Bull. Korean Math. Soc. **61** (2024), No. 5, pp. 1241–1252
https://doi.org/10.4134/BKMS.b230474
pISSN: 1015-8634 / eISSN: 2234-3016

iw-SPLIT MODULES

Xiaoying Wu

Abstract. In this paper, the notions of *iw*-split modules and *iw*-split dimension are introduced, and some equivalent characterizations of these notions are given. With the help of *iw*-split modules and *iw*-split dimensions, new characterizations of DW rings, semi-simple rings, and Dedekind domains are given. More precisely, it is shown that R is a DW ring if and only if every *iw*-split module is an injective module; while R is a semi-simple ring if and only if every R-module is an *iw*-split module; and R is a Dedekind domain if and only if every factor module of an *iw*-split module is *iw*-split.

1. Introduction

Throughout this paper, R denotes a commutative ring with identity 1 and ${}_R\mathfrak{M}$ be the category of R-modules. Recall from [4] that an ideal J of R is called a Glaz-Vasconcelos ideal (a GV-ideal for short) if J is finitely generated and the natural homomorphism $\varphi : R \to J^* = \mathrm{Hom}_R(J, R)$ is an isomorphism, denoted by $J \in \mathrm{GV}(R)$. Let M be an R-module. Define

$$\mathrm{tor}_{\mathrm{GV}}(M) = \{x \in M \mid Jx = 0 \text{ for some } J \in \mathrm{GV}(R)\}.$$

Thus $\mathrm{tor}_{\mathrm{GV}}(M)$ is a submodule of M. One calls M GV-torsion (resp., GV-torsionfree) if $\mathrm{tor}_{\mathrm{GV}}(M) = M$ (resp., $\mathrm{tor}_{\mathrm{GV}}(M) = 0$). A GV-torsionfree module M is called a w-module if $\mathrm{Ext}^1_R(R/J, M) = 0$ for all $J \in \mathrm{GV}(R)$. For any GV-torsionfree module M,

$$M_w = \{x \in E(M) \mid Jx \subseteq M \text{ for some } J \in \mathrm{GV}(R)\}$$

is a w-submodule of $E(M)$ containing M and is called the w-envelope of M, where $E(M)$ denotes the injective envelope of M. It is clear that a GV-torsionfree module M is a w-module if and only if $M_w = M$. Let M and N be R-modules and let $f : M \to N$ be a homomorphism (see [4]). Then, f is called a w-monomorphism (resp., a w-epimorphism, a w-isomorphism) if $f_{\mathfrak{m}} : M_{\mathfrak{m}} \to$

Received August 16, 2023; Revised March 4, 2024; Accepted March 25, 2024.
2020 *Mathematics Subject Classification.* 13D05, 13D07, 13F05.
Key words and phrases. w-split module, *iw*-split module, *iw*-split dimension, Dedekind domain.

$N_{\mathfrak{m}}$ is a monomorphism (resp., an epimorphism, an isomorphism) for any maximal w-ideal $\mathfrak{m}$ of R. A sequence $A \to B \to C$ of R-modules and homomorphisms is called w-exact if the sequence $A_{\mathfrak{m}} \to B_{\mathfrak{m}} \to C_{\mathfrak{m}}$ is exact for any maximal w-ideal $\mathfrak{m}$ of R. Let M be an R-module and set $L(M) = (M/\mathrm{tor}_{\mathrm{GV}}(M))_w$. Then M is said to be w-projective if $\mathrm{Ext}_R^1(L(M), N)$ is a GV-torsion module for any torsion-free w-module N. From the definition, it is clear that projective modules and GV-torsion modules are w-projective. An R-module E is said to be w-injective if for any w-exact sequence $0 \to A \to B \to C \to 0$, the induced sequence $0 \to \mathrm{Hom}_R(C, L(E)) \to \mathrm{Hom}_R(B, L(E)) \to \mathrm{Hom}_R(A, L(E)) \to 0$ is also w-exact. For unexplained terminologies and notations, we refer to [4].

