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EXTENDED INVERSE THEOREMS FOR RESTRICTED SUMSET IN INTEGERS

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Abstract. Let h and k be positive integers such that $h \leq k$. Let $A = \{a_0, a_1, \ldots, a_{k-1}\}$ be a nonempty finite set of k integers. The *h-fold sumset*, denoted by hA, is a set of integers that can be expressed as a sum of h elements (not necessarily distinct) of A. The *restricted h-fold sumset*, denoted by $h^\wedge A$, is a set of integers that can be expressed as a sum of h distinct elements of A. The characterization of the underlying set for small deviation from the minimum size of the sumset is called an *extended inverse problem*. Freiman studied such a problem and proved a theorem for $2A$, which is known as *Freiman's* $3k-4$ *theorem*. Very recently, Tang and Xing, and Mohan and Pandey studied some more extended inverse problems for the sumset hA, where $h \geq 2$. In this article, we prove some extended inverse theorems for sumsets $2^\wedge A$, $3^\wedge A$ and $4^\wedge A$. In particular, we classify the set(s) A for which $|2^\wedge A| = 2k-2$, $|2^\wedge A| = 2k-1$, and $|2^\wedge A| = 2k$. Furthermore, we also classify set(s) A when $|3^\wedge A| = 3k-7$, $|3^\wedge A| = 3k-6$, and $|4^\wedge A| = 4k-14$.

1. Introduction

Let $\mathbb{N}$ and $\mathbb{Z}$ be sets of positive integers and all integers, respectively. Let $A = \{a_0, a_1, \ldots, a_{k-1}\}$ be a nonempty finite set of k integers with $a_0 < a_1 < a_2 < \cdots < a_{k-1}$ and h be a positive integer. The *h-fold sumset* and the *restricted h-fold sumset*, denoted by hA and $h^\wedge A$, respectively, are defined as

$$hA := \left\{ \sum_{i=0}^{k-1} \lambda_i a_i : \lambda_i \in \mathbb{N} \cup \{0\} \text{ for } i = 0, 1, \ldots, k-1 \text{ with } \sum_{i=0}^{k-1} \lambda_i = h \right\},$$

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and

$$h^{\wedge}A := \left\{ \sum_{i=0}^{k-1} \lambda_i a_i : \lambda_i \in \{0,1\} \text{ for } i = 0, 1, \ldots, k-1 \text{ with } \sum_{i=0}^{k-1} \lambda_i = h \right\}.$$

In hA, each a_i appears at most h times as a summand, but in $h^{\wedge}A$, each a_i can appear at most once, so $h \leq k$ in $h^{\wedge}A$.

Throughout the paper, $|A|$ denotes the cardinality of the set A. For integers α and β, let

$$\alpha * A := \{\alpha a : a \in A\},$$
$$A + \beta := \{a + \beta : a \in A\},$$

and

$$[\alpha, \beta] := \{\alpha, \alpha + 1, \ldots, \beta\}.$$

For $\alpha > \beta$, we assume $[\alpha, \beta] = \emptyset$. For a given real number x, $\lfloor x \rfloor$ denotes the floor function of x. By $\min(A)$ and $\max(A)$, we mean the smallest and largest element of the set A, respectively. The greatest common divisor of the integers $x_1, x_2, \ldots, x_k$ is denoted by $d\,(x_1, x_2, \ldots, x_k)$. Let $A = \{a_0, a_1, \ldots, a_{k-1}\}$ be a set of integers with $a_0 < a_1 < \cdots < a_{k-1}$. Then, we define

$$d(A - a_0) := d(a_1 - a_0, a_2 - a_0, \ldots, a_{k-1} - a_0)$$

and

$$A^{(N)} := \left\{ a_i' = \frac{a_i - a_0}{d(A - a_0)} : a_i \in A \right\}.$$

Here $\min(A^{(N)}) = 0$ and $d(A^{(N)}) = 1$. The set $A^{(N)}$ is called the normal form of A. Note also that

$$h\left((\alpha * A) + \beta\right) = (\alpha * (hA)) + h\beta \text{ and } h^{\wedge}\left((\alpha * A) + \beta\right) = (\alpha * (h^{\wedge}A)) + h\beta.$$

So $|hA|$ and $|h^{\wedge}A|$ are translation and dilation invariant of the set A.

A *direct problem* associated with a sumset is to determine the minimum cardinality and properties of the sumset, and the *inverse problem* is the characterization of the underlying set for the minimum cardinality of the sumset. Problems associated with sumsets in groups have been the topic of study for more than two centuries. In 1813, Cauchy [1] studied sumset $A + B$, where A and B are nonempty subsets of residue classes modulo a prime p. Later, it was rediscovered by Davenport in 1935, so the result is known as *Cauchy-Davenport Theorem* [2]. The precise statement of Cauchy-Davenport Theorem is as follows.

Theorem 1.1 ([12, Theorem 2.3])**.** *Let A and B be nonempty finite subsets of $\mathbb{Z}_p$. Then*

$$|A + B| \geq \min\{|A| + |B| - 1,\ p\}.$$

Much later, Nathanson studied direct and inverse problems for hA and $h^{\wedge}A$.

Theorem 1.2 ([12, Theorem 1.4, Theorem 1.6])**.** *Let $h \geq 1$ and A be a nonempty finite set of integers. Then*

$$|hA| \geq h\,|A| - h + 1.$$

This lower bound is best possible. Furthermore, if $|hA|$ attains this lower bound with $h \geq 2$, then A is an arithmetic progression.

Theorem 1.3 ([12, Theorem 1.9, Theorem 1.10])**.** *Let A be a nonempty finite set of integers, and let $1 \leq h \leq |A|$. Then*

$$|h^\wedge A| \geq h\,|A| - h^2 + 1.$$

This lower bound is best possible. Furthermore, if $|h^\wedge A| = h\,|A| - h^2 + 1$ with $|A| \geq 5$ and $2 \leq h \leq |A| - 2$, then A is an arithmetic progression.

Characterization of the underlying set for small deviation from the minimum size of the sumset is called an *extended inverse problem.* The following results, Theorem 1.4 and Theorem 1.5, are some direct and extended inverse theorem for $2A$ proved by Freiman [3].

Theorem 1.4 ([3, Theorem 1.10])**.** *Let $k \geq 3$ and $A = \{a_0, a_1, \ldots, a_{k-1}\}$ be a set of integers such that $0 = a_0 < a_1 < \cdots < a_{k-1}$ with $d(A) = 1$.*

(a) *If $a_{k-1} \leq 2k - 3$, then $|2A| \geq k + a_{k-1}$.*
(b) *If $a_{k-1} \geq 2k - 2$, then $|2A| \geq 3k - 3$.*

Theorem 1.5 ([3, Theorem 1.9])**.** *Let A be a finite set of $k \geq 3$ integers. If $|2A| = 2k - 1 + b < 3k - 3$, then A is a subset of an arithmetic progression of length at most $k + b$.*

Lev [8] generalized Theorem 1.4 for hA. Recently, Tang and Xing [16] proved some extended inverse theorems for hA. Mohan and Pandey [11] extended the work of Tang and Xing and studied all possible inverse theorems for $|hA|$ when $hk - h + 1 < |hA| \leq hk + 2h + 1$. Some extended inverse theorems have been studied in abelian groups [5–7].

To the best of our knowledge, there is no analogous extended inverse theorem for $h^\wedge A$. Freiman conjectured (in personal communication with Lev) and also independently by Lev [9] the following conjecture.

Conjecture 1.6. Let $A = \{a_0, a_1, \ldots, a_{k-1}\}$ be a set of $k > 7$ integers such that $0 = a_0 < a_1 < \cdots < a_{k-1}$ with $d(A) = 1$. Then

$$|2^\wedge A| \geq \min\{a_{k-1}, 2k-5\} + k - 2 = \begin{cases} a_{k-1} + k - 2 & \text{if } a_{k-1} \leq 2k - 5 \\ 3k - 7 & \text{if } a_{k-1} \geq 2k - 4. \end{cases}$$

The lower bounds are tight, as letting $A = \{0, 1, \ldots, k-3\} \cup \{a_{k-1} - 1, a_{k-1}\}$, we get $2^\wedge A = \{1, 2, \ldots, 2k-7\} \cup \{a_{k-1} - 1, \ldots, a_{k-1} + k - 3\} \cup \{2a_{k-1} - 1\}$. Freiman *et al.* [4] did some work in proving Conjecture 1.6 but Theorem 1.7 by Lev [9] is close to Conjecture 1.6.

Theorem 1.7. *Let $A = \{a_0, a_1, \ldots, a_{k-1}\}$ be a set of $k \geq 3$ integers such that $0 = a_0 < a_1 < \cdots < a_{k-1}$ with $d(A) = 1$. Then*

$$|2^\wedge A| \geq \begin{cases} a_{k-1} + k - 2 & \textit{if } a_{k-1} \leq 2k - 5 \\ (\theta + 1)k - 6 & \textit{if } a_{k-1} \geq 2k - 4, \end{cases}$$

where $\theta = \dfrac{(1+\sqrt{5})}{2}$ *is the 'golden mean'.*

Some other important results on restricted sumset can be seen in [14], [10], [13], and [15]. In this article, we prove the extended inverse theorem for $2^\wedge A$, Theorem 1.8 and consequently, we prove extended inverse theorems for $3^\wedge A$ and $4^\wedge A$, Theorem 1.9 and Theorem 1.10, respectively. In Section 2, we have proved Theorem 1.8. Section 3, proofs of Theorem 1.9 and Theorem 1.10 are given. The precise statement of the main results of this paper is as follows.

Theorem 1.8. *Let $k \geq 7$ be a positive integer. Let A be a finite set of k nonnegative integers with $\min(A) = 0$ and $d(A) = 1$. Then*

(1) $|2^\wedge A| = 2k-2$ *if and only if* $A = [0,k]\setminus\{x\}$, *where* $x \in \{1, 2, k-2, k-1\}$.
(2) *For* $k \geq 9$, $|2^\wedge A| = 2k-1$ *if and only if* $A = [0,k+1] \setminus \{x,y\}$, *where* $\{x,y\}$ *is one of the sets* $\{1,2\}$, $\{k-1,k\}$, $\{2,3\}$, $\{k-2,k-1\}$, $\{1,3\}$, $\{k-2,k\}$, $\{1,4\}$, $\{k-3,k\}$, $\{1,k\}$, $\{1,k-1\}$, $\{2,k\}$, $\{2,k-1\}$, *and* $\{i,k+1\}$, *where* $3 \leq i \leq k-3$.
(3) *For* $k \geq 11$, $|2^\wedge A| = 2k$ *if and only if* $A = [0,k+2] \setminus \{x,y,z\}$, *where* $\{x,y,z\}$ *is one of the sets* $\{3,4,k+2\}$, $\{k-3,k-2,k+2\}$, $\{2,4,k+2\}$, $\{k-3,k-1,k+2\}$, $\{2,5,k+2\}$, $\{k-4,k-1,k+2\}$, $\{1,2,3\}$, $\{k-1,k,k+1\}$, $\{2,3,4\}$, $\{k-2,k-1,k\}$, $\{1,2,4\}$, $\{k-2,k,k+1\}$, $\{1,2,k+1\}$, $\{1,k,k+1\}$, $\{1,3,4\}$, $\{k-2,k-1,k+1\}$, $\{1,2,5\}$, $\{k-3,k,k+1\}$, $\{1,2,k\}$, $\{2,k,k+1\}$, $\{2,3,k\}$, $\{2,k-1,k\}$, $\{1,2,6\}$, $\{k-4,k,k+1\}$, $\{2,3,k+1\}$, $\{1,k-1,k\}$, $\{1,3,5\}$, $\{k-3,k-1,k+1\}$, $\{1,3,k\}$, $\{2,k-1,k+1\}$, $\{1,3,k+1\}$, $\{1,k-1,k+1\}$, $\{1,4,6\}$, $\{k-4,k-2,k+1\}$, $\{1,4,k\}$, $\{2,k-2,k+1\}$, $\{1,4,k+1\}$, $\{1,k-2,k+1\}$, $\{i,j,k+2\}$, *where* $i \in \{1,2\}$ *with* $i+4 \leq j \leq k-2$, *and* $\{i,j,k+2\}$, *where* $3 \leq i \leq j-4$ *with* $j \in \{k-1,k\}$.

Theorem 1.9. *Let A be a finite set of k nonnegative integers with $\min(A) = 0$ and $d(A) = 1$. Then*

(1) *For* $k \geq 10$, $|3^\wedge A| = 3k-7$ *if and only if* $A = [0,k] \setminus \{x\}$, *where* $x \in \{1,k-1\}$.
(2) *For* $k \geq 12$, $|3^\wedge A| = 3k-6$ *if and only if* $A = [0,k+1] \setminus \{x,y\}$, *where* $\{x,y\}$ *is one of the sets* $\{2,k+1\}$, $\{3,k+1\}$, $\{k-3,k+1\}$, $\{k-2,k+1\}$, $\{1,2\}$, $\{k-1,k\}$, *and* $\{1,k\}$.

