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DOMINATION PRESERVING LINEAR OPERATORS OVER SEMIRINGS

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Suppose $\mathfrak{K}$ is a field and $\mathcal{M}$ is the set of all $m \times n$ matrices over $\mathfrak{K}$. If T is a linear operator on $\mathcal{M}$ and f is a function defined on $\mathcal{M}$, then T preserves f if $f(T(A)) = f(A)$ for all $A \in \mathcal{M}$.

Let $\mathcal{M}$ be the set of all $m \times n$ matrices over a semiring $\mathcal{S}$. In 1991, Beasley and Pullman characterized the linear operator on $\mathcal{M}$ that preserve the term rank. In particular, they obtained the following theorem about a term rank preserver over a semiring.

THEOREM. A. [2]. *If $\mathcal{S}$ is any semiring, then the followings are equivalent for any linear operator T on $\mathcal{M} = \mathcal{M}_{m,n}(\mathcal{S})$*

(i) *T is a (P, Q, B) operator.*
(ii) *T preserves term rank.*
(iii) *T preserves term rank 1 and term rank 2.*
(iv) *T strongly preserves term rank 1.*
(v) *T is nonsingular and preserves term rank 1 (when $\mathcal{S}$ is a field).*

The above theorem is very useful for characterization of various linear preservers on $\mathcal{M}_{m,n}(\mathcal{S})$. In fact, Beasley and Pullman [1] obtained the characterization of permanent preserver and rook–polynomial preserver by using (P, Q, B)-operator. Also, Beasley, G. Y. Lee and S. G. Lee [3,4] characterized the linear operators on the real matrices which preserve the value of an assignment function of each matrix by using a term rank preserver.

In this paper, we prove that T is a nonsigular domination preserver and $T(A^t) = T(A)^t$ for $A \in \mathcal{M}_{2,2}(\mathcal{S})$ if and only if T is a term rank

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preserver on $\mathcal{M}_{m,n}(\mathcal{S})$. Then, we shall have some useful tools that characterize linear preserving operators on $\mathcal{M}_{m,n}(\mathcal{S})$.

We start with some definitions. A *semiring* is a binary system $(\mathcal{S}, +, \times)$ such that $(\mathcal{S}, +)$ is an abelian monoid (identity 0), $(\mathcal{S}, \times)$ is a monoid (identity 1), $\times$ distributes over $+$, $0 \times s = s \times 0 = 0$ for all s in $\mathcal{S}$, and $1 \neq 0$. Usually $\mathcal{S}$ denotes the system and $\times$ is denoted by juxtaposition.

Here are some examples of semirings which occur in combinatorics. Let $\mathbb{B}$ be any Boolean algebra, then $(\mathbb{B}, \cup, \cap)$ is a semiring. Let $\mathbb{F}$ be the real interval $[0,1]$, then $(\mathbb{F}, \max, \min)$ is a semiring. If $\mathbb{P}$ is any subring of $\mathbb{R}$, the reals, and $\mathbb{P}^+$ denotes the non-negative members of $\mathbb{P}$, then $\mathbb{P}^+$ is a semiring.

Algebraic terms such as *unit* and *zero divisor* are defined for semirings as they are for rings.

The *linearity* of operators is defined as for vector space over fields.

Let $\mathcal{M}_{m,n}(\mathcal{S})$ denote the set of all $m \times n$ matrices over $\mathcal{S}$. The $m \times n$ matrix of $1's$ is denoted $J_{m,n}$. Let E_{ij} denote the (0,1)-matrix whose only nonzero entry is in the (i,j) position. A *cell* is a multiple of E_{ij} for some (i,j), so that the set of cells is the set

$$\{\alpha_{ij}E_{ij} : \alpha_{ij} \in \mathcal{S},\ 1 \leq i \leq m,\ \ 1 \leq j \leq n\}.$$

A linear operators over $\mathcal{S}$ is completely determined by its behavior on the set of cells in $\mathcal{M}_{m,n}(\mathcal{S})$.

From now on we will assume that $2 \leq m \leq n$ unless specified otherwise, and let $\mathcal{M} = \mathcal{M}_{m,n}(\mathcal{S})$ a fixed semiring $\mathcal{S}$.