It is well known that semi-simple rings can be characterized by either projective modules or injective modules, i.e., R is a semi-simple ring if and only if every R-module is a projective module, if and only if every R-module is an injective module; see for example [4, Theorem 7].

Moreover, semi-simple rings are characterized via w-operation. Namely, R is a semi-simple ring if and only if every R-module is a w-projective module (see Fanggui Wang and Hwankoo Kim [3, Theorem 3.15]), and if and only if every R-module is a w-injective module (see Almahdi and Assaad [1, Theorem 2.12]).

Recently, in 2020, Fanggui Wang and Lei Qiao introduced the concepts of w-split short exact sequences and w-split modules. A short exact sequence of R-modules $\xi : 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is said to be w-split if there exist $J = (d_1, \ldots, d_n) \in \mathrm{GV}(R)$ and $h_1, \ldots, h_n \in \mathrm{Hom}_R(C, B)$ such that $gh_k = d_k\mathbf{1}_C$, $k = 1, \ldots, n$. Equivalently, there exist $q_1, \ldots, q_n \in \mathrm{Hom}_R(B, A)$ such that $q_k f = d_k\mathbf{1}_A$, $k = 1, \ldots, n$. An R-module M is said to be w-split if there is a w-split short exact sequence of R-modules $0 \to \ker(g) \to F \xrightarrow{g} M \to 0$, where F is a projective module, equivalently, $\mathrm{Ext}_R^1(M, N)$ is GV-torsion for all R-modules N (see [6]). Wang and Qiao showed that R is semi-simple if and only if every R-module is a w-split module. They choose to define "w-split" in the sense of projectivity. A natural question is whether an alternative kind of "w-split" corresponding to injectivity may also lead to a characterization of semi-simple rings.

In this paper, we introduce "w-split" modules in the sense of injectivity, which we call *iw*-split modules. After showing some equivalent descriptions of *iw*-split modules, we give a new characterization of semi-simple rings as: R is a semi-simple ring if and only if every R-module is *iw*-split. Additionally, the notion of *iw*-split dimension is introduced. With the help of *iw*-split modules and *iw*-split dimension, new characterizations of DW rings and Dedekind domains are given. More precisely, it is shown that R is a DW ring if and only if every *iw*-split module is an injective module; while R is a Dedekind domain if and only if every factor module of an *iw*-split module is *iw*-split.

2. iw-split modules and iw-split dimension

We begin this section by introducing the concept of iw-split modules.

Definition 2.1. An R-module N is said to be iw-split if there is a w-split short exact sequence of R-modules

$$0 \to N \to E \to C \to 0$$

with E injective.

According to the definition of iw-split modules, it is clear that an injective module is iw-split. The converse is not necessarily true (see Example 2.12).

Proposition 2.2. *Let $\xi : 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a w-split exact sequence, J be a GV-ideal associated with ξ. Let M be an R-module. Then:*

1. *$J\alpha \subseteq \mathrm{Im}(g_*)$ for any $\alpha \in \mathrm{Hom}_R(M, C)$. Hence $J\mathrm{Hom}_R(M, C) \subseteq \mathrm{Im}(g_*)$.*
2. *If $\mathrm{Ext}^1_R(M, B)$ is a GV-torsion module, then $\mathrm{Ext}^1_R(M, A)$ is a GV-torsion module.*
3. *If $\mathrm{Ext}^1_R(M, B) = 0$, then $J\mathrm{Ext}^1_R(M, A) = 0$, that is, $\mathrm{Ext}^1_R(M, A)$ is an R/J-module.*

Proof. Consider the induced exact sequence

$$0 \to \mathrm{Hom}_R(M, A) \to \mathrm{Hom}_R(M, B) \xrightarrow{g_*} \mathrm{Hom}_R(M, C) \to \mathrm{Ext}^1_R(M, A) \to \mathrm{Ext}^1_R(M, B).$$

(1) Since $\alpha \in \mathrm{Hom}_R(M, C)$, we have $\alpha q_k \in \mathrm{Hom}_R(M, B)$ and $d_k\alpha = g_*(\alpha q_k)$. So $J\alpha \subseteq \mathrm{Im}(g_*)$.