Theorem 1.10. *Let $k \geq 12$ be a positive integer and A be a finite set of k nonnegative integers with $\min(A) = 0$ and $d(A) = 1$. Then $|4^\wedge A| = 4k-14$ if and only if $A = [0,k] \setminus \{x\}$, where $x \in \{1,k-1\}$.*

2. Proof of Theorem 1.8

Lemma 2.1. *Let b be a positive integer and $k \geq \left\lfloor \dfrac{5b+15}{3} \right\rfloor + 1$ be an integer. Let A be a nonempty finite set of k integers such that $|2^\wedge A| = 2k-3+b$. Then*

$$A^{(N)} = [0,k+b-1] \setminus \{x_1,\ldots,x_b\}, \text{ where } 1 \leq x_1 < \cdots < x_b \leq k+b-1.$$

Proof. Since the cardinality of sumset is translation and dilation invariant, we have $A = \{a_0, a_1, \ldots, a_{k-1}\}$ with $0 = a_0 < a_1 < \cdots < a_{k-1}$ and $d(A) = 1$. Now, if $a_{k-1} \geq 2k-4$, then by Theorem 1.7,

$$2k - 3 + b = |2^\wedge A| \geq 2.6k - 6.$$

This gives $k \leq \dfrac{5b+15}{3} < \left\lfloor \dfrac{5b+15}{3} \right\rfloor + 1$, which is a contradiction. Therefore $a_{k-1} \leq 2k-5$. Again by Theorem 1.7,

$$2k - 3 + b = |2^\wedge A| \geq a_{k-1} + k - 2.$$

This gives $a_{k-1} \leq k+b-1$. Hence $A^{(N)} = [0, k+b-1] \setminus \{x_1, \ldots, x_b\}$, where $1 \leq x_1 < \cdots < x_b \leq k+b-1$. This completes the proof of the lemma. □

In Proposition 2.2, we find $|2^\wedge A|$ when $A = \{a_0, a_1, \ldots, a_{k-1}\} = [0,k] \setminus \{x\}$, where $0 \leq a_0 < a_1 < \cdots < a_{k-1} \leq k$ and $0 \leq x \leq k$. Since the cardinality of sumset is translation invariant, therefore we assume $a_0 = 0$ and $x \geq 1$. Note also that, if $x = k$, then $A = \{a_0, a_1, \ldots, a_{k-1}\} = [0, k-1]$, and this gives $|2^\wedge A| = 2k-3$. Therefore, we assume $a_{k-1} = k$ and $x \leq k-1$.

Proposition 2.2. *Let $k \geq 5$ be a positive integer and $A = [0,k] \setminus \{x\}$, where $1 \leq x \leq k-1$. Then the following hold.*

(1) *If $x \in \{1, 2, k-2, k-1\}$, then $|2^\wedge A| = 2k-2$.*
(2) *If $3 \leq x \leq k-3$, then $|2^\wedge A| = 2k-1$.*

Proof. (1) If $x = 1$, then $A = \{0\} \cup [2,k]$. So $2^\wedge A = [2, 2k-1]$. If $x = 2$, then $A = \{0,1\} \cup [3,k]$. So $2^\wedge A = \{1\} \cup [3, 2k-1]$. If $x \in \{k-1, k-2\}$, then we write $A = [0,k] \setminus \{x\} = k - \{[0,k] \setminus \{y\}\}$, where $y \in \{1,2\}$. So $|2^\wedge A| = |2^\wedge([0,k] \setminus \{y\})|$, where $y \in \{1,2\}$. Hence, in each case $|2^\wedge A| = 2k-2$.

(2) If $3 \leq x \leq k-3$, then $A = [0, x-1] \cup [x+1, k]$. Clearly

$$[1, x-1] \cup [x+1, k+x-1] \cup [k+x+1, 2k-1] \subseteq 2^\wedge A.$$

Since $x - 1 > 1$ and $x + 1 < k - 1$, we have $x = x - 1 + 1 \in 2^\wedge A$ and $k + x = (k-1) + (x+1) \in 2^\wedge A$. So $2^\wedge A = [1, 2k-1]$, giving $|2^\wedge A| = 2k-1$. □

Next, we find $|2^\wedge A|$ when $A = \{a_0, a_1, \ldots, a_{k-1}\} = [0, k+1] \setminus \{x, y\}$, where $0 = a_0 < a_1 < \cdots < a_{k-1} \leq k+1$ and $1 \leq x < y \leq k+1$. Note that, if $y = k+1$, then $A = \{a_0, a_1, \ldots, a_{k-1}\} = [0,k] \setminus \{x\}$, where $1 \leq x \leq k$ and this was already considered in Proposition 2.2. Therefore, we assume $y \leq k$ and $a_{k-1} = k+1$.

Proposition 2.3. *Let $k \geq 5$ be a positive integer and $A = [0, k+1] \setminus \{x, x+1\}$, where $1 \leq x \leq k-1$.*

(1) *If $x \in \{1, k-1\}$, then $|2^\wedge A| = 2k-1$.*
(2) *If $x \in \{2, k-2\}$, then* $|2^\wedge A| = \begin{cases} 2k-1 & \text{if } k \geq 6 \\ 8 & \text{if } k = 5. \end{cases}$

(3) *If* $x \in \{3, k-3\}$, *then* $|2^\wedge A| = \begin{cases} 2k & \text{if } k \geq 7 \\ 11 & \text{if } k = 6 \\ 8 & \text{if } k = 5. \end{cases}$

(4) *If* $x \in [4, k-4]$, *then* $|2^\wedge A| = 2k+1$.

Proof. (1) If $x = 1$, then $A = \{0\} \cup [3, k+1]$. Note that

$$[3, k+1] \cup [k+4, 2k+1] \subseteq 2^\wedge A.$$

Since $k \geq 5$, so $k+2 = k-1+3 \in 2^\wedge A$ and $k+3 \in 2^\wedge A$. Therefore $2^\wedge A = [3, 2k+1]$. If $x = k-1$, then $A = [0, k-2] \cup \{k+1\} = k+1-(\{0\} \cup [3, k+1])$. Since $|2^\wedge A|$ is translation invariant, we have $|2^\wedge A| = |2^\wedge(\{0\} \cup [3, k+1])|$. Hence, in both the cases $|2^\wedge A| = 2k-1$.

(2) If $x = 2$, then $A = \{0, 1\} \cup [4, k+1]$. Clearly

$$\{1\} \cup [4, k+2] \cup [k+4, 2k+1] \subseteq 2^\wedge A.$$

Also, if $k \geq 6$, then $k+3 = k-1+4 \in 2^\wedge A$. Therefore

$$2^\wedge A = \begin{cases} \{1\} \cup [4, 2k+1] & \text{if } k \geq 6 \\ \{1, 4, 5, 6, 7, 9, 10, 11\} & \text{if } k = 5. \end{cases}$$

If $x = k-2$, then $A = [0, k-3] \cup \{k, k+1\} = k+1-(\{0, 1\} \cup [4, k+1])$. So $|2^\wedge A| = |2^\wedge(\{0, 1\} \cup [4, k+1])|$. Hence, in both the cases

$$|2^\wedge A| = \begin{cases} 2k-1 & \text{if } k \geq 6 \\ 8 & \text{if } k = 5. \end{cases}$$

(3) If $x = 3$, then $A = \{0, 1, 2\} \cup [5, k+1]$. Clearly

$$\{1, 2, 3\} \cup [5, k+3] \cup [k+6, 2k+1] \subseteq 2^\wedge A.$$

Note that, if $k \geq 7$, then $k+4 = k-1+5 \in 2^\wedge A$, and if $k \geq 6$, then $k+5 \in 2^\wedge A$. Therefore

$$2^\wedge A = \begin{cases} \{1, 2, 3\} \cup [5, 2k+1] & \text{if } k \geq 7 \\ \{1, 2, 3, 5, 6, 7, 8, 9, 11, 12, 13\} & \text{if } k = 6 \\ \{1, 2, 3, 5, 6, 7, 8, 11\} & \text{if } k = 5. \end{cases}$$

If $x = k-3$, then $A = [0, k-4] \cup \{k-1, k, k+1\} = k+1-(\{0, 1, 2\} \cup [5, k+1])$. So $|2^\wedge A| = |2^\wedge(\{0, 1, 2\} \cup [5, k+1])|$. Hence, in both the cases

$$|2^\wedge A| = \begin{cases} 2k & \text{if } k \geq 7 \\ 11 & \text{if } k = 6 \\ 8 & \text{if } k = 5. \end{cases}$$

(4) If $x \in [4, k-4]$, then $A = [0, x-1] \cup [x+2, k+1]$. Clearly $2^\wedge A = [1, 2k+1]$ and therefore $|2^\wedge A| = 2k+1$. □

Proposition 2.4. *Let* $k \geq 5$ *be a positive integer and* $A = [0, k+1] \setminus \{x, x+2\}$, *where* $1 \leq x \leq k-2$.

(1) *If* $x \in \{1, k-2\}$, *then* $|2^\wedge A| = 2k-1$.

(2) *If* $x \in \{2, k-3\}$, *then* $|2^\wedge A| = \begin{cases} 2k & \text{if } k \geq 6 \\ 9 & \text{if } k = 5. \end{cases}$

(3) *If* $x \in [3, k-4]$, *then* $|2^\wedge A| = 2k+1$.

Proof. (1) If $x = 1$, then $A = \{0, 2\} \cup [4, k+1]$. It is easy to see that

$$2^\wedge A = \{2\} \cup [4, 2k+1].$$

If $x = k-2$, then $A = [0, k-3] \cup \{k-1, k+1\} = k+1-(\{0, 2\} \cup [4, k+1])$. So $|2^\wedge A| = |2^\wedge(\{0, 2\} \cup [4, k+1])|$. Hence, in both the cases $|2^\wedge A| = 2k-1$.

(2) If $x = 2$, then $A = \{0, 1, 3\} \cup [5, k+1]$. Clearly $\{1\} \cup [3, k+4] \cup [k+6, 2k+1] \subseteq 2^\wedge A$. Note that, if $k \geq 6$, then $k+5 \in 2^\wedge A$. Therefore

$$2^\wedge A = \begin{cases} \{1\} \cup [3, 2k+1] & \text{if } k \geq 6 \\ \{1, 3, 4, 5, 6, 7, 8, 9, 11\} & \text{if } k = 5. \end{cases}$$

If $x = k-3$, then $A = [0, k-4] \cup \{k-2, k, k+1\} = k+1-(\{0, 1, 3\} \cup [5, k+1])$. So $|2^\wedge A| = |2^\wedge(\{0, 1, 3\} \cup [5, k+1])|$. Hence, in both the cases

$$|2^\wedge A| = \begin{cases} 2k & \text{if } k \geq 6 \\ 9 & \text{if } k = 5. \end{cases}$$

(3) If $x \in [3, k-4]$, then $A = [0, x-1] \cup \{x+1\} \cup [x+3, k+1]$. It follows that $2^\wedge A = [1, 2k+1]$ and $|2^\wedge A| = 2k+1$. □

Proposition 2.5. *Let* $k \geq 5$ *be a positive integer and* $A = [0, k+1] \setminus \{x, x+3\}$, *where* $1 \leq x \leq k-3$.

(1) *If* $x \in \{1, k-3\}$, *then*

$$|2^\wedge A| = \begin{cases} 2k-1 & \text{if } k \geq 6 \\ 8 & \text{if } k = 5. \end{cases}$$

(2) *If* $x \in \{2, k-4\}$ *with* $k \geq 6$, *then*

$$|2^\wedge A| = \begin{cases} 2k & \text{if } k \geq 7 \\ 11 & \text{if } k = 6. \end{cases}$$

(3) *If* $x \in [3, k-5]$, *then* $|2^\wedge A| = 2k+1$.

Proof. (1) If $x = 1$, then $A = \{0, 2, 3\} \cup [5, k+1]$. Clearly

$$\{2, 3\} \cup [5, k+4] \cup [k+6, 2k+1] \subseteq 2^\wedge A.$$

Note that, if $k \geq 6$, then $k+5 \in 2^\wedge A$. Therefore

$$2^\wedge A = \begin{cases} \{2, 3\} \cup [5, 2k+1] & \text{if } k \geq 6 \\ \{2, 3, 5, 6, 7, 8, 9, 11\} & \text{if } k = 5. \end{cases}$$

If $x = k-3$, then $A = [0, k-4] \cup \{k-2, k-1, k+1\} = k+1-(\{0, 2, 3\} \cup [5, k+1])$. Hence, in both the cases

$$|2^\wedge A| = \begin{cases} 2k-1 & \text{if } k \geq 6 \\ 8 & \text{if } k = 5. \end{cases}$$

(2) If $x = 2$, then $A = \{0, 1, 3, 4\} \cup [6, k+1]$. It is easy to see that

$$\{1\} \cup [3, k+2] \cup \{k+4, k+5\} \subseteq 2^\wedge A.$$

Since $k \geq 6$, we have $k + 3 \in 2^\wedge A$ and $[k + 7, 2k + 1] \subseteq 2^\wedge A$. Also, if $k \geq 7$, then $k + 6 \in 2^\wedge A$. Therefore

$$2^\wedge A = \begin{cases} \{1\} \cup [3, 2k+1] & \text{if } k \geq 7 \\ \{1, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13\} & \text{if } k = 6. \end{cases}$$

If $x = k - 4$, then $A = [0, k - 5] \cup \{k - 3, k - 2, k, k + 1\} = k + 1 - (\{0, 1, 3, 4\} \cup [6, k + 1])$. Hence, in both the cases

$$|2^\wedge A| = \begin{cases} 2k & \text{if } k \geq 7 \\ 11 & \text{if } k = 6. \end{cases}$$

(3) If $x \in [3, k-5]$, then $A = [0, x-1] \cup \{x+1, x+2\} \cup [x+4, k+1]$. It is easy to see that $2^\wedge A = [1, 2k+1]$ and $|2^\wedge A| = 2k+1$. □

Proposition 2.6. *Let $k \geq 5$ be a positive integer and $A = [0, k+1] \setminus \{x, y\}$, where $1 \leq x < y \leq k$ and $y - x \geq 4$.*

(1) *If $\{x, y\}$ is one of the sets $\{1, k\}$ and $\{2, k-1\}$, then $|2^\wedge A| = 2k-1$.*
(2) *If $k \geq 6$ and $\{x, y\}$ is one of the sets $\{1, k-1\}$ and $\{2, k\}$, then $|2^\wedge A| = 2k-1$.*
(3) *If $x \in \{1, 2\}$ and $x + 4 \leq y \leq k-2$, then $|2^\wedge A| = 2k$.*
(4) *If $3 \leq x \leq y-4$ and $y \in \{k, k-1\}$, then $|2^\wedge A| = 2k$.*
(5) *If $3 \leq x \leq y-4 \leq k-6$, then $|2^\wedge A| = 2k+1$.*

Proof. (1) If $\{x, y\} = \{1, k\}$, then $A = \{0\} \cup [2, k-1] \cup \{k+1\}$. It follows that

$$[2, k-1] \cup \{k+1\} \cup [k+3, 2k] \subseteq 2^\wedge A.$$

Since $k \geq 5$, we have $k = k-2+2 \in 2^\wedge A$ and $k+2 = k-1+3 \in 2^\wedge A$. Therefore $2^\wedge A = [2, 2k]$. So $|2^\wedge A| = 2k-1$.

