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ALMOST RIGIDITY OF CONVEX HYPERSURFACES VIA THE EXTINCTION TIME OF MEAN CURVATURE FLOW

Xian-Tao Huang

Abstract. We prove that if a compact convex hypersurface of $\mathbb{R}^{n+1}$ has almost maximal extinction time when it is evolved by the mean curvature flow, then it must be nearly round in the C^0-norm.

1. Introduction

Denote the sphere S^n by Σ. Let $F_0 : \Sigma \to \mathbb{R}^{n+1}$ be a smooth embedding such that $\Sigma_0 = F_0(\Sigma)$ is a convex hypersurface of $\mathbb{R}^{n+1}$. Consider a one-parameter family of smooth embedding $F : \Sigma \times [0, T) \to \mathbb{R}^{n+1}$ solving the mean curvature flow with initial value F_0, i.e.,

$$(1)\qquad \begin{cases} \frac{\partial}{\partial t} F = -H\nu, \\ F(x, 0) = F_0(x). \end{cases}$$

Though out this note, $\nu(x, t)$, $H(x, t)$ and $A(x, t)$ denote the outer unit normal, the mean curvature and the second fundamental form of $\Sigma_t = F_t(\Sigma)$ at $F_t(x) = F(x, t)$ respectively. By the famous paper of Huisken ([6]), Σ_t remains convex, and the flow exists on a maximal time interval, which is denoted by $[0, T_e)$, such that Σ_t shrink to a point as $t \uparrow T_e$. Recall that under the mean curvature flow, the mean curvature satisfies the following equation:

$$(2)\qquad \frac{\partial H}{\partial t} = \triangle H + |A|^2 H.$$

Define $\omega : [0, T_e) \to \mathbb{R}$ by

$$(3)\qquad \omega(t) = \min_{x \in \Sigma} H(x, t).$$

Then $\omega(t)$ satisfies

$$(4)\qquad \frac{d\omega(t)}{dt} \geq \frac{\omega^3(t)}{n}.$$

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If we assume Σ_0 satisfies $H \geq n$, i.e., $\omega(0) \geq n$, then we obtain

$$\omega(t) \geq \frac{n}{\sqrt{1-2nt}}, \tag{5}$$

thus $T_e \leq \frac{1}{2n}$. By the strong maximum principle, $T_e = \frac{1}{2n}$ holds if and only if Σ_0 is a round sphere of radius 1.

In this short note, we prove that, if the extinction time T_e is very close to $\frac{1}{2n}$, then Σ_0 is nearly round.

Though out this note, for any $r > 0$, we denote by $S^n(r) = \{x \in \mathbb{R}^{n+1} \mid |x| = r\}$, $B^{n+1}(r) = \{x \in \mathbb{R}^{n+1} \mid |x| < r\}$.

Theorem 1.1. *For any $\eta > 0$, there exists $\tau > 0$ such that if $F_0 : \Sigma \to \mathbb{R}^{n+1}$ is an embedding satisfying*

(A) $\Sigma_0 = F_0(\Sigma)$ *is a convex hypersurface in $\mathbb{R}^{n+1}$, $H \geq n$ on Σ_0;*

and the mean curvature flow with initial value F_0 has extinction time $T_e > \frac{1}{2n} - \tau$, then there exists a vector $v \in \mathbb{R}^{n+1}$ such that $v + \Sigma_0$ is η-close to the unit sphere $S^n(1)$ in the C^0-norm.

Let $\Sigma_0 \subset \mathbb{R}^{n+1}$ be a convex compact hypersurface with the origin 0 contained in the interior of the domain enclosed by Σ_0, we say Σ_0 is η-close to $S^n(1)$ in the C^0-norm if, when we express Σ_0 as the graph of a function $u : S^n(1) \to \mathbb{R}^+$ via the polar coordinate, we have $|u - 1| < \eta$.

The C^0-closeness in Theorem 1.1 is the optimal conclusion on the regularity, which can be seen from the following example. Let Σ_0 be a smooth convex hypersurface lying in the interior of $B^{n+1}(1)$ and suppose Σ_0 is δ-close to $S^n(1)$ in the C^0-norm. It is easy to see $H \geq n$ on Σ_0. Since $\Sigma'_0 = S^n(1-\delta)$ lies in the domain bounded by Σ_0, under the mean curvature flow, Σ'_t still lies in the domain bounded by Σ_t, thus the extinction time of Σ_0 satisfies $T_e > \frac{1}{2n} - \delta_1$, where $\delta_1 \to 0$ as $\delta \to 0$.