(2) Set $L := \mathrm{Cok}(g_*)$. It is obtained from (1) that L is an R/J-module, and thus it is a GV-torsion module. It follows from the exact sequence $0 \to L \to \mathrm{Ext}^1_R(M, A) \to \mathrm{Ext}^1_R(M, B)$ that $\mathrm{Ext}^1_R(M, A)$ is a GV-torsion module.

(3) When $\mathrm{Ext}^1_R(M, B) = 0$, we get that $\mathrm{Ext}^1_R(M, A) = \mathrm{Cok}(g_*)$ is an R/J-module. □

Theorem 2.3. *The following statements are equivalent for an R-module N.*

1. *N is an iw-split module.*
2. *$\mathrm{Ext}^1_R(M, N)$ is GV-torsion for all R-modules M.*
3. *$\mathrm{Ext}^k_R(M, N)$ is GV-torsion for all R-modules M and for all $k \geq 1$.*
4. *If $0 \to A \to B \to C \to 0$ is an exact sequence, then the sequence*
$$0 \to \mathrm{Hom}_R(C, N) \to \mathrm{Hom}_R(B, N) \to \mathrm{Hom}_R(A, N) \to 0$$
is a w-exact sequence.
5. *Every exact sequence of the form $\eta : 0 \to N \to B \to C \to 0$ is w-split.*
6. *For any R-module monomorphism $f : A \to B$ and for each homomorphism $\alpha : A \to N$, there exist $J = (d_1, \ldots, d_n) \in \mathrm{GV}(R)$ and homomorphisms $q_k : B \to N$ such that $q_k f = d_k\alpha$, $k = 1, \ldots, n$.*
7. *There exists $J \in \mathrm{GV}(R)$ such that $\mathrm{Ext}^1_R(M, N)$ is an R/J-module for any R-module M.*

Proof. (1)⇒(2) Let $\xi : 0 \to N \xrightarrow{f} E \xrightarrow{g} C \to 0$ be a w-split exact sequence of R-modules with E injective. There exist $J = (d_1, \ldots, d_n) \in \mathrm{GV}(R)$ and homomorphisms $h_k \in \mathrm{Hom}_R(C, E)$ such that $gh_k = d_k\mathbf{1}_C$, $k = 1, \ldots, n$. Let $\alpha \in \mathrm{Hom}_R(M, C)$. Define $\beta_k = h_k\alpha$. Then $g\beta_k = gh_k\alpha = d_k\alpha$, $k = 1, \ldots, n$, so $J\alpha \subseteq \mathrm{Im}(g_*)$. Therefore g_* is a w-epimorphism. Thus

$$0 \to \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(M, E) \xrightarrow{g_*} \mathrm{Hom}_R(M, C) \to 0$$

is a w-exact sequence. Since

$$0 \to \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(M, E) \to \mathrm{Hom}_R(M, C) \to \mathrm{Ext}^1_R(M, N) \to 0$$

is an exact sequence, it follows that $\mathrm{Ext}^1_R(M, N)$ is GV-torsion.

(2)⇒(4) This is trivial.

(4)⇒(6) Since $f^* : \mathrm{Hom}_R(B, N) \to \mathrm{Hom}_R(A, N)$ is a w-epimorphism, then $\mathrm{Hom}_R(A, N)/\mathrm{Im}(f^*)$ is GV-torsion, so there exist $J = (d_1, \ldots, d_n) \in \mathrm{GV}(R)$ such that $J\alpha \subseteq \mathrm{Im}(f^*)$. Thus there are $q_k \in \mathrm{Hom}_R(B, N)$ such that $q_kf = d_k\alpha$, $k = 1, \ldots, n$.

(6)⇒(5) This follows from setting $A = N$ in (6).