If $\{x, y\} = \{2, k-1\}$, then $A = \{0, 1\} \cup [3, k-2] \cup \{k, k+1\}$. It is easy see that $2^\wedge A = \{1\} \cup [3, 2k-1] \cup \{2k+1\}$ and $|2^\wedge A| = 2k-1$.

(2) If $\{x, y\} = \{1, k-1\}$, then $A = \{0\} \cup [2, k-2] \cup \{k, k+1\}$. It follows that

$$[2, k-2] \cup [k, 2k-1] \cup \{2k+1\} \subseteq 2^\wedge A.$$

Note that $2k \notin 2^\wedge A$, and if $k \geq 6$, then $k-1 = k-3+2 \in 2^\wedge A$. Therefore $2^\wedge A = [2, 2k-1] \cup \{2k+1\}$. If $\{x, y\} = \{2, k\}$, then $A = \{0, 1\} \cup [3, k-1] \cup \{k+1\} = k+1-(\{0\} \cup [2, k-2] \cup \{k, k+1\})$. Therefore, in both the cases $|2^\wedge A| = 2k-1$.

(3) If $x \in \{1, 2\}$ and $x + 4 \le y \le k - 2$, then $A = [0, x - 1] \cup [x + 1, y - 1] \cup [y + 1, k + 1]$. It is easy to see that

$$2^\wedge A = [1, 2k + 1] \setminus \{x\}, \text{ where } x \in \{1, 2\}.$$

Therefore $|2^\wedge A| = 2k$.

(4) If $3 \le x \le y - 4$ and $y \in \{k, k-1\}$, then $A = [0, x-1] \cup [x+1, y-1] \cup [y+1, k+1] = k+1-([0, x_0 - 1] \cup [x_0 + 1, y_0 - 1] \cup [y_0 + 1, k+1])$ where $x_0 \in \{1, 2\}$ and $x_0 + 4 \le y_0 \le k - 2$. Therefore, by the previous case $|2^\wedge A| = 2k$.

(5) If $3 \le x < y \le k - 2$, then $A = [0, x - 1] \cup [x + 1, y - 1] \cup [y + 1, k + 1]$. It is easy to see that $2^\wedge A = [1, 2k + 1]$. So $|2^\wedge A| = 2k + 1$. □

Next we find $|2^\wedge A|$ when $A = [0, k+2] \setminus \{x, y, z\}$ with $1 \le x < y < z \le k+1$. Proposition 2.7 is based on the case when x, y, and z are consecutive integers.

Proposition 2.7. *Let $k \ge 5$ be a positive integer and $A = [0, k + 2] \setminus \{x, x + 1, x + 2\}$, where $1 \le x \le k - 1$.*

(1) If $x \in \{1, k - 1\}$, then

$$|2^\wedge A| = \begin{cases} 2k & \text{if } k \ge 6 \\ 9 & \text{if } k = 5. \end{cases}$$

(2) If $x \in \{2, k - 2\}$, then

$$|2^\wedge A| = \begin{cases} 2k & \text{if } k \ge 7 \\ 11 & \text{if } k = 6 \\ 8 & \text{if } k = 5. \end{cases}$$

(3) If $x \in \{3, k - 3\}$ with $k \ge 6$, then

$$|2^\wedge A| = \begin{cases} 2k + 1 & \text{if } k \ge 8 \\ 14 & \text{if } k = 7 \\ 11 & \text{if } k = 6. \end{cases}$$

(4) If $x \in \{4, k - 4\}$ with $k \ge 8$, then

$$|2^\wedge A| = \begin{cases} 2k + 2 & \text{if } k \ge 9 \\ 17 & \text{if } k = 8. \end{cases}$$

(5) If $x \in [5, k - 5]$ with $k \ge 10$, then $|2^\wedge A| = 2k + 3$.

Proof. (1) If $x = 1$, then $A = \{0\} \cup [4, k + 2]$. Clearly

$$[4, k + 2] \cup [k + 4, 2k + 3] \subseteq 2^\wedge A.$$

Note that, if $k \ge 6$, then $k + 3 = (k - 1) + 4 \in 2^\wedge A$. Therefore

$$2^\wedge A = \begin{cases} [4, 2k + 3] & \text{if } k \ge 6 \\ \{4, 5, 6, 7, 9, 10, 11, 12, 13\} & \text{if } k = 5. \end{cases}$$

If $x = k-1$, then $A = [0, k-2] \cup \{k+2\} = k+2-(\{0\} \cup [4, k+2])$. Hence, in both the cases

$$|2^\wedge A| = \begin{cases} 2k & \text{if } k \geq 6 \\ 9 & \text{if } k = 5. \end{cases}$$

(2) If $x = 2$, then $A = \{0, 1\} \cup [5, k+2]$. Clearly

$$\{1\} \cup [5, k+3] \cup [k+6, 2k+3] \subseteq 2^\wedge A.$$

Note that, if $k \geq 6$, then $k+5 \in 2^\wedge A$, and if $k \geq 7$, then $k+4 = k-1+5 \in 2^\wedge A$. Therefore

$$2^\wedge A = \begin{cases} \{1\} \cup [5, 2k+3] & \text{if } k \geq 7 \\ \{1, 5, 6, 7, 8, 9, 11, 12, 13, 14, 15\} & \text{if } k = 6 \\ \{1, 5, 6, 7, 8, 11, 12, 13\} & \text{if } k = 5. \end{cases}$$

If $x = k-2$, then $A = [0, k-3] \cup \{k+1, k+2\} = k+2-(\{0, 1\} \cup [5, k+2])$. Hence, in both the cases

$$|2^\wedge A| = \begin{cases} 2k & \text{if } k \geq 7 \\ 11 & \text{if } k = 6 \\ 8 & \text{if } k = 5. \end{cases}$$

(3) If $x = 3$, then $A = \{0, 1, 2\} \cup [6, k+2]$. It follows that

$$\{1, 2, 3\} \cup [6, k+4] \cup [k+7, 2k+3] \subseteq 2^\wedge A.$$

Note that, if $k \geq 8$, then $k+5 = k-1+6 \in 2^\wedge A$, and if $k \geq 7$, then $k+6 \in 2^\wedge A$. Therefore

$$2^\wedge A = \begin{cases} \{1, 2, 3\} \cup [6, 2k+3] & \text{if } k \geq 8 \\ \{1, 2, 3, 6, 7, 8, 9, 10, 11, 13, 14, 15, 16, 17\} & \text{if } k = 7 \\ \{1, 2, 3, 6, 7, 8, 9, 10, 13, 14, 15\} & \text{if } k = 6. \end{cases}$$

If $x = k-3$, then $A = [0, k-4] \cup \{k, k+1, k+2\} = k+2-(\{0, 1, 2\} \cup [6, k+2])$. Hence, in both the cases

$$|2^\wedge A| = \begin{cases} 2k+1 & \text{if } k \geq 8 \\ 14 & \text{if } k = 7 \\ 11 & \text{if } k = 6. \end{cases}$$

(4) If $x = 4$, then $A = \{0, 1, 2, 3\} \cup [7, k+2]$. It follows that

$$\{1, 2, 3, 4, 5\} \cup [7, k+5] \cup [k+7, 2k+3] \subseteq 2^\wedge A.$$

Note that, if $k \geq 9$, then $k+6 = k-1+7 \in 2^\wedge A$. Therefore

$$2^\wedge A = \begin{cases} [1, 5] \cup [7, 2k+3] & \text{if } k \geq 9 \\ [1, 5] \cup [7, 13] \cup [15, 19] & \text{if } k = 8. \end{cases}$$

If $x = k-4$, then $A = [0, k-5] \cup [k-1, k+2] = k+2-(\{0,1,2,3\} \cup [7, k+2])$. Hence, in both the cases

$$|2^\wedge A| = \begin{cases} 2k+2 & \text{if } k \geq 9 \\ 17 & \text{if } k = 8. \end{cases}$$

(5) If $x \in [5, k-5]$, then $A = [0, x-1] \cup [x+3, k+2]$. It is easy to see that $2^\wedge A = [1, 2k+3]$ and $|2^\wedge A| = 2k+3$. □

Proposition 2.8 and Proposition 2.9 are based on the case when exactly two of x, y, z are consecutive integers. In Proposition 2.8, we consider the case when $A = [0, k+2] \setminus \{x, y, z\}$, where $y = x+1$ and $1 < x < y < z \leq k+1$. Note that, if $A = [0, k+2] \setminus \{x, y, z\}$ with $y = x+1$, then $A' = k+2-A = [0, k+2] \setminus \{x_0, y_0, z_0\}$ where $y_0 = z_0 - 1$. Therefore, it is sufficient to find $|2^\wedge A|$ when x and y are consecutive. Hence, we obtain Proposition 2.9 from the Proposition 2.8 by replacing A to $(k+2) - A$.

Proposition 2.8. *Let $k \geq 5$ be a positive integer and $A = [0, k+2] \setminus \{x, y, z\}$, where $y = x+1$ and $1 < x < y < z \leq k+1$.*

(1) *If $\{x, y, z\} = \{1, 2, 4\}$, then $|2^\wedge A| = 2k$.*

(2) *If $\{x, y, z\}$ is one of the sets $\{1, 2, k+1\}$ and $\{k-2, k-1, k+1\}$, then*

$$|2^\wedge A| = \begin{cases} 2k & \text{if } k \geq 6 \\ 9 & \text{if } k = 5. \end{cases}$$

(3) *If $\{x, y, z\} = \{1, 2, 5\}$, then* $|2^\wedge A| = \begin{cases} 2k & \text{if } k \geq 6 \\ 8 & \text{if } k = 5. \end{cases}$

(4) *If $k \geq 6$ and $\{x, y, z\}$ is one of the sets $\{1, 2, k\}$ and $\{2, 3, k\}$, then*

$$|2^\wedge A| = \begin{cases} 2k & \text{if } k \geq 7 \\ 11 & \text{if } k = 6. \end{cases}$$

(5) *If $k \geq 7$ and $\{x, y, z\} = \{1, 2, 6\}$, then $|2^\wedge A| = 2k$.*

(6) *If $k \geq 6$ and $\{x, y, z\} = \{2, 3, k+1\}$, then $|2^\wedge A| = 2k$.*

(7) *If $\{x, y, z\} = \{2, 3, 5\}$, then* $|2^\wedge A| = \begin{cases} 2k+1 & \text{if } k \geq 6 \\ 9 & \text{if } k = 5. \end{cases}$

(8) *If $\{x, y, z\} = \{2, 3, 6\}$, then* $|2^\wedge A| = \begin{cases} 2k+1 & \text{if } k \geq 7 \\ 2k-1 & \text{if } k = 5 \text{ or } 6. \end{cases}$

(9) *If $k \geq 7$ and $\{x, y, z\} = \{k-3, k-2, k+1\}$, then $|2^\wedge A| = 2k+1$.*

(10) *If $k \geq 8$ and $\{x, y, z\}$ is one of the sets $\{k-4, k-3, k\}$, $\{2, 3, i\}$ where $7 \leq i \leq k-1$, and $\{1, 2, i\}$ where $7 \leq i \leq k-1$, then $|2^\wedge A| = 2k+1$.*

(11) *If $k \geq 6$ and $\{x, y, z\}$ is one of the sets $\{k-3, k-2, k\}$ and $\{3, 4, k+1\}$, then*

$$|2^\wedge A| = \begin{cases} 2k+1 & \text{if } k \geq 7 \\ 12 & \text{if } k = 6. \end{cases}$$

(12) *If* $\{x, y, z\} = \{3, 4, k\}$ *with* $k \geq 6$*, then*

$$|2^\wedge A| = \begin{cases} 2k + 1 & \textit{if } k \geq 8 \\ 2k & \textit{if } k = 6 \textit{ or } 7. \end{cases}$$

(13) *If* $k \geq 7$ *and* $\{x, y, z\}$ *is one of the sets* $\{3, 4, i\}$ *where* $6 \leq i \leq k-1$*,* $\{i, i+1, k\}$ *where* $4 \leq i \leq k-5$*, and* $\{i, i+1, k+1\}$ *where* $4 \leq i \leq k-4$*, then* $|2^\wedge A| = 2k + 2$*.*

(14) *If* $\{x, y, z\} = \{i, i+1, j\}$ *where* $4 \leq i \leq j-3 \leq k-4$*, then* $|2^\wedge A| = 2k+3$*.*

Proof. (1) If $\{x, y, z\} = \{1, 2, 4\}$, then $A = \{0, 3\} \cup [5, k + 2]$. It follows that $2^\wedge A = \{3\} \cup [5, 2k + 3]$. So $|2^\wedge A| = 2k$.