The *pattern*, $\overline{A}$, of a matrix A in $\mathcal{M}$ is the (0,1)-matrix whose (i,j)th entry is 0 if and only if $a_{ij} = 0$. We will also assume that $\overline{A}$ is in $\mathcal{M}_{m,n}(\mathbb{B})$, where $\mathbb{B}$ denotes the Boolean algebra of two elements $(\{0,1\}, +, \times)$ where $+$ is $\cup$ and $\times$ is $\cap$.

If A and B are in $\mathcal{M}$, we say that B *dominates* A (written $B \geq A$ or $A \leq B$) if $b_{ij} = 0$ implies $a_{ij} = 0$ for all i, j. We write $B > A$ if $B \geq A$ and $A \not\geq B$ where $A \not\geq B$ if and only if $\overline{A} \neq \overline{B}$. Note that $A \leq B$ iff $\overline{A} \leq \overline{B}$, and that $\overline{A + B} \leq \overline{A} + \overline{B}$ for all A and B.

If T is a linear operator on $\mathcal{M}$, let $\overline{T}$, its pattern, be the linear operator on $\mathcal{M}_{m,n}(\mathbb{B})$ denoted by $\overline{T}(\overline{\alpha_{ij}E_{ij}}) = \overline{T(\alpha_{ij}E_{ij})}$ for all (i,j). Then $\overline{T(A)} \leq \overline{T}(\overline{A})$ for all $A \in \mathcal{M}$.

An important concept in the combinatorial theory of matrices is that of the term rank of a matrix. The *term rank* of A, $t(A)$, is the minimum number of lines (rows or columns) which contain all the non-zero entries of A. Evidently the term rank of a matrix is the term rank of its pattern, i.e.,

$$t(A) = t(\overline{A}).$$

If P and Q are $m \times m$ and $n \times n$ permutation matrices, resp., B is an $m \times n$ matrix in $\mathcal{M}$ over $\mathcal{S}$ none of whose entries is a zero divisor or zero, then T is a (P, Q, B)-*operator* if

(i) $T(X) = P(X \circ B)Q$ for all X in $\mathcal{M}$ or
(ii) $m = n$ and $T(X) = P(X^t \circ B)Q$ for all $X \in \mathcal{M}$.

Let T be a linear operator on $\mathcal{M}$ such that if $A \leq B$ then $T(A) \leq T(B)$. We call T a *domination preserving operator* on $\mathcal{M}$. From now on we will assume that T is a domination preserving linear operator on $\mathcal{M}$.

Remark. Let $\mathcal{M}$ be the set of 2×2 matrices with entries from $\mathbb{B}$, the boolean algebra of two elements. Consider the following linear operator $T : \mathcal{M} \to \mathcal{M}$, where T is given by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{whenever} \quad a, b, c, d \in \mathbb{B}.$$

Then T is a domination preserving operator since if

$$\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \leq \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$$

then

$$T\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \leq T\left(\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

i.e., $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \leq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Since T sends $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ to the zero matrix, T is not nonsingular. For example, let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

Then, we know neither $A \leq B$ nor $B \leq A$. But $T(A) \leq T(B)$. So, if T is singular then T is not much of interest. Therefore, we will assume that domination preserving linear operator T is nonsingular, from now on.

The number of nonzero entries of a matrix A is denoted by $|A|$.

LEMMA 1. *The linear operator T is bijective on the set of cells.*

Proof. Since T is nonsingular, $|T(X)| \geq 1$ for all nonzero matrix X in $\mathcal{M}$. Let $C_1, C_2, \cdots, C_{mn}$ are cells. Suppose that $|T(C)| \geq 2$ for some cell C. Without loss of generality, let $C = C_1$ and $|T(C_1)| \geq 2$. Let $M_1 = \overline{C}_1$. Then $|T(M_1)| \geq 2$. Let

$$M_j = \begin{cases} M_{j-1}, & \text{if } T(\overline{C}_j) \leq T(M_{j-1}); \\ M_{j-1} + \overline{C}_j, & \text{if } T(\overline{C}_j) \nleq T(M_{j-1}) \end{cases}$$

for $j = 2, 3, \ldots, mn$. Then $|M_j| \leq |M_{j-1}| + 1$ for all $2 \leq j \leq mn$. If equality hold for every $2 \leq j \leq mn$, then $|T(M_j)| \geq j + 1$ since $C_j \nleq M_{j-1}$ and $|T(M_1)| \geq 2$. In particular, $|T(M_{mn})| \geq mn + 1$, which is impossible. Thus $|M_{mn}| \leq mn - 1$ and there exists j such that $M_j = M_{j-1}$ and $T(\overline{C}_j) \leq T(M_{j-1})$. Then, for the j,

$$T(J) = T(J \setminus \overline{C}_j).$$

Since T is nonsingular and $J > J \setminus \overline{C}_j$, this is a contradiction. Therefore, $T(C)$ is a cell.