A similar property as in Theorem 1.1 holds for solutions of Ricci flow, see Theorem 1.1 in Bamler and Maximo's paper [1]. Theorem 1.1 is motivated by [1], and its proof follows the ideas in [1] closely. There are also many almost rigidity type theorems for Riemannian manifolds, see e.g. [2, 3] etc.

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2. Proof of Theorem 1.1

Before the proof, we fix some notations. We denote by $T_0 = \frac{1}{4n}$, $\rho(t) = \frac{n}{\sqrt{1-2nt}}$. For the mean curvature flow solution $F : \Sigma \times [0, T) \to \mathbb{R}^{n+1}$, we denote by g_t the induced metric on Σ by the embedding F_t, and by d_t the distance induced by g_t. We use $B_{g_t}(x, r)$ to denote the geodesic ball with respect to the metric g_t. If we consider a sequence of solutions of mean curvature flow $F_i : \Sigma \times [0, T) \to \mathbb{R}^{n+1}$, we use the notations $F_{i,t}$, $H_i(x, t)$, $A_i(x, t)$, $g_{i,t}$, $d_{i,t}$ etc. to emphasize parameter i.

We recall the Harnack inequality for convex mean curvature flow due to Hamilton ([5]), which is very important in the argument of this note:

Proposition 2.1. *Let $F : \Sigma \times (0, T) \to \mathbb{R}^{n+1}$ be a convex solution to the mean curvature flow. Then for any $0 < t_1 < t_2 < T$, we have*

$$H(x_1, t_1) \le \sqrt{\frac{t_2}{t_1}} \exp\left(\frac{d_{t_1}^2(x_1, x_2)}{4(t_2 - t_1)}\right) H(x_2, t_2). \tag{6}$$

In the remaining part of this paper, we go to the proof of Theorem 1.1.

Lemma 2.2. *For any small positive number δ, there exists $\tau = \tau(n, \delta) > 0$ such that if F_t is a mean curvature flow with initial value F_0 satisfying* (A) *and has extinction time $T_e > \frac{1}{2n} - \tau$, then $\omega(t) < \rho(t) + \delta$ holds for every $t \in (0, T_0]$.*

Proof. Given $\delta > 0$, suppose there is some $\bar{t} \in (0, T_0]$ such that $\omega(\bar{t}) \ge \rho(\bar{t}) + \delta$, then by (4), for any $t \ge \bar{t}$, it holds

$$\omega(t) \ge \left(\frac{1}{(\rho(\bar{t}) + \delta)^{-2} - \frac{2}{n}(t - \bar{t})}\right)^{\frac{1}{2}}. \tag{7}$$

Thus

$$\begin{aligned} T_e &\le \bar{t} + \frac{n}{2}(\rho(\bar{t}) + \delta)^{-2} \\ &= \bar{t} + \frac{1 - 2n\bar{t}}{2n} \frac{1}{(1 + \frac{\sqrt{1-2n\bar{t}}}{n}\delta)^2} \\ &\le \bar{t} + \frac{1 - 2n\bar{t}}{2n} \frac{1}{1 + C_1\delta} \\ &\le \bar{t} + \frac{1 - 2n\bar{t}}{2n}(1 - C_2\delta) \\ &\le \frac{1}{2n} - C_3\delta, \end{aligned} \tag{8}$$

where C_1, C_2, C_3 are positive numbers depending only on n, and we assume δ is sufficiently small, and use $\bar{t} \in (0, \frac{1}{4n}]$ in the last three inequalities. In other word, if $T_e > \frac{1}{2n} - C_3\delta$, then $\omega(t) < \rho(t) + \delta$ holds for every $t \in (0, T_0]$. □

The following lemma is similar to Lemma 2.3 of [1], and the proof here is a modification of [1].