(2) If $\{x, y, z\} = \{1, 2, k+1\}$, then $A = \{0\} \cup [3, k] \cup \{k+2\}$. It follows that

$$[3, k] \cup [k + 2, 2k + 2] \subseteq 2^\wedge A.$$

Note that, if $k \geq 6$, then $k + 1 = (k - 2) + 3 \in 2^\wedge A$. Therefore

$$2^\wedge A = \begin{cases} [3, 2k + 2] & \text{if } k \geq 6 \\ \{3, 4, 5, 7, 8, 9, 10, 11, 12\} & \text{if } k = 5. \end{cases}$$

If $\{x, y, z\} = \{k - 2, k - 1, k + 1\}$, then $A = [0, k - 3] \cup \{k, k + 2\}$. It follows that

$$[1, k - 2] \cup [k, 2k - 1] \cup \{2k + 2\} \subseteq 2^\wedge A.$$

Note that, if $k \geq 6$, then $k - 1 = k - 3 + 2 \in 2^\wedge A$. Therefore

$$2^\wedge A = \begin{cases} [1, 2k - 1] \cup \{2k + 2\} & \text{if } k \geq 6 \\ \{1, 2, 3, 5, 6, 7, 8, 9, 12\} & \text{if } k = 5. \end{cases}$$

Hence, in both the cases

$$|2^\wedge A| = \begin{cases} 2k & \text{if } k \geq 6 \\ 9 & \text{if } k = 5. \end{cases}$$

(3) If $\{x, y, z\} = \{1, 2, 5\}$, then $A = \{0, 3, 4\} \cup [6, k + 2]$. It follows that

$$\{3, 4\} \cup [6, k + 2] \cup \{k + 4, k + 5, k + 6\} \cup [k + 8, 2k + 3] \subseteq 2^\wedge A.$$

Note that, if $k \geq 6$, then $\{k + 3, k + 7\} \subseteq 2^\wedge A$. Therefore

$$2^\wedge A = \begin{cases} \{3, 4\} \cup [6, 2k + 3] & \text{if } k \geq 6 \\ \{3, 4, 6, 7, 9, 10, 11, 13\} & \text{if } k = 5. \end{cases}$$

So

$$|2^\wedge A| = \begin{cases} 2k & \text{if } k \geq 6 \\ 8 & \text{if } k = 5. \end{cases}$$

(4) If $\{x, y, z\} = \{1, 2, k\}$ with $k \geq 6$, then $A = \{0\} \cup [3, k-1] \cup \{k+1, k+2\}$. It follows that

$$[3, k - 1] \cup [k + 1, 2k + 1] \cup \{2k + 3\} \subseteq 2^\wedge A.$$

Note that if $k \geq 7$, then $k = k - 3 + 3 \in 2^\wedge A$. Therefore

$$2^\wedge A = \begin{cases} [3, 2k+1] \cup \{2k+3\} & \text{if } k \geq 7 \\ \{3,4,5,7,8,9,10,11,12,13,15\} & \text{if } k = 6. \end{cases}$$

If $\{x, y, z\} = \{2, 3, k\}$ with $k \geq 6$, then $A = \{0, 1\} \cup [4, k-1] \cup \{k+1, k+2\}$. Clearly

$$\{1\} \cup [4, k+3] \cup [k+5, 2k+1] \cup \{2k+3\} \subseteq 2^\wedge A.$$

Note that, if $k \geq 7$, then $k + 4 = k - 1 + 5 \in 2^\wedge A$. Therefore

$$2^\wedge A = \begin{cases} \{1\} \cup [4, 2k+1] \cup \{2k+3\} & \text{if } k \geq 7 \\ \{1,4,5,6,7,8,9,11,12,13,15\} & \text{if } k = 6. \end{cases}$$

Hence, in each of the cases

$$|2^\wedge A| = \begin{cases} 2k & \text{if } k \geq 7 \\ 11 & \text{if } k = 6. \end{cases}$$

(5) If $\{x, y, z\} = \{1, 2, 6\}$ with $k \geq 7$, then $A = \{0, 3, 4, 5\} \cup [7, k+2]$. It is easy to see that $2^\wedge A = \{3, 4, 5\} \cup [7, 2k+3]$. So $|2^\wedge A| = 2k$.

(6) If $\{x, y, z\} = \{2, 3, k+1\}$ with $k \geq 6$, then $A = \{0, 1\} \cup [4, k] \cup \{k+2\}$. It is easy to see that $2^\wedge A = \{1\} \cup [4, 2k+2]$. So $|2^\wedge A| = 2k$.

(7) If $\{x, y, z\} = \{2, 3, 5\}$, then $A = \{0, 1, 4\} \cup [6, k+2]$. It follows that

$$\{1\} \cup [4, k+3] \cup \{k+5, k+6\} \cup [k+8, 2k+3] \subseteq 2^\wedge A.$$

Note that, if $k \geq 6$, then $k+4$ and $k+7 = k+1+6$ are in $2^\wedge A$. Therefore

$$2^\wedge A = \begin{cases} \{1\} \cup [4, 2k+3] & \text{if } k \geq 6 \\ \{1,4,5,6,7,8,10,11,13\} & \text{if } k = 5. \end{cases}$$

So

$$|2^\wedge A| = \begin{cases} 2k+1 & \text{if } k \geq 6 \\ 9 & \text{if } k = 5. \end{cases}$$

(8) If $\{x, y, z\} = \{2, 3, 6\}$, then $A = \{0, 1, 4, 5\} \cup [7, k+2]$. It follows that

$$\{1\} \cup [4, k+3] \cup \{k+6, k+7\} \subseteq 2^\wedge A.$$

Note that, if $k \geq 7$, then

$$\{k+4, k+5\} \cup [k+8, 2k+3] \subseteq 2^\wedge A.$$

Therefore

$$2^\wedge A = \begin{cases} \{1\} \cup [4, 2k+3] & \text{if } k \geq 7 \\ \{1,4,5,6,7,8,9,11,12,13,15\} & \text{if } k = 6 \\ \{1,4,5,6,7,8,9,11,12\} & \text{if } k = 5. \end{cases}$$

So

$$|2^\wedge A| = \begin{cases} 2k+1 & \text{if } k \geq 7 \\ 11 & \text{if } k = 6 \\ 9 & \text{if } k = 5. \end{cases}$$

(9) If $\{x, y, z\} = \{k-3, k-2, k+1\}$ with $k \geq 7$, then $A = [0, k-4] \cup \{k-1, k, k+2\}$. It follows that $2^\wedge A = [1, 2k-1] \cup \{2k+1, 2k+2\}$. So $|2^\wedge A| = 2k+1$.

(10) If $\{x, y, z\} = \{k-4, k-3, k\}$ with $k \geq 8$, then $A = [0, k-5] \cup \{k-2, k-1, k+1, k+2\}$. It follows that $2^\wedge A = [1, 2k-3] \cup \{2k-1, 2k, 2k+1, 2k+3\}$.

If $\{x, y, z\} = \{2, 3, i\}$, where $7 \leq i \leq k-1$ with $k \geq 8$, then $A = \{0, 1\} \cup [4, i-1] \cup [i+1, k+2]$. It follows that $2^\wedge A = \{1\} \cup [4, 2k+3]$.

If $\{x, y, z\} = \{1, 2, i\}$, where $7 \leq i \leq k-1$, then $A = \{0\} \cup [3, i-1] \cup [i+1, k+2]$. It follows that $2^\wedge A = [3, 2k+3]$.

Hence, in each of the cases $|2^\wedge A| = 2k+1$.

(11) If $\{x, y, z\} = \{k-3, k-2, k\}$ with $k \geq 6$, then $A = [0, k-4] \cup \{k-1, k+1, k+2\}$. It follows that

$$[1, k-3] \cup [k-1, 2k-2] \cup \{2k, 2k+1, 2k+3\} \subseteq 2^\wedge A.$$

Note that, if $k \geq 7$, then $k-2 = k-4+2 \in 2^\wedge A$. Therefore

$$2^\wedge A = \begin{cases} [1, 2k-2] \cup \{2k, 2k+1, 2k+3\} & \text{if } k \geq 7 \\ \{1, 2, 3, 5, 6, 7, 8, 9, 10, 12, 13, 15\} & \text{if } k = 6. \end{cases}$$

If $\{x, y, z\} = \{3, 4, k+1\}$ with $k \geq 6$, then $A = \{0, 1, 2\} \cup [5, k] \cup \{k+2\}$. It follows that

$$\{1, 2, 3\} \cup [5, k+5] \cup [k+7, 2k+2] \subseteq 2^\wedge A.$$

Note that, if $k \geq 7$, then $k+6 \in 2^\wedge A$. Therefore

$$2^\wedge A = \begin{cases} \{1, 2, 3\} \cup [5, 2k+2] & \text{if } k \geq 7 \\ \{1, 2, 3, 5, 6, 7, 8, 9, 10, 11, 13, 14\} & \text{if } k = 6. \end{cases}$$

Hence, in both the cases

$$|2^\wedge A| = \begin{cases} 2k+1 & \text{if } k \geq 7 \\ 12 & \text{if } k = 6. \end{cases}$$

(12) If $\{x, y, z\} = \{3, 4, k\}$ with $k \geq 6$, then $A = \{0, 1, 2\} \cup [5, k-1] \cup \{k+1, k+2\}$. It follows that

$$\{1, 2, 3\} \cup [5, k+4] \cup [k+6, 2k+1] \cup \{2k+3\} \subseteq 2^\wedge A.$$

Note that, if $k \geq 8$, then $k+5 = k-1+6 \in 2^\wedge A$. Therefore

$$2^\wedge A = \begin{cases} \{1, 2, 3\} \cup [5, 2k+1] \cup \{2k+3\} & \text{if } k \geq 8 \\ \{1, 2, 3, 5, 6, 7, 8, 9, 10, 12, 13, 15\} & \text{if } k = 6 \\ \{1, 2, 3, 5, 6, 7, 8, 9, 10, 11, 13, 14, 15, 17\} & \text{if } k = 7. \end{cases}$$

So

$$|2^\wedge A| = \begin{cases} 2k+1 & \text{if } k \geq 8 \\ 2k & \text{if } k = 6 \text{ or } 7. \end{cases}$$

(13) If $\{x, y, z\} = \{3, 4, i\}$, where $6 \leq i \leq k-1$, then $A = \{0, 1, 2\} \cup [5, i-1] \cup [i+1, k+2]$. It follows that $2^\wedge A = \{1, 2, 3\} \cup [5, 2k+3]$.

If $\{x, y, z\} = \{i, i+1, k\}$, where $4 \leq i \leq k-5$, then $A = [0, i-1] \cup [i+2, k-1] \cup \{k+1, k+2\}$. It follows that $2^\wedge A = [1, 2k+1] \cup \{2k+3\}$.

If $\{x, y, z\} = \{i, i+1, k+1\}$, where $4 \leq i \leq k-4$, then $A = [0, i-1] \cup [i+2, k] \cup \{k+2\}$. It follows that $2^\wedge A = [1, 2k+2]$.

Hence, in each of the cases, $|2^\wedge A| = 2k+2$.