Now, let $i \neq j$, i.e., $C_i \neq C_j$. Suppose that $T(C_i) = T(C_j)$. Then, $\overline{T(C_i) + T(C_j)}$ is either $\overline{T(C_i)}$ or $\overline{T(C_j)}$ and

$$\begin{aligned} \overline{T(J)} &= \overline{T[J \setminus (C_i + C_j) + C_i + C_j]} \\ &= \overline{T[J \setminus (C_i + C_j)] + T(C_i) + T(C_j)} \\ &= \overline{T[J \setminus (C_i + C_j)] + T(C_i)} \\ &= \overline{T(J \setminus C_j)}. \end{aligned}$$

But $J > J \setminus C_j$. Therefore, $T(C_i) \neq T(C_j)$ by nonsingularity of T. ■

The following lemma 2 gives some domination properties for permutation and transposition.

LEMMA 2. *For $A, B \in \mathcal{M}$, if $A \leq B$ then*
(i) *$PAQ \leq PBQ$ for any $m \times m$, $n \times n$ permutation matrices P and Q, respectively.*
(ii) $A^t \leq B^t$.

Proof. The proof is straight forward. ■

REMARK. Let $A \leq B$ for $A, B \in \mathcal{M}$. Then, we can possibly choose a matrix X in $\mathcal{M}$ such that $A + X \not\leq B + X$. For example, let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \ B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \text{ and } X = \begin{bmatrix} 0 & -1 \\ 0 & -1 \end{bmatrix}.$$

Then $A \leq B$ and $A + X \geq B + X$. Thus, if T is a domination preserving operator, then $T(A)$ does not have a form $X + Y$, $X, Y \in \mathcal{M}$, in general.

We note that the domination can be varied with multiplication of (invertible) matrices, in general. That is, $UA \geq UB$ for $A \leq B$. For example, let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}.$$

Then $A \leq B$. We can choose an (invertible) matrix $U = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$. Then

$$UA = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \geq \begin{bmatrix} 1 & 5 \\ 2 & 0 \end{bmatrix} = UB.$$

Therefore, the domination preserving operator T does not have a form $T(A) = X + Y$ and $T(A) = UAV$ for some matrices X, Y, U, V, in general.

LEMMA 3. *For $A \in \mathcal{M}$, there exist $m \times m$, $n \times n$ permutation matrices U, V, respectively, such that*

$$T(PAQ) = UT(A)V$$

for some $m \times m$, $n \times n$ permutation matrices P and Q, resp..

Proof. Since T is bijective on the set of cells, there exists a bijective map f on indices set. Let $T(E_{ij}) = E_{rs}$. Then $T(PE_{ij}Q) = T(E_{\sigma(i)\tau(j)})$

where σ, τ are permutations with respect to P and Q, respectively. Since f is bijective on indices set,

$$f(\sigma(i), \tau(j)) = (\delta\sigma(i), \rho\tau(j))$$

for some permutations δ, ρ. Therefore, there exist $m \times m$, $n \times n$ permutation matrices U, V, resp., such that $T(PAQ) = UT(A)V$. ■

A matrix M in $\mathcal{M}$ is a *monomial* if the pattern of M is a column permutation of $[I_m \vdots \mathbf{0}_{m,n-m}]$ where I_m is the $m \times m$ identity matrix and $\mathbf{0}_{m,n-m}$ is the $m \times (n-m)$ zero matrix. In particular, if $m = n$ then $\overline{M}$ is a permutation matrix. If $L \leq M$ and M is a monomial, then we call L a *submonomial* matrix.

LEMMA 4. *Let $T(A^t) = T(A)^t$ for $A \in \mathcal{M}_{2,2}(\mathcal{S})$. There exists a monomial matrix $M \in \mathcal{M}$ such that $T(M)$ is a monomial.*

Proof. Let A be a monomial matrix with $t(T(A)) = k$. If $k = m$, then the proof is completed.