Lemma 2.3. *There exists a positive constant $K = K(n)$ such that, for any $t_2 \in (0, T_0]$, there exists $t_1 \in (\frac{t_2}{2}, t_2)$ depending only on t_2 and n such that, let $F : \Sigma \times [0, T_0) \to \mathbb{R}^{n+1}$ be a mean curvature flow with initial value satisfying* (A)*, if there exists a point $\bar{x} \in \Sigma$ such that $H(\bar{x}, t_2) < \rho(t_2) + 1$, then there exists a bounded nonnegative Lipschitz function $u(x, t)$ defined on $\Sigma \times [t_1, t_2)$ satisfying*:

(a) $(\frac{d}{dt^-} - \Delta_{g_t})u(x, t) \le 0$ *in the barrier sense*;
(b) $\forall t \in [t_1, t_2)$, $u(x, t)$ *is supported in* $B_{g_t}(\bar{x}, K\sqrt{t_2 - t})$;

(c) $0 \le u(\cdot,t) \le 1$ *and* $u(\bar{x},t) = \sqrt{t_2 - t}$.

Proof. Let $\varphi : [0,\infty) \to [0,\infty)$ be a fixed smooth function with $\varphi = 1$ on $[0,1]$, $\varphi \ge \frac{1}{2}$ on $[0,2]$, $\varphi = 0$ on $[3,\infty)$, $\varphi' \le 0$ on $[0,\infty)$, and $\varphi'' \ge 0$ on $[2,\infty)$. Then we can always choose $\alpha = \alpha(n) > 0$ (in fact, α also depends on the fixed function φ) sufficiently small such that

$$-\frac{1}{2}\varphi(r) + \frac{1}{2}r\varphi'(r) \le 4\alpha r\varphi''(r) + 2\alpha n\varphi'(r) \tag{9}$$

holds for every $r \ge 0$. One can easily check that such α always exists, see the Appendix of [1].

Then we fix $K = K(n)$ such that $\alpha K^2 > 3$.

By the Harnack inequality (6) and the convexity of the hypersurfaces, for any $t \in (\frac{t_2}{2}, t_2)$ and $x \in B_{g_t}(\bar{x}, 2K\sqrt{t_2 - t})$, it holds

$$|A(x,t)| \le H(x,t) \le \sqrt{\frac{t_2}{t}}\exp\Big(\frac{d_t^2(x,\bar{x})}{4(t_2-t)}\Big)(\rho(t_2)+1) \le C(n,t_2). \tag{10}$$

For any $x, y \in B_{g_t}(\bar{x}, K\sqrt{t_2-t})$, let $\gamma : [0, d_t(x,y)] \to B_{g_t}(\bar{x}, 2K\sqrt{t_2-t})$ be a path connecting x and y which is a shortest geodesic with respect to g_t and parametrized by arc length. Let $L_{\tilde{t}}$ be the length of γ with respect to $g_{\tilde{t}}$, where $\tilde{t}$ is in a small neighborhood of t, then we have

$$\begin{aligned} \frac{d}{d\tilde{t}}\Big|_{\tilde{t}=t} L_{\tilde{t}} &= \int_0^{d_t(x,y)} \Big(\frac{\partial}{\partial\tilde{t}}\Big|_{\tilde{t}=t}\sqrt{g_{\tilde{t}}(\frac{d\gamma}{ds},\frac{d\gamma}{ds})}\Big)ds \\ &= -\int_0^{d_t(x,y)} HA(\frac{d\gamma}{ds},\frac{d\gamma}{ds}) \ge -C^2 d_t(x,y), \end{aligned} \tag{11}$$

where we use (10) in the last inequality. Thus by (11),

$$\begin{aligned} \frac{d}{d\tilde{t}^-}\Big|_{\tilde{t}=t} d_{\tilde{t}}(x,y) &= \lim_{\tilde{t}\to t^-}\frac{d_{\tilde{t}}(x,y) - d_t(x,y)}{\tilde{t}-t} \\ &\ge \lim_{\tilde{t}\to t^-}\frac{L_{\tilde{t}} - L_t}{\tilde{t}-t} \ge -C^2 d_t(x,y). \end{aligned} \tag{12}$$

Since $t \mapsto d_t(x,y)$ is a Lipschitz function, it is differentiable for almost every t, and at those differentiable point, it holds

$$\frac{\partial}{\partial t}d_t(x,y) = \frac{d}{dt^-}d_t(x,y) \ge -C^2 d_t(x,y). \tag{13}$$

The function $u(x,t)$ will be chosen to have the form

$$u(x,t) = \sqrt{t_2-t}\,\varphi\big(\alpha\frac{d_t^2(\bar{x},x)}{t_2-t}\big). \tag{14}$$

It is easy to see that u satisfies (b) and (c).