(14) If $\{x, y, x\} = \{i, i+1, j\}$, where $4 \leq i \leq j-3 \leq k-4$, then $A = [0, i-1] \cup [i+2, j-1] \cup [j+1, k+2]$. It easy to see that $2^\wedge A = [1, 2k+3]$. So $|2^\wedge A| = 2k+3$. □

Proposition 2.9. *Let $k \geq 5$ be a positive integer and $A = [0, k+2] \setminus \{x, y, z\}$, where $z = y+1$ and $1 \leq x < y < z \leq k+1$.*

(1) *If $\{x, y, z\} = \{k-2, k, k+1\}$, then $|2^\wedge A| = 2k$.*

(2) *If $\{x, y, z\}$ is one of the sets $\{1, k, k+1\}$ and $\{1, 3, 4\}$, then*

$$|2^\wedge A| = \begin{cases} 2k & \text{if } k \geq 6 \\ 9 & \text{if } k = 5. \end{cases}$$

(3) *If $\{x, y, z\} = \{k-3, k, k+1\}$, then* $|2^\wedge A| = \begin{cases} 2k & \text{if } k \geq 6 \\ 8 & \text{if } k = 5. \end{cases}$

(4) *If $k \geq 6$ and $\{x, y, z\}$ is one of the sets $\{2, k, k+1\}$ and $\{2, k-1, k\}$, then*

$$|2^\wedge A| = \begin{cases} 2k & \text{if } k \geq 7 \\ 11 & \text{if } k = 6. \end{cases}$$

(5) *If $\{x, y, z\} = \{k-4, k, k+1\}$ with $k \geq 7$, then $|2^\wedge A| = 2k$.*

(6) *If $\{x, y, z\} = \{1, k-1, k\}$ with $k \geq 6$, then $|2^\wedge A| = 2k$.*

(7) *If $\{x, y, z\} = \{k-3, k-1, k\}$, then*

$$|2^\wedge A| = \begin{cases} 2k+1 & \text{if } k \geq 6 \\ 9 & \text{if } k = 5. \end{cases}$$

(8) *If $\{x, y, z\} = \{k-4, k-1, k\}$,*

$$|2^\wedge A| = \begin{cases} 2k+1 & \text{if } k \geq 7 \\ 2k-1 & \text{if } k = 5 \text{ or } 6. \end{cases}$$

(9) *If $\{x, y, z\} = \{1, 4, 5\}$, then $|2^\wedge A| = 2k+1$.*

(10) *If $\{x, y, z\}$ is one of the sets $\{2, 5, 6\}$, $\{i, k-1, k\}$ where $3 \leq i \leq k-5$, and $\{i, k, k+1\}$ where $3 \leq i \leq k-5$, then $|2^\wedge A| = 2k+1$.*

(11) *If* $\{x, y, z\}$ *is one of the sets* $\{2, 4, 5\}$ *and* $\{1, k-2, k-1\}$ *with* $k \geq 6$, *then*

$$|2^\wedge A| = \begin{cases} 2k+1 & \text{if } k \geq 7 \\ 12 & \text{if } k = 6. \end{cases}$$

(12) *If* $\{x, y, z\} = \{2, k-2, k-1\}$ *with* $k \geq 6$, *then*

$$|2^\wedge A| = \begin{cases} 2k+1 & \text{if } k \geq 8 \\ 2k & \text{if } k = 6 \text{ or } 7. \end{cases}$$

(13) *If* $\{x, y, z\}$ *is one of the sets* $\{i, k-2, k-1\}$ *where* $3 \leq i \leq k-4$, $\{2, i, i+1\}$ *where* $6 \leq i \leq k-3$, *and* $\{1, i, i+1\}$ *where* $5 \leq i \leq k-3$, *then* $|2^\wedge A| = 2k+2$.

(14) *If* $\{x, y, z\} = \{i, j, j+1\}$, *where* $3 \leq i \leq j-2 \leq k-5$, *then* $|2^\wedge A| = 2k+3$.

In Proposition 2.10, we consider the case when $A = [0, k+2] \setminus \{x, y, z\}$, where $1 \leq x < y < z \leq k+1$ with $y - x = 2$ and $z - y \geq 3$. Therefore, we get Proposition 2.11 from Proposition 2.10 by replacing A to $k+2-A$.

Proposition 2.10. *Let* $k \geq 5$ *be a positive integer and* $A = [0, k+2] \setminus \{x, y, z\}$, *where* $1 \leq x < y < z \leq k+1$ *with* $y - x = 2$ *and* $z - y \geq 3$.

(1) *If* $\{x, y, z\} = \{1, 3, k\}$ *with* $k \geq 6$, *then* $|2^\wedge A| = 2k$.

(2) *If* $\{x, y, z\} = \{1, 3, k+1\}$, *then* $|2^\wedge A| = \begin{cases} 2k & \text{if } k \geq 6 \\ 8 & \text{if } k = 5. \end{cases}$

(3) *If* $\{x, y, z\} = \{k-4, k-2, k+1\}$ *with* $k \geq 7$, *then* $|2^\wedge A| = 2k$.

(4) *If* $\{x, y, z\} = \{2, 4, k+1\}$ *with* $k \geq 6$, *then* $|2^\wedge A| = \begin{cases} 2k+1 & \text{if } k \geq 7 \\ 11 & \text{if } k = 6. \end{cases}$

(5) *If* $k \geq 7$ *and* $\{x, y, z\}$ *is one of the sets* $\{2, 4, k\}$ *and* $\{1, 3, i\}$ *where* $6 \leq i \leq k-1$, *then* $|2^\wedge A| = 2k+1$.

(6) *If* $\{x, y, z\}$ *is one of the sets* $\{i, i+2, k\}$ *where* $3 \leq i \leq k-5$, $\{i, i+2, k+1\}$ *where* $3 \leq i \leq k-5$, *and* $\{2, 4, i\}$ *where* $7 \leq i \leq k-1$, *then* $|2^\wedge A| = 2k+2$.

(7) *If* $\{x, y, z\} = \{i, i+2, j\}$, *where* $3 \leq i \leq j-5 \leq k-6$, *then* $|2^\wedge A| = 2k+3$.

Proof. (1) If $\{x, y, z\} = \{1, 3, k\}$, then $A = \{0, 2\} \cup [4, k-1] \cup \{k+1, k+2\}$. It follows that

$$\{2\} \cup [4, k-1] \cup [k+1, 2k+1] \cup \{2k+3\} \subseteq 2^\wedge A.$$

Since $k \geq 6$, we have $k = k-2+2 \in 2^\wedge A$. Therefore $2^\wedge A = \{2\} \cup [4, 2k+1] \cup \{2k+3\}$. So $|2^\wedge A| = 2k$.

(2) If $\{x, y, z\} = \{1, 3, k+1\}$, then $A = \{0, 2\} \cup [4, k] \cup \{k+2\}$. It follows that

$$\{2\} \cup [4, k+2] \cup \{k+4\} \cup [k+6, 2k+2] \subseteq 2^\wedge A.$$

Note that, if $k \geq 6$, then $k+3 = k-1+4$ and $k+5$ are in $2^\wedge A$. Therefore

$$|2^\wedge A| = \begin{cases} \{2\} \cup [4, 2k+2] & \text{if } k \geq 6 \\ \{2, 4, 5, 6, 7, 9, 11, 12\} & \text{if } k = 5. \end{cases}$$

So

$$|2^\wedge A| = \begin{cases} 2k & \text{if } k \geq 6 \\ 8 & \text{if } k = 5. \end{cases}$$

(3) If $\{x, y, z\} = \{k-4, k-2, k+1\}$ with $k \geq 7$, then $A = [0, k-5] \cup \{k-3, k-1, k, k+2\}$. It is easy to see that $2^\wedge A = [1, 2k-3] \cup \{2k-1, 2k+1, 2k+2\}$. So $|2^\wedge A| = 2k$.

(4) If $\{x, y, z\} = \{2, 4, k+1\}$ with $k \geq 6$, then $A = \{0, 1, 3\} \cup [5, k] \cup \{k+2\}$. It follows that

$$\{1\} \cup [3, k+3] \cup \{k+5\} \cup [k+7, 2k+2] \subseteq 2^\wedge A.$$

Note that, if $k \geq 7$, then $k+4 = k-1+5$ and $k+6$ are in $2^\wedge A$. Therefore

$$2^\wedge A = \begin{cases} \{1\} \cup [3, 2k+2] & \text{if } k \geq 7 \\ \{1, 3, 4, 5, 6, 7, 8, 9, 11, 13, 14\} & \text{if } k = 6. \end{cases}$$

So

$$|2^\wedge A| = \begin{cases} 2k+1 & \text{if } k \geq 7 \\ 11 & \text{if } k = 6. \end{cases}$$

(5) If $\{x, y, z\} = \{2, 4, k\}$ with $k \geq 7$, then $A = \{0, 1, 3\} \cup [5, k-1] \cup \{k+1, k+2\}$. It is easy to see that $2^\wedge A = \{1\} \cup [3, 2k+1] \cup \{2k+3\}$. So $|2^\wedge A| = 2k+1$.

If $\{x, y, z\} = \{1, 3, i\}$, where $6 \leq i \leq k-1$, then $A = \{0, 2\} \cup [4, i-1] \cup [i+1, k+2]$. It follows that

$$\{2\} \cup [4, i-1] \cup [i+1, 2k+3] \subseteq 2^\wedge A.$$

Since $i-2 \geq 4$, we have $i = i-2+2 \in 2^\wedge A$. Therefore $2^\wedge A = \{2\} \cup [4, 2k+3]$. So $|2^\wedge A| = 2k+1$.

(6) If $\{x, y, z\} = \{i, i+2, k\}$, where $3 \leq i \leq k-5$, then $A = [0, i-1] \cup \{i+1\} \cup [i+3, k-1] \cup \{k+1, k+2\}$. It follows that $2^\wedge A = [1, 2k+1] \cup \{2k+3\}$. So $|2^\wedge A| = 2k+2$.

If $\{x, y, z\} = \{i, i+2, k+1\}$, where $3 \leq i \leq k-5$, then $A = [0, i-1] \cup \{i+1\} \cup [i+3, k] \cup \{k+2\}$. It is easy to see that $2^\wedge A = [1, 2k+2]$. So $|2^\wedge A| = 2k+2$.

If $\{x, y, z\} = \{2, 4, i\}$, where $7 \leq i \leq k-1$, then $A = \{0, 1, 3\} \cup [5, i-1] \cup [i+1, k+2]$. It is easy to see that $2^\wedge A = \{1\} \cup [3, 2k+3]$. So $|2^\wedge A| = 2k+2$.

(7) If $\{x, y, z\} = \{i, i+2, j\}$, where $3 \leq i \leq j-5 \leq k-6$, then $A = [0, i-1] \cup \{i+1\} \cup [i+3, j-1] \cup [j+1, k+2]$. It is easy to see that $2^\wedge A = [1, 2k+3]$. So $|2^\wedge A| = 2k+3$. □

Proposition 2.11. *Let $k \geq 5$ be a positive integer and $A = [0, k+2] \setminus \{x, y, z\}$, where $1 \leq x < y < z \leq k+1$ with $y - x \geq 3$ and $z - y = 2$.*

(1) *If $\{x, y, z\} = \{2, k-1, k+1\}$ with $k \geq 6$, then $|2^\wedge A| = 2k$.*

(2) *If* $\{x,y,z\} = \{1, k-1, k+1\}$*, then* $|2^\wedge A| = \begin{cases} 2k & \text{if } k \geq 6 \\ 8 & \text{if } k = 5. \end{cases}$

(3) *If* $k \geq 7$ *and* $\{x,y,z\} = \{1,4,6\}$*, then* $|2^\wedge A| = 2k$*.*

(4) *If* $\{x,y,z\} = \{1, k-2, k\}$ *with* $k \geq 6$*, then*

$$|2^\wedge A| = \begin{cases} 2k+1 & \text{if } k \geq 7 \\ 11 & \text{if } k = 6. \end{cases}$$

(5) *If* $k \geq 8$ *and* $\{x,y,z\}$ *is one of the sets* $\{2, k-2, k\}$*, and* $\{i, k-1, k+1\}$ *where* $3 \leq i \leq k-4$*, then* $|2^\wedge A| = 2k+1$*.*

(6) *If* $\{x,y,z\}$ *is one of the sets* $\{2, i, i+2\}$*, where* $5 \leq i \leq k-3$*,* $\{1, i, i+2\}$*, where* $5 \leq i \leq k-3$*, and* $\{i, k-2, k\}$*, where* $3 \leq i \leq k-5$*, then* $|2^\wedge A| = 2k+2$*.*

(7) *If* $\{x,y,z\} = \{i, j, j+2\}$*, where* $3 \leq i \leq j-3 \leq k-6$*, then* $|2^\wedge A| = 2k+3$*.*

Proposition 2.12. *Let* $k \geq 5$ *be a positive integer and* $A = [0, k+2] \setminus \{x,y,z\}$*, where* $1 \leq x < y < z \leq k+1$ *with* $y - x = 2$ *and* $z - y = 2$*.*

(1) *If* $\{x,y,z\}$ *is one of the sets* $\{1,3,5\}$ *and* $\{k-3, k-1, k+1\}$*, then*

$$|2^\wedge A| = \begin{cases} 2k & \text{if } k \geq 6 \\ 9 & \text{if } k = 5. \end{cases}$$

(2) *If* $k \geq 6$ *and* $\{x,y,z\}$ *is one of the sets* $\{2,4,6\}$ *and* $\{k-4, k-2, k\}$*, then*

$$|2^\wedge A| = \begin{cases} 2k+2 & \text{if } k \geq 7 \\ 13 & \text{if } k = 6. \end{cases}$$

(3) *If* $\{x,y,z\} = \{i, i+2, i+4\}$*, where* $3 \leq i \leq k-5$*, then* $|2^\wedge A| = 2k+3$*.*

Proof. (1) If $\{x,y,z\} = \{1,3,5\}$, then $A = \{0,2,4\} \cup [6, k+2]$. We have

$$\{2,4\} \cup [6, k+6] \cup [k+8, 2k+3] \subseteq 2^\wedge A.$$

Note that, if $k \geq 6$, then $k+7 = k+1+6 \in 2^\wedge A$. Therefore

$$2^\wedge A = \begin{cases} \{2,4\} \cup [6, 2k+3] & \text{if } k \geq 6 \\ \{2,4,6,7,8,9,10,11,13\} & \text{if } k = 5. \end{cases}$$

