{PSL}_3(2), \quad \mathrm{PSL}_2(9) \cong \mathrm{A}_6,$$
$$\mathrm{PSL}_4(2) \cong \mathrm{A}_8, \quad \mathrm{PSp}_4(3) \cong \mathrm{PSU}_4(2).$$

For a positive integer n, the set of prime divisors of n is denoted by $\pi(n)$, and if G is a finite group, $\pi(G) := \pi(|G|)$, where $|G|$ is the order of G. We denote the set of elements' orders of G by $\omega(G)$ (known as spectrum of G). Recall that $\mathsf{nse}(G)$ is the set of the numbers elements in G with the same order. In other word, $\mathsf{nse}(G)$ consists of the number $\mathsf{m}_i(G)$ of elements of order i in G for $i \in \omega(G)$. Also, we denote a Sylow p-subgroup of G by G_p and the number of Sylow p-subgroups of G by $\mathsf{n}_p(G)$. The prime graph $\Gamma(G)$ of a finite group G is a graph whose vertex set is $\pi(G)$, and two distinct vertices u and v are adjacent if and only if $uv \in \omega(G)$. Assume further that $\Gamma(G)$ has $t(G)$ connected components $\pi_i(G)$ for $i = 1, 2, \ldots, t(G)$. In the case where G is of even order, we always assume that $2 \in \pi_1(G)$, and $\pi_1(G)$ is said to be the even component of G. Also we denote by $\omega_i(G)$ the subset of $\omega(G)$ consisting of all the numbers such that their prime divisors are in $\pi_i(G)$. Further, the largest element in each $\omega_i(G)$ is called the order component of G.

2. Preliminaries

In this section, we give some useful results which will be used in the proof of Theorem 1.1.

Lemma 2.1 ([3, Theorem 2]). *Let G be a Frobenius group of even order with kernel K and complement H. Then the following statements hold*:

(a) *K is a nilpotent group*;
(b) *$|H|$ divides $|K|-1$*;
(c) *$t(G)=2$, $\pi(H)$ and $\pi(K)$ are the connected components of $\Gamma(G)$.*

Lemma 2.2 ([3, Theorem 2]). *Let G be a 2-Frobenius group of even order. Then the following statements hold*:

(a) *$t(G)=2$, $\pi_1(G)=\pi(H)\cup\pi(G/K)$, and $\pi_2(G)=\pi(K/H)$*;
(b) *G/K and K/H are cyclic groups, $|G/K|$ divides $|\mathsf{Aut}(K/H)|$, $\gcd(|G/K|,|K/H|)=1$ and $|G/K|<|K/H|$*;
(c) *H is a nilpotent group and G is a solvable group.*

Lemma 2.3 ([10, Lemma 3 and Theorem A]). *Let G be a finite group with $t(G)\geqslant 2$. Then one of the following statements holds*:

(a) *G is a Frobenius group*;
(b) *G is a 2-Frobenius group*;
(c) *G has a normal series $1\trianglelefteq H\trianglelefteq K\trianglelefteq G$ such that H and G/K are π_1-groups, K/H is a non-abelian simple group, H is a nilpotent group, $|G/K|$ divides $|\mathsf{Out}(K/H)|$, $t(K/H)\geqslant t(G)$, and for any $i\in\{2,\ldots,t(G)\}$, there exists $j\in\{2,\ldots,t(K/H)\}$ such that $\pi_i(G)=\pi_j(K/H)$.*

Lemma 2.4 ([6, Page 4]). *Let G be a finite group, and let n be a positive integer dividing $|G|$. If $G(n)=\{g\in G \mid g^n=1\}$, then n divides $|G(n)|$.*

In what follows, φ is the *Euler totient* function. The proof of the following result is straightforward by Lemma 2.4.