Suppose that $k < m$. Then, there exists a submonomial matrix B such that $B \leq A$ and $t(B) = t(T(B)) = k$. Since B is a submonomial with $t(B) = k$ and $t(T(B)) = k$, $T(B)$ is a submonomial matrix. Since $T(B)$ is a submonomial, there exist permutation matrices P, Q such that $T(B) = PBQ$. So, without loss of generality, let $T(B) = B = I_k \oplus \mathbf{0}_{m-k,n-k}$ and $P = I_k \oplus P'$, $Q = I_k \oplus Q'$ where P' and Q' are $(m-k) \times (m-k)$, $(n-k) \times (n-k)$ permutation matrices, resp.. Since $T(B)$ is a submonomial, there exists a submonomial matrix D such that $t(D) = m - k$ and $T(B) + D$ is a monomial matrix. Thus, $T(B) + D = PAQ$. That is, $D = P(A \setminus B)Q$. If $T(D)$ is a submonomial matrix, then $T(D) = PDQ$ and $T(B+D) = P(B+D)Q$ is a monomial matrix. Thus, if $T(D)$ is a submonomial matrix, we can construct a monomial matrix that whose image is a monomial matrix.

Now, suppose that $T(D)$ is not submonomial for any $1 \leq k < m$. For k, we can choose the $k = m - 2$. Then

$$\begin{aligned} T(D) &= \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \oplus \mathbf{0}_{m-2,n-2}, \text{ or;} \\ &= \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \oplus \mathbf{0}_{m-2,n-2}. \end{aligned}$$

Without loss of generality, we may assume that

$$T(D) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \oplus \mathbf{0}_{m-2,n-2} \text{ and } D = I_2 \oplus \mathbf{0}_{m-2,n-2}$$

Then, we only consider the linear preserving operator T on $\mathcal{M}_{2,2}(\mathcal{S})$.

Since T is bijective on the set of cells, without loss of generality, let $T(E_{11}) = E_{11}$. Then $T(E_{22}) = E_{12}$. Also, we may assume that $T(E_{21}) = E_{21}$. Then $T(E_{12}) = E_{22}$. Since $T(A^t) = T(A)^t$ for $A \in \mathcal{M}_{2,2}(\mathcal{S})$,

$$T\left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}^t\right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^t = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$\neq \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = T\left(\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\right) = T\left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}^t\right).$$

Therefore, there exists a monomial matrix $M \in \mathcal{M}$ such that $T(M)$ is a monomial matrix. ■

THEOREM 5. *Let $T(A^t) = T(A)^t$ for $A \in \mathcal{M}_{2,2}(\mathcal{S})$ and T be a domination preserving operator on $\mathcal{M}$. Then T preserves term rank 1 and term rank 2.*

Proof. First, we prove that T preserves term rank 1.

Suppose that T is not a term rank 1 preserver. Without loss of generality, let $T(E_{pq}) = E_{ij}$ and $T(E_{pv}) = E_{rs}$, $i \neq r$, $j \neq s$. Then, there exists a matrix M such that $|M| = m$, $E_{pq} + E_{pv} \leq M$ and $T(M)$ is a monomial. That is,

$$T(E_{pq} + E_{pv}) = E_{ij} + E_{rs} \leq T(M).$$

By Lemma 3 and Lemma 4, T preserves monomial matrices on $\mathcal{M}$. Thus, this is a contradiction and hence T preserves term rank 1.

Now, suppose that T is not a term rank 2 preserver. Then, there exist i, j, r, s such that

$$T(E_{ij} + E_{rs}) = E_{pq} + E_{pv}, \quad i \neq r,\ j \neq s.$$

Since T preserves term rank 1, this is a contradiction. Therefore, T preserves term rank 1 and term rank 2. ■

An immediate consequence of the above Theorem 5 is the following:

THEOREM 6. *If $\mathcal{S}$ is any semiring, then the following are equivalent for any linear operator T on $\mathcal{M}$.*

(i) *T is a (P, Q, B) operator.*
(ii) *T preserves term rank.*
(iii) *T preserves term rank 1 and term rank 2.*
(iv) *T strongly preserves term rank 1.*
(v) *T is nonsingular and preserves term rank 1 (when $\mathcal{S}$ is $\mathcal{S}$ a field).*
(vi) *T is nonsigular and preserves domination with $T(A^t) = T(A)^t$ for $A \in \mathcal{M}_{2,2}(\mathcal{S})$.*

Since, by above Theorem A and Theorem 5, the Theorem 6 is obvious, we state it without proof.

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