If $\{x,y,z\} = \{k-3, k-1, k+1\}$, then $A = [0, k-4] \cup \{k-2, k, k+2\} = k+2-(\{0,2,4\} \cup [6, k+2])$. It follows that $|2^\wedge A| = |2^\wedge(\{0,2,4\} \cup [6, k+2])|$. Hence, in both the cases

$$|2^\wedge A| = \begin{cases} 2k & \text{if } k \geq 6 \\ 9 & \text{if } k = 5. \end{cases}$$

(2) If $\{x,y,z\} = \{2,4,6\}$, then $A = \{0,1,3,5\} \cup [7, k+2]$. We have

$$\{1\} \cup [3, k+3] \cup \{k+5, k+7\} \subseteq 2^\wedge A.$$

Since $k \geq 6$, we have $k+4 = k+1+3 \in 2^\wedge A$, $k+6 = k+1+5 \in 2^\wedge A$ and $[k+9, 2k+3] \subseteq 2^\wedge A$. Note also that, if $k \geq 7$, then $k+8 \in 2^\wedge A$. Therefore

$$2^\wedge A = \begin{cases} \{1\} \cup [3, 2k+3] & \text{if } k \geq 7 \\ \{1,3,4,5,6,7,8,9,10,11,12,13,15\} & \text{if } k = 6. \end{cases}$$

If $\{x,y,z\} = \{k-4, k-2, k\}$, then $A = [0, k-5] \cup \{k-3, k-1, k+1, k+2\} = k+2-(\{0,1,3,5\} \cup [7, k+2])$. It follows that $|2^\wedge A| = |2^\wedge(\{0,1,3,5\} \cup [7, k+2])|$. Hence, in both the cases

$$|2^\wedge A| = \begin{cases} 2k+2 & \text{if } k \geq 7 \\ 13 & \text{if } k = 6. \end{cases}$$

(3) If $\{x,y,z\} = \{i, i+2, i+4\}$, where $3 \leq i \leq k-5$, then $A = [0, i-1] \cup \{i+1, i+3\} \cup [i+5, k+2]$. So $2^\wedge A = [1, 2k+3]$, which gives $|2^\wedge A| = 2k+3$. □

Proposition 2.13. *Let $k \geq 5$ be a positive integer and $A = [0, k+2] \setminus \{x,y,z\}$, where $1 \leq x < y < z \leq k+1$ with $y - x \geq 3$ and $z - y \geq 3$.*

(1) *If $k \geq 7$ and $\{x,y,z\}$ is one of the sets $\{1,4,k\}$ and $\{2, k-2, k+1\}$, then $|2^\wedge A| = 2k$.*

(2) *If $\{x,y,z\}$ is one of the sets $\{1,4,k+1\}$ and $\{1, k-2, k+1\}$ with $k \geq 6$, then*

$$|2^\wedge A| = \begin{cases} 2k & \text{if } k \geq 7 \\ 11 & \text{if } k = 6. \end{cases}$$

(3) *If $k \geq 8$ and $\{x,y,z\}$ is one of the sets $\{1, i, k+1\}$ where $5 \leq i \leq k-3$, $\{1,4,i\}$ where $7 \leq i \leq k-1$, $\{i, k-2, k+1\}$ where $3 \leq i \leq k-5$, $\{1, i, k\}$ where $5 \leq i \leq k-3$, $\{2, i, k+1\}$ where $5 \leq i \leq k-3$, and $\{2, i, k\}$ where $5 \leq i \leq k-3$, then $|2^\wedge A| = 2k+1$.*

(4) *If $\{x,y,z\}$ is one of the sets $\{2, i, j\}$ where $5 \leq i \leq j-3 \leq k-4$, $\{i, j, k\}$ where $3 \leq i \leq j-3 \leq k-6$, $\{1, i, j\}$ where $5 \leq i \leq j-3 \leq k-4$, and $\{i, j, k+1\}$ where $3 \leq i \leq j-3 \leq k-6$, then $|2^\wedge A| = 2k+2$.*

(5) *If $3 \leq x \leq y-3 \leq z-6 \leq k-7$, then $|2^\wedge A| = 2k+3$.*

Proof. (1) If $\{x,y,z\} = \{1,4,k\}$ with $k \geq 7$, then $A = \{0,2,3\} \cup [5, k-1] \cup \{k+1, k+2\}$. It is easy to see that

$$2^\wedge A = \{2,3\} \cup [5, 2k+1] \cup \{2k+3\}.$$

Therefore $|2^\wedge A| = 2k$.

If $\{x,y,z\} = \{2, k-2, k+1\}$ with $k \geq 7$, then $A = \{0,1\} \cup [3, k-3] \cup \{k-1, k, k+2\} = k+2-(\{0,2,3\} \cup [5, k-1] \cup \{k+1, k+2\})$. So by the previous case $|2^\wedge A| = 2k$.

(2) If $\{x,y,z\} = \{1,4,k+1\}$ with $k \geq 6$, then $A = \{0,2,3\} \cup [5,k] \cup \{k+2\}$. It is easy to see that $\{2,3\} \cup [5, k+5] \cup [k+7, 2k+2] \subseteq 2^\wedge A$. Note also that,

if $k \geq 7$, then $k+6 \in 2^\wedge A$. Therefore

$$2^\wedge A = \begin{cases} \{2,3\} \cup [5, 2k+2] & \text{if } k \geq 7 \\ \{2,3,5,6,7,8,9,10,11,13,14\} & \text{if } k = 6. \end{cases}$$

So

$$|2^\wedge A| = \begin{cases} 2k & \text{if } k \geq 7 \\ 11 & \text{if } k = 6. \end{cases}$$

If $\{x,y,z\} = \{1, k-2, k+1\}$ with $k \geq 6$, then $A = \{0\} \cup [2, k-3] \cup \{k-1, k, k+2\} = k+2-(\{0,2,3\} \cup [5,k] \cup \{k+2\})$. So by the previous case

$$|2^\wedge A| = \begin{cases} 2k & \text{if } k \geq 7 \\ 11 & \text{if } k = 6. \end{cases}$$

(3) If $\{x,y,z\} = \{1, i, k+1\}$, where $5 \leq i \leq k-3$, then $A = \{0\} \cup [2, i-1] \cup [i+1, k] \cup \{k+2\}$. It follows that $2^\wedge A = [2, 2k+2]$. So $|2^\wedge A| = 2k+1$.

If $\{x,y,z\} = \{1,4,i\}$, where $7 \leq i \leq k-1$, then $A = \{0,2,3\} \cup [5, i-1] \cup [i+1, k+2]$. It follows that $2^\wedge A = \{2,3\} \cup [5, 2k+3]$. So $|2^\wedge A| = 2k+1$.

If $\{x,y,z\} = \{i, k-2, k+1\}$, where $3 \leq i \leq k-5$, then $A = [0, i-1] \cup [i+1, k-3] \cup \{k-1, k, k+2\} = k+2-(\{0,2,3\} \cup [5, j-1] \cup [j+1, k+2])$ where $7 \leq j \leq k-1$. So by the previous case $|2^\wedge A| = 2k+1$.

If $\{x,y,z\} = \{1, i, k\}$, where $5 \leq i \leq k-3$, then $A = \{0\} \cup [2, i-1] \cup [i+1, k-1] \cup \{k+1, k+2\}$. It is easy to see that $2^\wedge A = [2, 2k+1] \cup \{2k+3\}$. So $|2^\wedge A| = 2k+1$.

If $\{x,y,z\} = \{2, i, k+1\}$, where $5 \leq i \leq k-3$, then $A = \{0,1\} \cup [3, i-1] \cup [i+1, k] \cup \{k+2\} = k+2-(\{0\} \cup [2, i_0-1] \cup [i_0+1, k-1] \cup \{k+1, k+2\})$ where $5 \leq i_0 \leq k-3$. So by the previous case $|2^\wedge A| = 2k+1$.

If $\{x,y,z\} = \{2, i, k\}$, where $5 \leq i \leq k-3$, then $A = \{0,1\} \cup [3, i-1] \cup [i+1, k-1] \cup \{k+1, k+2\}$. Therefore $2^\wedge A = \{1\} \cup [3, 2k+1] \cup \{2k+3\}$. Hence, in each of the cases $|2^\wedge A| = 2k+1$.

(4) If $\{x,y,z\} = \{2, i, j\}$ where $5 \leq i \leq j-3 \leq k-4$, then $A = \{0,1\} \cup [3, i-1] \cup [i+1, j-1] \cup [j+1, k+2]$. Therefore $2^\wedge A = \{1\} \cup [3, 2k+3]$, which gives $|2^\wedge A| = 2k+2$.

If $\{x,y,z\} = \{i, j, k\}$ with $3 \leq i \leq j-3 \leq k-6$, then $A = [0, i-1] \cup [i+1, j-1] \cup [j+1, k-1] \cup \{k+1, k+2\} = k+2-(\{0,1\} \cup [3, i_0-1] \cup [i_0+1, j_0-1] \cup [j_0+1, k+2])$, where $5 \leq i_0 \leq j_0-3 \leq k-4$. So by the previous case $|2^\wedge A| = 2k+2$.

If $\{x,y,z\} = \{1, i, j\}$, where $5 \leq i \leq j-3 \leq k-4$, then $A = \{0\} \cup [2, i-1] \cup [i+1, j-1] \cup [j+1, k+2]$. It is easy to see that $2^\wedge A = [2, 2k+3]$. So $|2^\wedge A| = 2k+2$.

If $\{x,y,z\} = \{i, j, k+1\}$, where $3 \leq i \leq j-3 \leq k-6$, then $A = [0, i-1] \cup [i+1, j-1] \cup [j+1, k] \cup \{k+2\} = k+2-(\{0\} \cup [2, i_0-1] \cup [i_0+1, j_0-1] \cup [j_0+1, k+2])$, where $5 \leq i_0 \leq j_0-3 \leq k-4$. So by the previous case $|2^\wedge A| = 2k+2$.

(5) If $3 \le x \le y-3 \le z-6 \le k-7$, then $A = [0, x-1] \cup [x+1, y-1] \cup [y+1, z-1] \cup [z+1, k+2]$. Therefore $2^\wedge A = [1, 2k+3]$, which gives $|2^\wedge A| = 2k+3$. □

Proof of Theorem 1.8. Lemma 2.1 and Proposition 2.2 together prove (1). Similarly, we can prove (2) and (3) using Lemma 2.1 and Propositions 2.2-2.13. □

3. Proof of Theorem 1.9 and Theorem 1.10

Lemma 3.1. *Let $k \ge 5$ and $h \ge 2$ be positive integers with $h \le k-1$. Let $A = \{a_0, a_1, \ldots, a_{k-1}\}$ be a finite set of k integers such that $0 = a_0 < a_1 < \cdots < a_{k-1}$ and $d(A) = 1$. If $A' = A \setminus \{a_1\}$ and $d(A') > 1$, then $|h^\wedge A| \ge (2h-1)k - 2h^2 + 2$.*

Proof. Since $d(A') > 1$, the sets $(h-1)^\wedge A' + a_1$ and $h^\wedge A'$ are disjoint subsets of $h^\wedge A$. Therefore, by Theorem 1.3, we have

$$\begin{aligned}|h^\wedge A| &\ge |(h-1)^\wedge A' + a_1| + |h^\wedge A'| \\ &\ge (h-1)(k-1) - (h-1)^2 + 1 + h(k-1) - h^2 + 1 \\ &= (2h-1)k - 2h^2 + 2.\end{aligned}$$

□

Lemma 3.2. *Let $k \ge 10$ and $A = \{a_0, a_1, \ldots, a_{k-1}\}$ be a finite set of k integers such that $0 = a_0 < a_1 < \cdots < a_{k-1}$. Let $A' = A \setminus \{a_1\}$, $d(A') = 1$ and $|2^\wedge A'| = 2(k-1) - 1$. Then*

$$|3^\wedge A| \in \{3k-7, 3k-6, 3k-5\}.$$

Moreover, the following hold.

(1) *If $|3^\wedge A| = 3k-7$, then $A = [0,k] \setminus \{x\}$, where $x \in \{1, k-1, \}$.*
(2) *If $|3^\wedge A| = 3k-6$, then $A = [0,k] \setminus \{x\}$, where $x \in \{2, 3, k-2\}$.*
(3) *If $|3^\wedge A| = 3k-5$, then $A = [0,k] \setminus \{4\}$.*

Proof. If $|2^\wedge(A')| = 2(k-1) - 1$, then by Theorem 1.8 (2), we have

$$A' = \{a_0, a_2, a_3, \ldots, a_{k-1}\} = [0,k] \setminus \{x, y\},$$

where $\{x,y\}$ is one of the sets $\{1,2\}$, $\{k-2,k-1\}$, $\{2,3\}$, $\{k-3,k-2\}$, $\{1,3\}$, $\{k-3,k-1\}$, $\{1,4\}$, $\{k-4,k-1\}$, $\{1,k-1\}$, $\{1,k-2\}$, $\{2,k-1\}$, $\{2,k-2\}$, and $\{i,k\}$ with $3 \le i \le k-4$. Therefore $A = A' \cup \{a_1\}$ with $a_2 \ge 2$. Now, consider the following cases.

If $\{x,y\}$ is one of the sets $\{k-2,k-1\}$, $\{2,3\}$, $\{k-3,k-2\}$, $\{k-3,k-1\}$, $\{k-4,k-1\}$, $\{2,k-1\}$, $\{2,k-2\}$, and $\{i,k\}$ where $3 \le i \le k-4$, then $a_2 = 1$, which is a contradiction.