Lemma 2.5. *Let G be a finite group, and let $i\in\omega(G)$. Then $\mathsf{m}_i(G)=k\varphi(i)$, where k is the number of cyclic subgroups of order i in G, and i divides $\sum_{j\mid i}\mathsf{m}_j(G)$. Moreover, if $i>2$, then $\mathsf{m}_i(G)$ is even.*

Lemma 2.6 ([11, Lemma 6]). *Let a,m,n be natural numbers. Then*

(a) $\gcd(a^m-1,a^n-1)=a^{\gcd(m,n)}-1$;

(b) $\gcd(a^m+1,a^n+1)=\begin{cases} a^{\gcd(m,n)}+1, & \text{if both } \frac{m}{\gcd(m,n)} \text{ and } \frac{n}{\gcd(m,n)} \text{ are odd};\\ \gcd(2,a+1), & \text{otherwise.}\end{cases}$

(c) $\gcd(a^m-1, a^n+1) = \begin{cases} a^{\gcd(m,n)}+1, \text{ if } \frac{m}{\gcd(m,n)} \text{ is even and } \frac{n}{\gcd(m,n)} \text{ is odd;} \\ \gcd(2, a+1), \text{ otherwise.} \end{cases}$

In particular, for every $a \geq 2$ *and* $m \geq 1$*, the inequality* $\gcd(a^m-1, a^m+1) \leq 2$ *holds.*

A group G is called a C_{pp}-group if the centralizers of its elements of order p in G are p-groups.

Lemma 2.7 ([4]). *Let* $p = 2^\alpha 3^\beta + 1$ *be a prime number. Then the finite simple* C_{pp}*-groups are as in* Table 2.

TABLE 2. Finite simple C_{pp}-groups

p	***Group***	***Conditions***
2	A_5, A_6	
2	$PSL_2(q)$, $PSL_3(2^2)$	q *Fermat or Mersenne prime*, $q = 2^n \geqslant 8$
2	$Sz(2^{2n+1})$	$n \geqslant 1$
3	A_5, A_6	
3	$PSL_2(q)$, $PSL_2(2^3)$, $PSL_3(2^2)$	$q = 3^{n+1}$, $q = 2 \cdot 3^n \pm 1$ *prime*, $n \geqslant 1$
5	A_5, A_6, A_7	
5	$PSL_2(q)$, $PSL_2(7^2)$, $PSL_3(2^2)$, $PSU_4(3)$, $PSp_4(3)$, $PSp_4(7)$	$q = 5^n$, $q = 2 \cdot 5^n \pm 1$ *prime*, $n \geqslant 1$
5	$Sz(2^3)$, $Sz(2^5)$	
5	M_{11}, M_{22}	
7	A_7, A_8, A_9	
7	$PSL_2(q)$, $PSL_2(2^3)$, $PSL_3(2^2)$, $PSU_3(3)$, $PSU_3(5)$, $PSU_3(19)$, $PSU_4(3)$, $PSU_6(2)$, $PSp_6(2)$, $P\Omega_8^+(2)$	$q = 7^n$, $q = 2 \cdot 7^n - 1$ *prime*, $n \geqslant 1$
7	$G_2(3)$, $G_2(19)$, $Sz(2^3)$	
7	M_{22}, J_1, J_2, HS	
13	A_{13}, A_{14}, A_{15}	
13	$PSL_2(q)$, $PSL_2(3^3)$, $PSL_2(5^2)$, $PSL_3(3)$, $PSL_4(3)$, $PSU_3(2^2)$, $PSU_3(23)$, $PSp_4(5)$, $PSp_6(3)$, $P\Omega_7(3)$, $P\Omega_8^+(3)$	$q = 13^n$, $q = 2 \cdot 13^n - 1$ *prime*, $n \geqslant 1$
13	$F_4(2)$, $G_2(2^2)$, $G_2(3)$, $Sz(2^3)$, ${}^3D_4(2)$, ${}^2E_6(2)$, ${}^2F_4(2)'$	
13	Suz, Fi_{22}	
17	A_{17}, A_{18}, A_{19}	
17	$PSL_2(q)$, $PSL_2(2^4)$, $PSp_4(4)$, $PSp_8(2)$, $P\Omega_8^-(2)$, $P\Omega_{10}^-(2)$	$q = 17^n$, $q = 2 \cdot 17^n \pm 1$ *prime*, $n \geqslant 1$
17	$F_4(2)$, ${}^2E_6(2)$	
17	J_3, He, Fi_{23}, Fi_{24}	
19	A_{19}, A_{20}, A_{21}	
19	$PSL_2(q)$, $PSL_3(7)$, $PSU_3(2^3)$	$q = 19^n$, $2 \cdot 19^n - 1$ *prime*, $n \geqslant 1$
19	$R(3^3)$, ${}^2E_6(2)$	
19	J_1, J_3, O′N, Th, HN	
37	A_{37}, A_{38}, A_{39}	
37	$PSL_2(q)$, $PSU_3(11)$	$q = 37^n$, $2 \cdot 37^n - 1$ *prime*, $n \geqslant 1$
37	$R(3^3)$, ${}^2F_4(2^3)$	
37	J_4, Ly	
73	A_{73}, A_{74}, A_{75}	
73	$PSL_2(q)$, $PSL_3(2^3)$, $PSU_3(3^2)$, $PSp_6(2^3)$	$q = 73^n$, $2 \cdot 73^n - 1$ *prime*, $n \geqslant 1$
73	$G_2(2^3)$, $G_2(3^2)$, $F_4(3)$, $E_6(2)$, $E_7(2)$, ${}^3D_4(3)$	
109	A_{109}, A_{110}, A_{111}	
109	$PSL_2(q)$	$q = 109^n, 2 \cdot 109^n - 1$ *prime*, $n \geqslant 1$
109	${}^2F_4(2^3)$	
2^m+1	A_p, A_{p+1}, A_{p+2}	$m = 2^s$
2^m+1	$PSL_2(q)$	$m = 2^s$, $q = 2^m$, $q = p^n$, $q = 2 \cdot p^n \pm 1$ *prime*, $s \geqslant n \geqslant 1$
2^m+1	$PSp_a(2^b)$	$m = 2^s$, $a = 2^{c+1}$, $b = 2^d$, $c \geqslant 1$, $c + d = s$
2^m+1	$P\Omega_{2(m+1)}^-(2)$	$m = 2^s$, $s \geqslant 1$
2^m+1	$P\Omega_a^-(2^b)$	$m = 2^s$, $a = 2^{c+1}$, $b = 2^d$, $c \geqslant 2$, $c + d = s$
2^m+1	$F_4(2^e)$	$4e = m = 2^s$, $e \geqslant 1$
other	A_p, A_{p+1}, A_{p+2}	
other	$PSL_2(q)$	$q = p^n$, $2 \cdot p^n - 1$ *prime* , $n \geqslant 1$