By direct computation,

$$(\frac{d}{dt^-} - \Delta_{g_t})u(x,t) \le -\frac{1}{2\sqrt{t_2-t}}\varphi + \alpha\frac{d_t^2}{(t_2-t)^{\frac{3}{2}}}\varphi' - 2\alpha C^2\frac{d_t^2}{\sqrt{t_2-t}}\varphi' \tag{15}$$

$$- 4\alpha^2 \frac{d_t^2}{(t_2-t)^{\frac{3}{2}}}\varphi'' - 2n\alpha \frac{1}{\sqrt{t_2-t}}\varphi'$$

holds in the barrier sense, where we use (12) and the Laplacian comparison Theorem (recall that (Σ, g_t) has positive sectional curvature). In (15), d_t is short for $d_t(\bar{x}, x)$.

Choose $t_1 \in (\frac{t_2}{2}, t_2)$ such that $(t_2 - t_1)C^2 < \frac{1}{4}$. Recall that it holds $\varphi' \le 0$ and (9), we have $(\frac{d}{dt^-} - \Delta_{g_t})u(x,t) \le 0$ on $\Sigma \times [t_1, t_2)$. □

Lemma 2.4. *Suppose K is the constant given in Lemma 2.3. For any $t_2 \in (0, T_0]$, suppose $t_1 \in (\frac{t_2}{2}, t_2)$ is given in Lemma 2.3. Given $\theta > 0$ small, there exists $\delta = \delta(\theta, t_2, n) \in (0,1)$ such that, suppose $F : \Sigma \times [0, T_0) \to \mathbb{R}^{n+1}$ is a mean curvature flow with initial value satisfying* (A)*, if there exists $\bar{x} \in \Sigma$ such that $H(\bar{x}, t_2) \le \rho(t_2) + \delta$, then there exists $y \in B_{g_{t_1}}(\bar{x}, K\sqrt{t_2 - t_1})$ satisfying $H(y, t_1) \le \rho(t_1) + \theta$.*

Proof. Suppose there exists a small θ such that, for any i there exists a mean curvature flow $F_i : \Sigma \times [0, T_0) \to \mathbb{R}^{n+1}$ with initial value satisfying (A), and there exists $\bar{x}_i \in \Sigma$ such that $\rho(t_2) \le H_i(\bar{x}_i, t_2) \le \rho(t_2) + \frac{1}{i}$, but $H_i(y, t_1) > \rho(t_1) + \theta$ for any $y \in B_{g_{i,t_1}}(\bar{x}_i, K\sqrt{t_2 - t_1})$.

Let $u_i(x,t) : \Sigma \times [t_1, t_2) \to \mathbb{R}^+$ be the functions constructed in Lemma 2.3. Thus

$$\begin{aligned}(16)\qquad &\frac{d}{dt^-}\big(H_i(x,t) - \theta u_i(x,t)\big) \\ &\ge \Delta\big(H_i(x,t) - \theta u_i(x,t)\big) + \frac{1}{n}H_i^3(x,t) \\ &\ge \Delta\big(H_i(x,t) - \theta u_i(x,t)\big) + \frac{1}{n}\big(H_i(x,t) - \theta u_i(x,t)\big)^3.\end{aligned}$$

Note that for any $x \in \Sigma$,

$$H_i(x, t_1) \ge \rho(t_1) + \theta u_i(x, t_1), \tag{17}$$

with the strict inequality holds for $x \in B_{g_{i,t_1}}(\bar{x}_i, K\sqrt{t_2 - t_1})$. Thus by the maximum principle, for any $t \in (t_1, t_2)$, we have

$$H_i(x,t) > \rho(t) + \theta u_i(x,t). \tag{18}$$

In particular,

$$H_i(\bar{x}_i, t) > \rho(t) + \theta\sqrt{t_2 - t}. \tag{19}$$

On the other hand, by Proposition 2.1, for any $t \in [t_1, t_2]$,

$$H_i(\bar{x}_i, t) \le \sqrt{\frac{t_2}{t}} H_i(\bar{x}_i, t_2) \le \sqrt{\frac{t_2}{t}}(\rho(t_2) + \frac{1}{i}), \tag{20}$$

hence

$$\rho(t) + \theta\sqrt{t_2 - t} < \sqrt{\frac{t_2}{t}}(\rho(t_2) + \frac{1}{i}) \tag{21}$$

holds for every $t \in (t_1, t_2]$ and any i.