If $\{x,y\} = \{1,3\}$, then $a_1 = 1$. It follows that $A = \{0,1,2\} \cup [4,k]$. Therefore $3^\wedge A = \{3\} \cup [5, 3k-3]$. So $|3^\wedge A| = 3k-6$.

If $\{x,y\} = \{1,4\}$, then $a_1 = 1$. It follows that $A = \{0,1,2,3\} \cup [5,k]$. Therefore $3^\wedge A = [3, 3k-3]$. So $|3^\wedge A| = 3k-5$.

If $\{x,y\} = \{1,k-1\}$, then $a_1 = 1$. It follows that $A = [0,k-2] \cup \{k\}$. Therefore $3^\wedge A = [3, 3k-5]$. So $|3^\wedge A| = 3k-7$.

If $\{x, y\} = \{1, k-2\}$, then $a_1 = 1$. It follows that $A = [0, k-3] \cup \{k-1, k\}$. Therefore $3^\wedge A = [3, 3k-4]$. So $|3^\wedge A| = 3k-6$.

If $\{x, y\} = \{1, 2\}$, then $a_1 \in \{1, 2\}$. It follows that $A = \{0, a_1\} \cup [3, k]$. Therefore

$$3^\wedge A = \begin{cases} [4, 3k-3] & \text{if } a_1 = 1 \\ [5, 3k-3] & \text{if } a_1 = 2. \end{cases}$$

So

$$|3^\wedge A| = \begin{cases} 3k-6 & \text{if } a_1 = 1 \\ 3k-7 & \text{if } a_1 = 2. \end{cases}$$

□

Lemma 3.3. *Let $k \geq 12$ and $A = \{a_0, a_1, \ldots, a_{k-1}\}$ be a finite set of k integers such that $0 = a_0 < a_1 < \cdots < a_{k-1}$. Let $A' = A \setminus \{a_1\}$, $d(A') = 1$, and $|2^\wedge A'| = 2(k-1)$. Then*

$$|3^\wedge A| \in \{3k-6, 3k-5, 3k-4, 3k-3, 3k-2\}.$$

Moreover, the following hold.

(1) *If $|3^\wedge A| = 3k-6$, then $A = [0, k+1] \setminus \{x, y\}$, where $\{x, y\}$ is one of the sets $\{1, 2\}$, $\{1, k\}$, and $\{k-1, k\}$.*
(2) *If $|3^\wedge A| = 3k-5$, then $A = [0, k+1] \setminus \{x, y\}$, where $\{x, y\}$ is one of the sets $\{1, 3\}$, $\{1, 4\}$, $\{2, k\}$, $\{1, k-1\}$, $\{3, k\}$, $\{k-2, k\}$, $\{k-3, k\}$, and $\{i, k+1\}$ where $5 \leq i \leq k-3$.*
(3) *If $|3^\wedge A| = 3k-4$, then $A = [0, k+1] \setminus \{x, y\}$, where $\{x, y\}$ is one of the sets $\{2, 3\}$, $\{2, 4\}$, $\{3, 4\}$, $\{1, 5\}$, $\{2, 5\}$, $\{2, k-1\}$, $\{1, 6\}$, $\{k-2, k-1\}$, $\{3, k-1\}$, and $\{4, k\}$.*
(4) *If $|3^\wedge A| = 3k-3$, then $A = [0, k+1] \setminus \{x, y\}$, where $\{x, y\}$ is one of the sets $\{2, 6\}$, $\{3, 5\}$, and $\{4, k-1\}$.*
(5) *If $|3^\wedge A| = 3k-2$, then $A = [0, k+1] \setminus \{4, 6\}$.*

Proof. If $|2^\wedge A'| = 2(k-1)$, then by Theorem 1.8 (3), $A' = [0, k+1] \setminus \{x, y, z\}$, where $\{x, y, z\}$ is one of the sets $\{3, 4, k+1\}$, $\{k-4, k-3, k+1\}$, $\{2, 4, k+1\}$, $\{k-4, k-2, k+1\}$, $\{2, 5, k+1\}$, $\{k-5, k-2, k+1\}$, $\{1, 2, 3\}$, $\{k-2, k-1, k\}$, $\{2, 3, 4\}$, $\{k-3, k-2, k-1\}$, $\{1, 2, 4\}$, $\{k-3, k-1, k\}$, $\{1, 2, k\}$, $\{1, k-1, k\}$, $\{1, 3, 4\}$, $\{k-3, k-2, k\}$, $\{1, 2, 5\}$, $\{k-4, k-1, k\}$, $\{1, 2, k-1\}$, $\{2, k-1, k\}$, $\{2, 3, k-1\}$, $\{2, k-2, k-1\}$, $\{1, 2, 6\}$, $\{k-5, k-1, k\}$, $\{2, 3, k\}$, $\{1, k-2, k-1\}$, $\{1, 3, 5\}$, $\{k-4, k-2, k\}$, $\{1, 3, k-1\}$, $\{2, k-2, k\}$, $\{1, 3, k\}$, $\{1, k-2, k\}$, $\{1, 4, 6\}$, $\{k-5, k-3, k\}$, $\{1, 4, k-1\}$, $\{2, k-3, k\}$, $\{1, 4, k\}$, $\{1, k-3, k\}$, $\{i, j, k+1\}$, where $i \in \{1, 2\}$ with $i+4 \leq j \leq k-3$, and $\{i, j, k+1\}$, where $3 \leq i \leq j-4$ with $j \in \{k-2, k-1\}$. Therefore $A = A' \cup \{a_1\}$ with $a_2 \geq 2$. Now, consider the following cases.

(1) $\{x, y, z\}$ is one of the sets $\{3, 4, k+1\}$, $\{k-4, k-3, k+1\}$, $\{2, 4, k+1\}$, $\{k-4, k-2, k+1\}$, $\{2, 5, k+1\}$, $\{k-5, k-2, k+1\}$, $\{k-2, k-1, k\}$, $\{2, 3, 4\}$, $\{k-3, k-2, k-1\}$, $\{k-3, k-1, k\}$, $\{k-3, k-2, k\}$, $\{k-4, k-1, k\}$, $\{2, k-1, k\}$, $\{2, 3, k-1\}$, $\{2, k-2, k-1\}$, $\{k-5, k-1, k\}$, $\{2, 3, k\}$, $\{k-4, k-2, k\}$, $\{2, k-2, k\}$

$\{k-5,k-3,k\}$, $\{2,k-3,k\}$, $\{2,j,k+1\}$ where $6 \le j \le k-3$, and $\{i,j,k+1\}$ where $3 \le i \le j-4$ with $j \in \{k-1,k-2\}$, then $a_2 = 1$, which is a contradiction.

(2) If $\{x,y,z\} = \{1,2,3\}$, then $a_1 \in \{1,2,3\}$. It follows that

$$A = \begin{cases} \{0,1\} \cup [4,k+1] & \text{if } a_1 = 1 \\ \{0,2\} \cup [4,k+1] & \text{if } a_1 = 2 \\ \{0\} \cup [3,k+1] & \text{if } a_1 = 3. \end{cases}$$

Therefore

$$3^\wedge A = \begin{cases} [5,3k] & \text{if } a_1 = 1 \\ [6,3k] & \text{if } a_1 = 2 \\ [7,3k] & \text{if } a_1 = 3. \end{cases}$$

So

$$|3^\wedge A| = \begin{cases} 3k-4 & \text{if } a_1 = 1 \\ 3k-5 & \text{if } a_1 = 2 \\ 3k-6 & \text{if } a_1 = 3. \end{cases}$$

(3) If $\{x,y,z\} = \{1,2,4\}$, then $a_1 \in \{1,2\}$. It follows that

$$A = \begin{cases} \{0,1,3\} \cup [5,k+1] & \text{if } a_1 = 1 \\ \{0,2,3\} \cup [5,k+1] & \text{if } a_1 = 2. \end{cases}$$

Therefore

$$3^\wedge A = \begin{cases} \{4\} \cup [6,3k] & \text{if } a_1 = 1 \\ \{5\} \cup [7,3k] & \text{if } a_1 = 2. \end{cases}$$

So

$$|3^\wedge A| = \begin{cases} 3k-4 & \text{if } a_1 = 1 \\ 3k-5 & \text{if } a_1 = 2. \end{cases}$$

(4) If $\{x,y,z\} = \{1,2,k\}$, then $a_1 \in \{1,2\}$. It follows that

$$A = \begin{cases} \{0,1\} \cup [3,k-1] \cup \{k+1\} & \text{if } a_1 = 1 \\ \{0\} \cup [2,k-1] \cup \{k+1\} & \text{if } a_1 = 2. \end{cases}$$

Therefore

$$3^\wedge A = \begin{cases} [4,3k-2] & \text{if } a_1 = 1 \\ [5,3k-2] & \text{if } a_1 = 2. \end{cases}$$

So

$$|3^\wedge A| = \begin{cases} 3k-5 & \text{if } a_1 = 1 \\ 3k-6 & \text{if } a_1 = 2. \end{cases}$$

(5) If $\{x,y,z\} = \{1,k-1,k\}$, then $a_1 = 1$. It follows that $A = [0,k-2] \cup \{k+1\}$. Therefore, $3^\wedge A = [3,3k-4]$ and $|3^\wedge A| = 3k-6$.

(6) If $\{x,y,z\} = \{1,3,4\}$, then $a_1 = 1$. It follows that $A = \{0,1,2\} \cup [5,k+1]$. Therefore $3^\wedge A = \{3\} \cup [6,3k]$ and $|3^\wedge A| = 3k-4$.

(7) If $\{x, y, z\} = \{1, 2, 5\}$, then $a_1 \in \{1, 2\}$. It follows that

$$A = \begin{cases} \{0, 1, 3, 4\} \cup [6, k+1] & \text{if } a_1 = 1 \\ \{0, 2, 3, 4\} \cup [6, k+1] & \text{if } a_1 = 2. \end{cases}$$

Therefore

$$3^\wedge A = \begin{cases} \{4, 5\} \cup [7, 3k] & \text{if } a_1 = 1 \\ [5, 3k] & \text{if } a_1 = 2. \end{cases}$$

So

$$|3^\wedge A| = 3k - 4.$$

(8) If $\{x, y, z\} = \{1, 2, k-1\}$, then $a_1 \in \{1, 2\}$. It follows that

$$A = \begin{cases} \{0, 1\} \cup [3, k-2] \cup \{k, k+1\} & \text{if } a_1 = 1 \\ \{0\} \cup [2, k-2] \cup \{k, k+1\} & \text{if } a_1 = 2. \end{cases}$$

Therefore

$$3^\wedge A = \begin{cases} [4, 3k-1] & \text{if } a_1 = 1 \\ [5, 3k-1] & \text{if } a_1 = 2. \end{cases}$$

So

$$|3^\wedge A| = \begin{cases} 3k - 4 & \text{if } a_1 = 1 \\ 3k - 5 & \text{if } a_1 = 2. \end{cases}$$

(9) If $\{x, y, z\} = \{1, 2, 6\}$, then $a_1 \in \{1, 2\}$. It follows that

$$A = \begin{cases} \{0, 1, 3, 4, 5\} \cup [7, k+1] & \text{if } a_1 = 1 \\ \{0, 2, 3, 4, 5\} \cup [7, k+1] & \text{if } a_1 = 2. \end{cases}$$

Therefore

$$3^\wedge A = \begin{cases} [4, 3k] & \text{if } a_1 = 1 \\ [5, 3k] & \text{if } a_1 = 2. \end{cases}$$

So

$$|3^\wedge A| = \begin{cases} 3k - 3 & \text{if } a_1 = 1 \\ 3k - 4 & \text{if } a_1 = 2. \end{cases}$$

(10) If $\{x, y, z\} = \{1, k-2, k-1\}$, then $a_1 = 1$. It follows that $A = [0, k-3] \cup \{k, k+1\}$ Therefore $3^\wedge A = [3, 3k-2]$ and $|3^\wedge A| = 3k - 4$.

(11) If $\{x, y, z\} = \{1, 3, 5\}$, then $a_1 = 1$. It follows that $A = \{0, 1, 2, 4\} \cup [6, k+1]$. Therefore $3^\wedge A = \{3\} \cup [5, 3k]$ and $|3^\wedge A| = 3k - 3$.

(12) If $\{x, y, z\} = \{1, 3, k-1\}$, then $a_1 = 1$. It follows that $A = \{0, 1, 2\} \cup [4, k-2] \cup \{k, k+1\}$. Therefore $3^\wedge A = \{3\} \cup [5, 3k-1]$ and $|3^\wedge A| = 3k - 4$.

(13) If $\{x, y, z\} = \{1, 3, k\}$, then $a_1 = 1$. It follows that $A = \{0, 1, 2\} \cup [4, k-1] \cup \{k+1\}$. Therefore $3^\wedge A = \{3\} \cup [5, 3k-2]$ and $|3^\wedge A| = 3k - 5$.

(14) If $\{x, y, z\} = \{1, k-2, k\}$, then $a_1 = 1$. It follows that $A = [0, k-3] \cup \{k-1, k+1\}$. Therefore $3^\wedge A = [3, 3k-3]$ and $|3^\wedge A| = 3k - 5$.