3. Proof of the main result

In this section, we prove Theorem 1.1. Here we set $\mathrm{F} := \mathrm{F}_4(q)$, where $q = 2^n$, and $p = q^4 + 1$. Let also G be a finite group with $\mathsf{nse}(G) = \mathsf{nse}(\mathrm{F})$ and $|G| = |\mathrm{F}| = q^{24}(q^{12} - 1)(q^8 - 1)(q^6 - 1)(q^2 - 1)$. In the following proposition, we determine the set of the number of elements in F with the same order.

Proposition 3.1. *Let* F *be the finite simple exceptional group* $\mathrm{F}_4(2^n)$*, where* $p = 2^{4n} + 1$ *is a prime number and* $n > 1$ *is a power of* 2. *Then the following properties hold*:

(a) $\mathsf{m}_p(\mathrm{F}) = (p - 1)|\mathrm{F}|/(8p)$;
(b) p *divides* $\mathsf{m}_i(\mathrm{F})$ *for all* $i \in \omega(\mathrm{F}) \setminus \{1, p\}$.

Proof. (a) Suppose that F_p is a Sylow p-subgroup of F. Since F_p is a cyclic group of order p, Lemma 2.5 implies that $\mathsf{m}_p(\mathrm{F}) = \varphi(p)\mathsf{n}_p(\mathrm{F}) = (p - 1)\mathsf{n}_p(\mathrm{F})$. We now obtain $\mathsf{n}_p(\mathrm{F})$. By [7], p is an isolated vertex of the prime graph $\Gamma(\mathrm{F})$ of F. Then $|\mathbf{C}_\mathrm{F}(\mathrm{F}_p)| = p$ and $|\mathbf{N}_\mathrm{F}(\mathrm{F}_p)| = kp$ for some positive integer k, and so k divides $p - 1$ because $\mathbf{N}_\mathrm{F}(\mathrm{F}_p)/\mathbf{C}_\mathrm{F}(\mathrm{F}_p) \lesssim \mathsf{Aut}(\mathrm{F}_p)$. On the other hand, by the Sylow's theorem, we have that $p \mid (8 - k)$. This follows that $k = 8$ and $\mathsf{n}_p(\mathrm{F}) = |\mathrm{F}|/(8p)$, as claimed.