Note that there exists a positive constant C depending on t_2 and n such that $|\sqrt{\frac{t_2}{t}}\rho(t_2) - \rho(t)| \leq C(t_2 - t)$ for every $t \in [t_1, t_2]$. Let $i \to \infty$ in (21), we have

$$\theta\sqrt{t_2 - t} \leq C(t_2 - t) \tag{22}$$

for every $t \in [t_1, t_2]$, which is a contradiction. □

Lemma 2.5. *Suppose that K is the constant given in Lemma 2.3. Given $t_2 \in (0, T_0]$, suppose $t_1 \in (\frac{t_2}{2}, t_2)$ is given in Lemma 2.3. Then for any $\epsilon > 0$, there exists a $\delta = \delta(\epsilon, t_2, n) \in (0, 1)$ satisfying the following property: suppose $F : \Sigma \times [0, T_0) \to \mathbb{R}^{n+1}$ is a mean curvature flow with initial value $F_0(\Sigma)$ satisfying* (A)*, and there exists some $\bar{x} \in \Sigma$ satisfying $H(\bar{x}, t_2) \leq \rho(t_2) + \delta$, then*

$$\max_{x\in\Sigma} |A(x, t_1) - \frac{1}{n}\rho(t_1)\mathrm{Id}| < \epsilon. \tag{23}$$

Proof. Suppose that on the contrary, there exists $\epsilon_0 > 0$ such that, for any i there exist a mean curvature flow $F_i : \Sigma \times [0, T_0) \to \mathbb{R}^{n+1}$ with initial value satisfying (A) and $\bar{x}_i \in \Sigma$ with $H_i(\bar{x}_i, t_2) \leq \rho(t_2) + \frac{1}{i}$, but the conclusion (23) fails for ϵ_0.

By Lemma 2.4, there exist points $y_i \in B_{g_{i,t_1}}(\bar{x}_i, K\sqrt{t_2 - t_1})$ such that $\rho(t_1) \leq H_i(y_i, t_1) < \rho(t_1) + \delta_i$ with $\delta_i \to 0$ as $i \to \infty$. Without loss of generality, we assume $F_{i,t_1}(y_i) = 0 \in \mathbb{R}^{n+1}$.

Now we fix $D = \frac{16n\pi}{\rho(t_1)}$. By (6), for any $x \in B_{g_{i,t_1}}(y_i, D)$,

$$H_i(x, t_1) \leq \sqrt{\frac{t_2}{t_1}} \exp\left(\frac{(K\sqrt{t_2 - t_1} + D)^2}{4(t_2 - t_1)}\right) H_i(\bar{x}_i, t_2) \leq C(t_2, n), \tag{24}$$

and for any $t \in [\frac{t_1}{2}, t_1]$,

$$H_i(x, t) \leq \sqrt{\frac{t_1}{t}} H_i(x, t_1) \leq C(t_2, n). \tag{25}$$

Thus by the convexity of the hypersurface, for any $(x, t) \in B_{g_{i,t_1}}(y_i, D) \times [\frac{t_1}{2}, t_1]$, we have $|A_i(x, t)| \leq H_i(x, t) \leq C(t_2, n)$. Furthermore, by the curvature estimate of mean curvature flow (see [4]), for any $k \geq 1$, $|\nabla^k A_i(x, t)|$ is uniformly bounded on $B_{g_{i,t_1}}(y_i, \frac{D}{2}) \times [\frac{3t_1}{4}, t_1]$.

Thus by the method in [7], we can prove that, after passing to a subsequence of $\{i\}$, there exist an open set $U \subset \Sigma$ and a sequence of diffeomorphisms $\phi_i : U \to U_i \subset \Sigma$, such that $F_i(\phi_i(x), t)$ converges smoothly to a solution of mean curvature flow $F_\infty(x, t) : U \times [\frac{3t_1}{4}, t_1] \to \mathbb{R}^{n+1}$ and satisfy: (a) $B_{g_{i,t_1}}(y_i, \frac{D}{4}) \subset U_i \subset B_{g_{i,t_1}}(y_i, \frac{D}{2})$; (b) $\phi_i^{-1}(y_i) \to x_\infty \in U$ and hence $F_{\infty,t_1}(x_\infty) = 0$; (c) $H_\infty(x, t) \geq \rho(t)$ for every $(x, t) \in U \times [\frac{3t_1}{4}, t_1]$, and $H_\infty(x_\infty, t_1) = \rho(t_1)$. By the strong maximum principle, we have $H_\infty(x,t) = \rho(t)$ and hence

$$A_\infty(x,t) = \frac{1}{n}\rho(t)\mathrm{Id}$$

for every $(x,t) \in U \times [\frac{3t_1}{4}, t_1]$. Thus

$$\max_{x\in B_{g_{i,t_1}}(y_i,\frac{D}{4})} |A_i(x,t_1) - \frac{1}{n}\rho(t_1)\mathrm{Id}| \to 0 \tag{26}$$

as $i \to \infty$.