(5)⇒(1) This follows easily from the definition of iw-split modules

(2)⇒(3) Let $0 \to A \to P \to M \to 0$ be an exact sequence with P projective and $k > 1$. Then $\mathrm{Ext}^k_R(M, N) \cong \mathrm{Ext}^{k-1}_R(C, N)$. By using induction on k, we see that $\mathrm{Ext}^k_R(M, N)$ is GV-torsion.

(3)⇒(2) Clear.

(1)⇒(7) Since N is an iw-split module, then there is a w-split exact sequence $\xi : 0 \to N \to E \to C \to 0$, where E is an injective module. Let J be a GV-ideal associated with ξ. By Proposition 2.2(3), $\mathrm{Ext}^1_R(M, N)$ is an R/J-module for any R-module M.

(7)⇒(2) Clear. □

Corollary 2.4. (1) *Let N be injective. Then N is an iw-split module.*

(2) *Let $0 \to A \to B \to C \to 0$ be an exact sequence with A an iw-split module. Then B is an iw-split module if and only if C is an iw-split module.*

(3) *Let M and N be any R-modules. Then $M \bigoplus N$ is an iw-split module if and only if M and N are iw-split modules. Therefore every direct summand of an iw-split module is iw-split.*

Proof. (1) Clear.

(2) Since $0 \to A \to B \to C \to 0$ be an exact sequence with A an iw-split module, then for any R-module M, there is an exact sequence

$$\mathrm{Ext}^1_R(M, A) \to \mathrm{Ext}^1_R(M, B) \to \mathrm{Ext}^1_R(M, C) \to \mathrm{Ext}^2_R(M, A).$$

Since A is an iw-split module, by Theorem 2.3, we see that $\mathrm{Ext}^1_R(M, A)$ and $\mathrm{Ext}^2_R(M, A)$ are GV-torsion. Therefore $\mathrm{Ext}^1_R(M, B)$ is GV-torsion if and only if $\mathrm{Ext}^1_R(M, C)$ is GV-torsion. By Theorem 2.3, we see that B is an iw-split module if and only if C is an iw-split module.

(3) Let K be an R-module. Since

$$\mathrm{Ext}_R^1(K, M \bigoplus N) = \mathrm{Ext}_R^1(K, M) \bigoplus \mathrm{Ext}_R^1(K, N),$$

then $\mathrm{Ext}_R^1(K, M \bigoplus N)$ is GV-torsion if and only if $\mathrm{Ext}_R^1(K, M)$ and $\mathrm{Ext}_R^1(K, N)$ are GV-torsion, so $M \bigoplus N$ is an iw-split module if and only if M and N are iw-split modules. □

The concept of w-injective modules is introduced by Fanggui Wang and Hwankoo Kim in [2] and some equivalent characterizations of w-injective modules are given. An R-module E is said to be w-injective if

$$0 \to \mathrm{Hom}_R(C, L(E)) \to \mathrm{Hom}_R(B, L(E)) \to \mathrm{Hom}_R(A, L(E)) \to 0$$

is w-exact for any w-exact sequence $0 \to A \to B \to C \to 0$. In [2], it is also shown that a w-module E is w-injective if and only if $\mathrm{Ext}_R^1(M, E)$ is GV-torsion for all R-modules M. Hence we have the following:

Corollary 2.5. *Let N be a w-module. Then N is a w-injective module if and only if N is an iw-split module.*

Recall from [4] that an R-module D is said to be a GV-divisible module if $JD = D$ for any $J \in \mathrm{GV}(R)$, equivalently, $(R/J) \bigotimes_R D = 0$. By the introduction of [7], we can easily get the following lemma.