(b) Let $i \in \omega(\mathrm{F}) \setminus \{1, p\}$. As p is an isolated vertex of $\Gamma(\mathrm{F})$, it follows that p does not divide i and $pi \notin \omega(\mathrm{F})$. Therefore, F_p acts fixed point freely by conjugation on the set of elements of order i, and this implies that $|\mathrm{F}_p| \mid \mathsf{m}_i(\mathrm{F})$, as desired. □

Proposition 3.2. *Let* F *be the finite simple exceptional group* $\mathrm{F}_4(2^n)$*, where* $p = 2^{4n} + 1$ *is a prime number and* $n > 1$ *is a power of* 2. *If* G *is a finite group with* $\mathsf{nse}(G) = \mathsf{nse}(\mathrm{F})$ *and* $|G| = |\mathrm{F}|$*, then the following properties of group* G *hold*:

(a) $\mathsf{m}_2(G) = \mathsf{m}_2(\mathrm{F})$;
(b) $\mathsf{m}_p(G) = \mathsf{m}_p(\mathrm{F})$;
(c) $\mathsf{n}_p(G) = \mathsf{n}_p(\mathrm{F})$;
(d) p *is an isolated vertex of* $\Gamma(G)$;
(e) $p \mid \mathsf{m}_i(G)$ *for all* $i \in \omega(G) \setminus \{1, p\}$.

Proof. (a) By Lemma 2.5, $\mathsf{m}_i(G)$ is odd if and only if $i = 1$ or 2, and so $\mathsf{m}_2(G) = \mathsf{m}_2(\mathrm{F})$.

(b) As $p \nmid \mathsf{m}_p(G)$ and $\mathsf{nse}(G) = \mathsf{nse}(\mathrm{F})$, Proposition 3.1 implies that $\mathsf{m}_p(G) = \mathsf{m}_p(\mathrm{F})$.

(c) We know that both G_p and F_p are cyclic groups of order p. So by part (b), we have that $\mathsf{m}_p(G) = \varphi(p)\mathsf{n}_p(G) = \varphi(p)\mathsf{n}_p(\mathrm{F}) = \mathsf{m}_p(\mathrm{F})$, which gives $\mathsf{n}_p(G) = \mathsf{n}_p(\mathrm{F})$.

(d) Assuming to the contrary that p is not an isolated vertex of $\Gamma(G)$. Then there exists $i \in \pi(G) - \{p\}$ such that $ip \in \omega(G)$. We now obtain $\mathsf{m}_{ip}(G)$. We know that $\mathsf{m}_{ip}(G) = \phi(ip)\mathsf{n}_p(G)k$, where k is the number of cyclic subgroups of order i in $\mathbf{C}_G(G_p)$, and since $\mathsf{n}_p(G) = \mathsf{n}_p(\mathrm{F})$, this implies that $\mathsf{m}_{ip}(G) =$

$(i-1)(p-1)|\mathrm{F}|k/(8p)$. Assume that $\mathsf{m}_{ip}(G)$ is coprime to p, that is to say, $p \nmid \mathsf{m}_{ip}(G)$. Then, by Proposition 3.1, $\mathsf{m}_{ip}(G) = \mathsf{m}_p(G)$, and so $i = 2$ and $k = 1$. Lemma 2.5 implies that p divides $\mathsf{m}_2(G) + \mathsf{m}_{2p}(G)$, and since $\mathsf{m}_2(G) = \mathsf{m}_2(\mathrm{F})$ and $p \mid \mathsf{m}_2(\mathrm{F})$, we deduce that p divides $\mathsf{m}_{2p}(G)$, which is a contradiction. Therefore, $p \mid \mathsf{m}_{ip}(G)$, and hence p divides $(i-1)k$. Thus the fact that $\mathsf{m}_{ip}(G) < |G|$ yields $p - 1 \leq 8$, but this is impossible as $p = 2^{4n} + 1$ and $n > 1$ is a power of 2. Therefore, p is an isolated vertex of $\Gamma(G)$.