By (26) and Gauss equations, for i sufficiently large, we have

$$\mathrm{Ric}_{g_{i,t_1}} \geq \frac{n-1}{4n^2}(\rho(t_1))^2$$

on $B_{g_{i,t_1}}(y_i, \frac{D}{4})$. Then by Myer's Theorem, $B_{g_{i,t_1}}(y_i, \frac{D}{4})$ is the whole Σ. Thus (26) contradicts to the assumption at the begin and we complete the proof. □

We complete the proof of Theorem 1.1 in the following.

Proof of Theorem 1.1. Suppose there exists $\eta > 0$ such that for any i, there exists a sequence of embeddings $F_{i,0} : \Sigma \to \mathbb{R}^{n+1}$ satisfying the assumption (A) and the mean curvature flow $F_i : \Sigma \times [0, T_i) \to \mathbb{R}^{n+1}$ with initial value $F_{i,0}$ has extinction time $T_i \to \frac{1}{2n}$ as $i \to \infty$, but there does not exist a vector $v \in \mathbb{R}^{n+1}$ such that $v + F_{i,0}(\Sigma)$ can be viewed as a graph over $S^n(1)$ with C^0-norm less than η.

We choose a sequence of times $t_{2,i} \in (0, T_0)$ such that $t_{2,i} \to 0$ as $i \to \infty$. By Lemma 2.2, there exists $x_i \in \Sigma$ such that $H_i(x_i, t_{2,i}) < \rho(t_{2,i}) + \delta_i$, where $\delta_i \to 0$ as $i \to \infty$.

Then by Lemmas 2.3-2.5, there exist times $t_{1,i} \in (\frac{t_{2,i}}{2}, t_{2,i})$ such that

$$\max_{x\in\Sigma} |A_i(x, t_{1,i}) - \frac{1}{n}\rho(t_{1,i})\mathrm{Id}| < \epsilon_i, \tag{27}$$

where $\epsilon_i \to 0$ as $i \to \infty$.

Then by a compactness argument as in [7], we conclude that, for every i there is a vector $v_i \in \mathbb{R}^{n+1}$ such that, $v_i + F_{i,t_{1,i}}(\Sigma)$ is η_i-close in the $C^{1,1}$-norm to $S^n(1)$ with $\eta_i \to 0$.

By Proposition 2.1 and (27), for $t \in (0, t_{1,i}]$ and $x \in \Sigma$, we have

$$H_i(x,t) \leq \sqrt{\frac{t_{1,i}}{t}} H_i(x, t_{1,i}) \leq \frac{C}{\sqrt{t}}. \tag{28}$$

Here and in the following, C denotes a positive constant depending only on n, but the values of C may change in different lines. By (28),

$$|F_i(x,0) - F_i(x,t_{1,i})| \leq \int_0^{t_{1,i}} |H_i(x,s)|ds \leq \int_0^{t_{1,i}} \frac{C}{\sqrt{s}} ds \leq C\sqrt{t_{1,i}}, \tag{29}$$

which implies that $F_{i,0}(\Sigma)$ is $C\sqrt{t_{1,i}}$ close to $F_{i,t_{1,i}}(\Sigma)$ in the Hausdorff distance. Hence $F_{i,0}(\Sigma) + v_i$ is $C(\sqrt{t_{1,i}} + \eta_i)$-close to $S^n(1)$ in the Hausdorff distance. Because $F_{i,0}(\Sigma)$ is convex, $F_{i,0}(\Sigma) + v_i$ is a graph over $S^n(1)$ with C^0-norm less than $C(\sqrt{t_{1,i}} + \eta_i)$, which contradicts to the assumption at the beginning of the proof. The proof is completed. □

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Xian-Tao Huang
School of Mathematics
Sun Yat-sen University
Guangzhou 510275, P. R. China
Email address: huangxt28@mail.sysu.edu.cn; hxiant@mail2.sysu.edu.cn