Lemma 2.6. (1) *Let E be an injective module. Then E is a GV-divisible module.*

(2) *Let $f : M \to N$ be an epimorphism and M be a GV-divisible module. Then N is a GV-divisible module.*

(3) *Let $\{D_i\}$ be a family of GV-divisible modules. Then $\bigoplus_i D_i$ is a GV-divisible module.*

(4) *Let $0 \to A \to B \to C \to 0$ be an exact sequence. If A and C are GV-divisible modules, then B is a GV-divisible module.*

Proposition 2.7. *Let N be an R-module.*

(1) *If D_1 and D_2 are GV-divisible submodules of N, then $D_1 + D_2$ is a GV-divisible submodule of N.*

(2) *If $\{D_i\}$ is an ascending chain on GV-divisible submodules of N, then $D := \bigcup_i D_i$ is a GV-divisible submodule of N.*

(3) *N has the largest GV-divisible submodule.*

Proof. (1) By the epimorphism $f : D_1 \bigoplus D_2 \to D_1 + D_2$ and Lemma 2.6, we can get the conclusion.

(2) Let $x \in D$ and $J \in \mathrm{GV}(R)$. Then there exists a subscript i such that $x \in D_i$. Thus $x \in JD_i \subseteq JD$. Hence $D = JD$.

(3) By Zorn's lemma, N has a maximal GV-divisible submodule. By Proposition 2.7(1), we see that N has only one maximal GV-divisible submodule. Therefore the maximal GV-divisible submodule is the largest GV-divisible submodule. □

In order to get the Theorem 2.10 and Example 2.11, next, we will introduce the concept of GV-reduced module.

Definition 2.8. Let N be an R-module. If the largest GV-divisible submodule of N is 0, then N is said to be a GV-reduced module.

Proposition 2.9. (1) *Every submodule of a GV-reduced module is still GV-reduced.*

(2) *An R-module N is a GV-reduced module if and only if* $\mathrm{Hom}_R(D, N) = 0$ *for any GV-divisible module D.*

(3) *Let D be the largest GV-divisible submodule of N. Then N/D is a GV-reduced module.*

(4) *Let $\{N_i\}$ be a family of R-modules. Then $\prod_i N_i$ is a GV-reduced module if and only if every N_i is a GV-reduced module, if and only if $\bigoplus_i N_i$ is a GV-reduced module.*

(5) *Let $J \in$ GV(R) and N be an R/J-module. Then N is a GV-reduced module.*

(6) *Let $J \in$ GV(R) and $N = \bigoplus_m R/J^m$. Then N is a GV-reduced module.*

Proof. (1) Clear.

(2) Let N be a GV-reduced module, D be a GV-divisible module and $f : D \to N$ be a homomorphism. By Lemma 2.6, $f(D)$ is a GV-divisible submodule of N, thus $f(D) = 0$, that is $f = 0$. Therefore $\mathrm{Hom}_R(D, N) = 0$.

Conversely, suppose that D is a non-zero GV-divisible submodule of N, then the inclusion mapping $\lambda : D \to N$ is a non-zero mapping, which is a contradiction. Thus $D = 0$. Hence N is a GV-reduced module.

(3) Let D be the largest GV-divisible submodule of N and N_0 be a submodule of N which contains D. Then $0 \to D \to N_0 \to N_0/D \to 0$ is an exact sequence. If N_0/D is a GV-divisible module, by Lemma 2.6, we can get that N_0 is a GV-divisible submodule of N. Thus $N_0 = D$. Therefore N/D is a GV-reduced module.

(4) By Proposition 2.9(2), we can get the conclusion.

(5) Let D be a GV-divisible submodule of N. Then $D = JD = 0$. Therefore N is a GV-reduced module.

(6) By Proposition 2.9(5), every R/J^m is a GV-reduced module. By Proposition 2.9(4), we see that N is a GV-reduced module. $\square$

Theorem 2.10. *Let N be an R-module.*

(1) *If there exists $J \in$ GV(R) such that $JN = 0$, then N is an iw-split module, that is, every R/J-module is iw-split.*

(2) *Let N be a GV-reduced module. If N is an iw-split module, then there exists $J \in$ GV(R) such that $JN = 0$.*

Proof. (1) Since $JN = 0$, then N is an R/J-module. So $\mathrm{Ext}^1_R(M, N)$ is an R/J-module for all R-modules M. By Theorem 2.3, we see that N is an *iw*-split module.