(e) It follows from part (d) that p is an isolated vertex of $\Gamma(G)$. Then $p \nmid i$ and $pi \notin \omega(G)$, and so G_p acts fixed point freely by conjugation on the set of elements of order i. Thus $|G_p|$ divides $\mathsf{m}_i(G)$, and hence p divides $\mathsf{m}_i(G)$ as claimed. □

Proposition 3.3. *Let* F *be the finite simple exceptional group* $\mathrm{F}_4(2^n)$*, where* $p = 2^{4n}+1$ *is a prime number and* $n > 1$ *is a power of* 2*. If* G *is a finite group with* $\mathsf{nse}(G) = \mathsf{nse}(\mathrm{F})$ *and* $|G| = |\mathrm{F}|$*, then* G *is neither a Frobenius group, nor a* 2*-Frobenius group.*

Proof. Suppose to the contrary that G is a Frobenius group with kernel K and complement H. Then by Lemma 2.1, $t(G) = 2$, $\pi(H)$ and $\pi(K)$ are the connected components of $\Gamma(G)$ and $|H|$ divides $|K| - 1$. Now by Proposition 3.2, p is an isolated vertex of $\Gamma(G)$, and hence either $|H| = p$ and $|K| = |G|/p$, or $|H| = |G|/p$ and $|K| = p$, with $p = 2^{4n} + 1$ prime. The latter case can be ruled out as $|H|$ must divide $|K| - 1$. Therefore, $|H| = p$ and $|K| = |G|/p$, and hence $p = 2^{4n} + 1$ divides 7, which is impossible.

Suppose to the contrary that G is a 2-Frobenius group. Then G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that G/H and K are Frobenius groups with kernels K/H and H, respectively. Since p is an isolated vertex of $\Gamma(G)$, we conclude that $|K/H| = p$. Thus Lemma 2.2 implies that $|G/K|$ divides $|\mathsf{Aut}(K/H)| = p-1$. Therefore, $2^n + 1$ divides $|H|$. As H is a nilpotent group, $H_t \rtimes L$ is a Frobenius group with kernel H_t and complement L, where L is the complement of Frobenius group K and $t \in \pi(2^n + 1)$. Therefore $p = 2^{4n} + 1$ divides $t - 1$, which is impossible. □

Proof of Theorem 1.1. Let F be the finite simple exceptional group $\mathrm{F}_4(2^n)$, where $p = 2^{4n} + 1$ is a prime number and $n > 1$ is a power of 2. Let also G be a finite group with $\mathsf{nse}(G) = \mathsf{nse}(\mathrm{F})$ and $|G| = |\mathrm{F}|$. By Propositions 3.2 and 3.3, the prime graph of G has at least two connected components and G is neither a Frobenius group nor a 2-Frobenius group. Thus Lemma 2.3 implies that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups, K/H is a non-abelian simple group, H is a nilpotent group and $|G/K|$ divides $|\mathsf{Out}(K/H)|$. Moreover, any odd order component of G is also an odd order component of K/H.

We first prove that K/H is isomorphic to F. Since p is an odd order component of G, Lemma 2.3 follows immediately that p is an odd order component of K/H. Thus K/H is a simple C_{pp}-group, and hence K/H is isomorphic to one of the groups recorded in Table 2. In what follows, we discuss the alternating and the classical cases and other cases can be treated in a similar manner.

(1) K/H is not isomorphic to alternating groups.

If $K/H \cong A_n$, then according to Table 2, $n \in \{p, p+1, p+2\}$. We know that $|K/H| \mid |G|$, so $q^4 - 2$ divides $q^{24}(q^{12}-1)(q^8-1)(q^6-1)(q^2-1)$, but since $q = 2^n$ with $n > 1$ power of 2, it is impossible by Lemma 2.6.

(2) K/H is not isomorphic to projective special linear groups.