(15) If $\{x, y, z\} = \{1, 4, 6\}$, then $a_1 = 1$. It follows that $A = \{0, 1, 2, 3, 5\} \cup [7, k+1]$. Therefore $3^\wedge A = [3, 3k]$ and $|3^\wedge A| = 3k - 2$.

(16) If $\{x, y, z\} = \{1, 4, k-1\}$, then $a_1 = 1$. It follows that $A = \{0, 1, 2, 3\} \cup [5, k-2] \cup \{k, k+1\}$. Therefore $3^\wedge A = [3, 3k-1]$ and $|3^\wedge A| = 3k-3$.

(17) If $\{x, y, z\} = \{1, 4, k\}$, then $a_1 = 1$. It follows that $A = \{0, 1, 2, 3\} \cup [5, k-1] \cup \{k+1\}$. Therefore $3^\wedge A = [3, 3k-2]$ and $|3^\wedge A| = 3k-4$.

(18) If $\{x, y, z\} = \{1, k-3, k\}$, then $a_1 = 1$. It follows that $A = [0, k-4] \cup \{k-2, k-1, k+1\}$. Therefore $3^\wedge A = [3, 3k-4] \cup \{3k-2\}$ and $|3^\wedge A| = 3k-5$.

(19) If $\{x, y, z\} = \{1, j, k+1\}$, where $5 \leq j \leq k-3$, then $a_1 = 1$. It follows that $A = [0, j-1] \cup [j+1, k]$. Therefore $3^\wedge A = [3, 3k-3]$ and $|3^\wedge A| = 3k-5$. □

Lemma 3.4. *Let $k \geq 8$ and $A = \{a_0, a_1, \ldots, a_{k-1}\}$ be a finite set of k integers such that $0 = a_0 < a_1 < \cdots < a_{k-1}$. Let $A' = A \setminus \{a_1\}$ and $d(A') = 1$. If $|3^\wedge A| > 3k-8$, then $|2^\wedge A'| \geq 2(k-1)-1$.*

Proof. Clearly $a_2 \geq 2$. Let $|2^\wedge(A')| \leq 2(k-1)-2$. By Theorem 1.3, we have

$$|2^\wedge(A')| \geq 2(k-1)-3.$$

Therefore $2(k-1)-3 \leq |2^\wedge(A')| \leq 2(k-1)-2$. Consider the following cases.

(1) If $|2^\wedge(A')| = 2(k-1)-3$, then by Theorem 1.3, $A' = [0, k-2]$. This gives $a_2 = 1$, which is a contradiction. Therefore $|2^\wedge(A')| \neq 2(k-1)-3$.

(2) If $|2^\wedge(A')| = 2(k-1)-2$, then by Theorem 1.8, $A' = [0, k-1] \setminus \{x\}$, where $x \in \{1, 2, k-3, k-2\}$. If $x \in \{2, k-3, k-2\}$, then $a_2 = 1$, which is a contradiction. If $x = 1$, then $a_2 = 2$ and $a_1 = 1$. This gives $A = [0, k-1]$ and $|3^\wedge A| = 3k-8$, which is also a contradiction. Therefore $|2^\wedge(A')| \neq 2(k-1)-2$. Hence $|2^\wedge A'| \geq 2(k-1)-1$. This completes the proof of the lemma. □

Lemma 3.5. *Let $k \geq 10$ and A be a finite set of k integers with $\min(A) = 0$ and $d(A) = 1$. Then $|3^\wedge A| = 3k-7$ if and only if $A = [0, k] \setminus \{x\}$, where $x \in \{1, k-1\}$.*

Proof. Let $A = [0, k] \setminus \{x\}$, where $x \in \{1, k-1\}$. Then

$$3^\wedge A = \begin{cases} [5, 3k-3] & \text{if } x = 1 \\ [3, 3k-5] & \text{if } x = k-1. \end{cases}$$

So

$$|3^\wedge A| = 3k-7.$$

Let $A = \{a_0, a_1, \ldots, a_{k-1}\}$ be a set of k integers such that $0 = a_0 < a_1 < \cdots < a_{k-1}$, $d(A) = 1$ and $|3^\wedge A| = 3k-7$. Then $a_2 \geq 2$. Set $A' = A \setminus \{a_1\}$ and $S = \{a_i + a_{k-2} + a_{k-1} : i \in [2, k-3]\}$. Note that $2^\wedge A' + a_1$ and S are disjoint subsets of $3^\wedge A$ because $\max(2^\wedge A' + a_1) < \min(S)$.

Claim 1: $d(A') = 1$.

If $d(A') > 1$, then Lemma 3.1 implies that

$$|3^\wedge A| \geq 5k-16 > 3k-7.$$

Therefore $d(A') = 1$.

Claim 2: $|2^\wedge(A')| \leq 2(k-1)-1$.

If $|2^\wedge A' + a_1| = |2^\wedge(A')| \geq 2(k-1)$, then

$$|3^\wedge A| \geq |2^\wedge(A') + a_1| + |S| \geq 2(k-1) + k - 4 = 3k - 6 > 3k - 7.$$

Therefore, $|2^\wedge(A')| \leq 2(k-1) - 1$. Using Lemma 3.4, we get $|2^\wedge(A')| = 2(k-1) - 1$ and again using Lemma 3.2, we get $A = [0, k] \setminus \{x\}$, where $x \in \{1, k-1\}$. This completes the proof of the lemma. □

Lemma 3.6. *Let $k \geq 12$ and A be a finite set of k integers with* $\min(A) = 0$ *and* $d(A) = 1$*. Then $|3^\wedge A| = 3k - 6$ if and only if $A = [0, k+1] \setminus \{x, y\}$, where $\{x, y\}$ is one of the sets $\{2, k+1\}$, $\{3, k+1\}$, $\{k-3, k+1\}$, $\{k-2, k+1\}$, $\{1, 2\}$, $\{k-1, k\}$, and $\{1, k\}$.*

Proof. If $A = [0, k+1] \setminus \{x, y\}$, where $\{x, y\}$ is one of the sets $\{2, k+1\}$, $\{3, k+1\}$, $\{k-3, k+1\}$, $\{k-2, k+1\}$, $\{1, 2\}$, $\{k-1, k\}$, and $\{1, k\}$, then it is easy to see $|3^\wedge A| = 3k - 6$. Now, we prove the converse part.

Let $A = \{a_0, a_1, \ldots, a_{k-1}\}$ where $0 = a_0 < a_1 < \cdots < a_{k-1}$ and $d(A) = 1$ such that $|3^\wedge A| = 3k - 6$. Then $a_2 \geq 2$. Set $A' = A \setminus \{a_1\}$ and $S = \{a_i + a_{k-2} + a_{k-1} : i \in [2, k-3]\}$. Note that $2^\wedge A' + a_1$ and S are disjoint subsets of $3^\wedge A$ because $\max(2^\wedge A' + a_1) < \min(S)$.

Claim 1: $d(A') = 1$.

If $d(A') > 1$, then Lemma 3.1 implies that

$$|3^\wedge A| \geq 5k - 16 > 3k - 6.$$

Therefore $d(A') = 1$.

Claim 2: $|2^\wedge(A')| \leq 2(k-1)$.

If $|2^\wedge A' + a_1| = |2^\wedge(A')| \geq 2(k-1) + 1$, then

$$|3^\wedge A| \geq |2^\wedge(A') + a_1| + |S| \geq 2(k-1) + 1 + k - 4 = 3k - 5 > 3k - 6.$$

Therefore, $|2^\wedge(A')| = |2^\wedge(A') + a_1| \leq 2(k-1)$. Using Lemma 3.4, we get

$$2(k-1) - 1 \leq |2^\wedge(A')| \leq 2(k-1).$$

Now, consider the following cases.

(1) If $|2^\wedge(A')| = 2(k-1) - 1$, then by Lemma 3.2, we have

$$A = [0, k] \setminus \{x\}, \text{ where } x \in \{2, 3, k-2\}.$$

Since $|3^\wedge A| = |3^\wedge(k - A)| = 3k - 6$. Therefore $A = [0, k] \setminus \{x\}$, where $x \in \{2, 3, k-3, k-2\}$.

(2) If $|2^\wedge(A')| = 2(k-1)$, then by Lemma 3.3, we have

$$A = [0, k+1] \setminus \{x, y\},$$

where $\{x, y\}$ is one of the sets $\{1, 2\}$, $\{1, k\}$, and $\{k-1, k\}$.

Thus, $|3^\wedge A| = 3k - 6$ if and only if $A = [0, k+1] \setminus \{x, y\}$, where $\{x, y\}$ is one of the sets $\{2, k+1\}$, $\{3, k+1\}$, $\{k-3, k+1\}$, $\{k-2, k+1\}$, $\{1, 2\}$, $\{k-1, k\}$, and $\{1, k\}$. This completes the proof of the lemma. □

Proof of Theorem 1.9. Combining Lemma 3.5 and Lemma 3.6, we get Theorem 1.9. □

Lemma 3.7. *Let $k \geq 10$ and $A = \{a_0, a_1, \ldots, a_{k-1}\}$ be a finite set of k integers such that $0 = a_0 < a_1 < \cdots < a_{k-1}$ and $d(A) = 1$. Let $A' = A \setminus \{a_1\}$. If $|4^\wedge A| > 4k - 15$, then $|3^\wedge A'| \geq 3(k-1) - 6$.*

Proof. The proof is similar to the proof of Lemma 3.4. So, we omit the details. □

Proof of Theorem 1.10. Let $A = \{a_0, a_1, \ldots, a_{k-1}\}$, where $0 = a_0 < a_1 < \cdots < a_{k-1}$ and $d(A) = 1$. Then $a_2 \geq 2$. Set $A' = A \setminus \{a_1\}$ and $S = \{a_i + a_{k-3} + a_{k-2} + a_{k-1} : i \in [2, k-4]\}$. Note that $3^\wedge A' + a_1$ and S are disjoint subsets of $4^\wedge A$ because $\max(3^\wedge A' + a_1) < \min(S)$.

Claim 1: $d(A') = 1$.
If $d(A') > 1$, then Lemma 3.1 implies that

$$|4^\wedge A| \geq 7k - 30 > 4k - 14.$$

Therefore $d(A') = 1$.

Claim 2: $|3^\wedge(A')| \leq 3(k-1) - 6$.
If $|3^\wedge A' + a_1| = |3^\wedge(A')| \geq 3(k-1) - 5$, then

$$|4^\wedge A| \geq |3^\wedge(A') + a_1| + |S| \geq 3(k-1) - 5 + k - 5 = 4k - 13 > 4k - 14.$$

Therefore, $|3^\wedge(A')| = |3^\wedge(A') + a_1| \leq 3(k-1) - 6$. Using Lemma 3.7, we have $|3^\wedge(A')| = 3(k-1) - 6$. Therefore, by Theorem 1.9, $A' = [0, k] \setminus \{x, y\}$, where $\{x, y\}$ is one of the sets $\{2, k\}$, $\{3, k\}$, $\{k-4, k\}$, $\{k-3, k\}$, $\{1, 2\}$, $\{k-2, k-1\}$ and $\{1, k-1\}$. Now consider the following cases.

(i) If $\{x, y\}$ is one of the sets $\{2, k\}$, $\{3, k\}$, $\{k-4, k\}$, $\{k-3, k\}$, and $\{k-2, k-1\}$, then $a_2 = 1$, which is a contradiction.

(ii) If $\{x, y\} = \{1, 2\}$, then $a_1 \in \{1, 2\}$. It follows that

$$A = \begin{cases} \{0, 1\} \cup [3, k] & \text{if } a_1 = 1, \\ \{0\} \cup [2, k] & \text{if } a_1 = 2. \end{cases}$$

Therefore

$$4^\wedge A = \begin{cases} [8, 4k-6] & \text{if } a_1 = 1 \\ [9, 4k-6] & \text{if } a_1 = 2. \end{cases}$$

So

$$|4^\wedge A| = \begin{cases} 4k - 13 & \text{if } a_1 = 1 \\ 4k - 14 & \text{if } a_1 = 2. \end{cases}$$

(iii) If $\{x, y\} = \{1, k-1\}$, then $a_1 = 1$. It follows that $A = [0, k-2] \cup \{k\}$. Therefore $4^\wedge A = [6, 4k-9]$. So $|4^\wedge A| = 4k - 14$.

Hence, $|4^\wedge A| = 4k - 14$ if and only if $A = [0, k] \setminus \{x\}$, where $x \in \{1, k-1\}$. This completes the proof of the theorem. □

4. Conclusion

On the basis of Theorem 1.8, Theorem 1.9, and Theorem 1.10, we propose Conjecture 4.1.

Conjecture 4.1. Let k be a large positive integer and h be a positive integer with $2 \leq h \leq k-2$. Let A be a finite set of k nonnegative integers with $\min(A) = 0$ and $d(A) = 1$.

(a) If $|h^\wedge A| = hk - h^2 + 2$, then $A \subset [0, k]$.
(b) If $|h^\wedge A| = hk - h^2 + 3$, then $A \subset [0, k+1]$.
(c) If $|h^\wedge A| = hk - h^2 + 4$, then $A \subset [0, k+2]$.

It is seen in this paper that (a) is true for $h = 2, 3, 4$, (b) is true for $h = 2, 3$, and (c) is true for $h = 2$ due to Theorem 1.8, Theorem 1.9, and Theorem 1.10, respectively.

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