(2) Let $\xi : 0 \to N \to E \to C \to 0$ be a w-split exact sequence of R-modules with E injective. By Lemma 2.6, E is a GV-divisible module. By Proposition 2.9 and N is a GV-reduced module, we have $\mathrm{Hom}_R(E, N) = 0$. So $0 \to \mathrm{Hom}_R(N, N) \to \mathrm{Ext}^1_R(C, N)$ is an exact sequence. By Theorem 2.3, there exists $J \in \mathrm{GV}(R)$ such that $\mathrm{Ext}^1_R(C, N)$ is an R/J-module. Thus $\mathrm{Hom}_R(N, N)$ is an R/J-module. Hence $J\,\mathbf{1}_N = 0$. Therefore $JN = 0$. $\square$

According to the definitions of iw-split modules and w-injective modules, it is clear that an iw-split module is w-injective. The converse is not necessarily true. Next, we will give an example of a w-injective module, which is not iw-split.

Example 2.11. Let K be a field, x and y be indeterminates, $R = K[x, y]$, $I = (x, y)$. Then

(1) $I \in \mathrm{GV}(R)$.
(2) $\bigcap_{m=1}^{\infty} I^m = 0$.
(3) Let $N = \bigcap_{m=1}^{\infty} R/I^m$. Then N is a w-injective module. However, N is not an iw-split module.

Proof. (1) Clear.

(2) If $\bigcap_{m=1}^{\infty} I^m \neq 0$, then there exists a polynomial $f \in \bigcap_{m=1}^{\infty} I^m$ and $f \neq 0$. Let $\deg(f) = s$. Then $f \notin I^{s+1}$, a contradiction. Therefore $f = 0$.

(3) Since N is a GV-torsion module, then N is w-injective. If N is an iw-split module, by Theorem 2.10, there exists $J \in \mathrm{GV}(R)$ such that $JN = 0$. So $J(R/I^m) = (J + I^m)/I^m = 0$ for any m. Hence for any m, we have $J \subseteq I^m$. By Example 2.11(2), $J = 0$, a contradiction. Therefore, N is not an iw-split module. $\square$

Example 2.12. Let R be an unique factorization domain and $u, v \in R$ be relatively prime. Then $J = (u, v) \in \mathrm{GV}(R)$. By Theorem 2.10, we can get R/J is iw-split. If $J \neq R$, then R/J is not a division module, therefore R/J is not an injective module.

In [1], Almahdi and Assaad introduced the concept of w-split dimension of modules. Let M be an R-module. Define $w\text{-sd}(M) \leq n$ if M has a w-split module resolution of length n. If no such finite resolution exists, then define $w\text{-sd}(M) = \infty$. Correspondingly, in the following we will define the concept of iw-split dimension of modules.

Definition 2.13. Let N be an R-module. If there exists an exact sequence

$$0 \to N \to E_0 \to \cdots \to E_{n-1} \to E_n \to 0,$$

where $E_0, E_1, \ldots, E_n$ are iw-split modules and n is the least such nonnegative integer, we call $iw\text{-sd}_R(N)$ the iw-split dimension of N, and we define $iw\text{-sd}_R(N) = n$. If no such finite resolution exists, then we define $iw\text{-sd}_R(N) = \infty$.

Clearly, we have $iw\text{-sd}_R(N) \leq \mathrm{id}_R(N)$ for all R-modules N.

The corresponding dimension of R-modules is used to measure how far away a module is from being a certain kind of module. For example, the projective dimension of an R-module is used to measure how far away the module is from being projective, and the injective dimension of an R-module is used to measure how far away the module is from being injective. So the iw-split dimension of R-modules measures how far away a module is from being an iw-split module. Hence we can get the following.

Example 2.14. Let N be an iw-split module. Then $0 \to N \to N \to 0$ is a resolution of N, thus $iw\text{-sd}_R(N) = 0$. Conversely, if $iw\text{-sd}_R(N) = 0$, then N is iw-split.