If K/H is isomorphic to $\mathrm{PSL}_2(q')$, then by Table 2, we have the following three cases to consider:

(2.1) $q' = q^4$. Since $|G/K|$ divides $|\mathsf{Out}(K/H)|$, and by Table 1, $|\mathsf{Out}(K/H)| = 4n$ and n is a power of 2, we deduce that $|G/K|$ is a divisor of $4n$ and $2(q^{12}-1)(q^6-1)(q^2-1)$ divides $|H|$. Thus for every $i \in \pi(q^4 - q^2 + 1)$, we have that $H_i = G_i$. This implies that $\mathsf{m}_i(H) = \mathsf{m}_i(G)$. On the other hand, H has only one Sylow i-subgroup since H is nilpotent. Thus $\mathsf{m}_i(H) \leqslant q^4 - q^2 + 1$, which is impossible because $\mathsf{m}_i(H) = \mathsf{m}_i(G)$ and $p \mid \mathsf{m}_i(G)$.

(2.2) $q' = p^k$. Then $p+1$ divides $|K/H|$, but $p+1$ does not divide $|G|$, which is a contradiction.

(2.3) $q' = 2 \cdot p^k \pm 1$. Then $k = 1$ because p^2 divides $|K/H|$ and $p^2 \nmid |G|$. So $|K/H| = 2p(2p \pm 1)(2p \pm 2)$, which is a contradiction as $2p \pm 1 \nmid |G|$.

(3) K/H is not isomorphic to projective symplectic groups.

By Table 2, $K/H \cong \mathrm{PSp}_a(2^b)$, where $a = 2^{c+1}$ and $b = 2^d$ with $c \geqslant 1$, $c + d = s$. If $c > 2$, then $2^{10b} - 1 \mid |K/H|$ but $2^{10b} - 1$ does not divide $|G|$, which is a contradiction. Thus $c = 1$ or 2. Since $|G/K|$ divides $|\mathsf{Out}(K/H)|$, $|\mathsf{Out}(K/H)| \mid 2b$ and b is a power of 2, we deduce that $|G/K|$ divides $2b$ and $q^4 - q^2 + 1 \mid |H|$. Thus for every $i \in \pi(q^4 - q^2 + 1)$, we have that $H_i = G_i$. This implies that $\mathsf{m}_i(H) = \mathsf{m}_i(G)$. Note that H is nilpotent. Thus H has only one Sylow i-subgroup, and so $\mathsf{m}_i(H) \leqslant q^4 - q^2 + 1$. This is impossible as $\mathsf{m}_i(H) = \mathsf{m}_i(G)$ and $p \mid \mathsf{m}_i(G)$.

(4) K/H is not isomorphic to simple groups of orthogonal type.

If $K/H \cong \mathrm{P}\Omega^-_{2(m+1)}(2)$, then $2q^4 + 1 \mid |K/H|$, which is a contradiction as $2q^4 + 1$ is not a divisor of $|G|$. If $K/H \cong \mathrm{P}\Omega^-_a(2^b)$, where $a = 2^{c+1} \geqslant 8$ and $b = 2^d$ with $c+d = s$, then since $(q^4)^{2^c-1} \mid |K/H|$, it follows that $2^c - 1 \leqslant 6$, and hence $c = 2$. Since also $|G/K|$ divides $|\mathsf{Out}(K/H)| = 2b$ and b is a power of 2, we deduce that $|G/K| \mid 2b$ and $q^4 - q^2 + 1 \mid |H|$. Thus for every $i \in \pi(q^4 - q^2 + 1)$, we have $H_i = G_i$. This implies that $\mathsf{m}_i(H) = \mathsf{m}_i(G)$, but H has only one Sylow i-subgroup. Thus $\mathsf{m}_i(H) \leqslant q^4 - q^2 + 1$, which is impossible as $\mathsf{m}_i(H) = \mathsf{m}_i(G)$ and $p \mid \mathsf{m}_i(G)$.

As noted above, by a similar argument, we conclude that K/H is isomorphic to $F_4(2^e)$ with $4e = 2^s$. Thus $q = 2^e$, and hence $|K/H| = |G|$. Therefore, $H = 1$, and consequently, $G = K = F_4(q)$. □

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