Proposition 2.15. *The following statements are equivalent for an R-module N and a nonnegative integer n.*

(1) $iw\text{-sd}_R(N) \leq n$.
(2) $\mathrm{Ext}_R^{n+1}(M, N)$ *is* GV*-torsion for all R-modules M.*
(3) *Let $0 \to N \to E_0 \to \cdots \to E_{n-1} \to E_n \to 0$ be an exact sequence, where $E_0, \ldots, E_{n-1}$ are iw-split modules. Then E_n is iw-split.*
(4) *Let $0 \to N \to E_0 \to \cdots \to E_{n-1} \to E_n \to 0$ be an exact sequence, where $E_0, \ldots, E_{n-1}$ are injective modules. Then E_n is iw-split.*

Proof. (1)⇒(2) By the definition of iw-split dimension, there exists an exact sequence

$$0 \to N \to E_0 \to \cdots \to E_{n-1} \to E_n \to 0,$$

where $E_0, E_1, \ldots, E_n$ are iw-split modules. Thus there is an exact sequence

$$\mathrm{Ext}_R^n(M, E_n) \to \mathrm{Ext}_R^{n+1}(M, N) \to \mathrm{Ext}_R^{n+1}(M, E_0).$$

Since E_0 and E_n are iw-split modules, by Theorem 2.3, we see that $\mathrm{Ext}_R^n(M, E_0)$ and $\mathrm{Ext}_R^n(M, E_n)$ are GV-torsion. Therefore $\mathrm{Ext}_R^{n+1}(M, N)$ is GV-torsion.

(2)⇒(3) Suppose that (2) holds. Thus there exists an exact sequence

$$\mathrm{Ext}_R^1(M, E_{n-1}) \to \mathrm{Ext}_R^1(M, E_n) \to \mathrm{Ext}_R^2(M, N)$$

for all R-modules M. Since $\mathrm{Ext}_R^1(M, E_{n-1})$ and $\mathrm{Ext}_R^2(M, N)$ are GV-torsion, then $\mathrm{Ext}_R^1(M, E_n)$ is GV-torsion. By Theorem 2.3, E_n is iw-split.

(3)⇒(4)⇒(1) This follows from Corollary 2.4(1). □

Proposition 2.16. *Let $0 \to A \to B \to C \to 0$ be an exact sequence. Then*:

(1) *If $iw\text{-sd}_R(B) < iw\text{-sd}_R(A)$, then $iw\text{-sd}_R(C) = iw\text{-sd}_R(A) - 1$.*
(2) *If $iw\text{-sd}_R(A) \leq iw\text{-sd}_R(B)$, then $iw\text{-sd}_R(B) = iw\text{-sd}_R(C)$.*

Proof. Let M be an R-module and $k \geq 0$.

(1) This is obtained directly from the exact sequence

$$\mathrm{Ext}_R^k(M, B) \to \mathrm{Ext}_R^k(M, C) \to \mathrm{Ext}_R^{k+1}(M, A) \to \mathrm{Ext}_R^{k+1}(M, B).$$

(2) This follows from the exact sequence

$$\mathrm{Ext}_R^k(M,A) \to \mathrm{Ext}_R^k(M,B) \to \mathrm{Ext}_R^k(M,C) \to \mathrm{Ext}_R^{k+1}(M,A). \qquad \square$$

Recall that an R-module M is called a w-flat module if the induced map $1 \bigotimes f : M \bigotimes_R A \to M \bigotimes_R B$ is a w-monomorphism for any w-monomorphism $f : A \to B$. The concept of w-flat dimension was introduced by Fanggui Wang and Lei Qiao in [5]. Let M be an R-module. Recall that $w\text{-fd}(M) \le n$ if there exists an exact sequence

$$0 \to F_n \to F_{n-1} \to \cdots \to F_0 \to M \to 0,$$

where $F_n, F_{n-1}, \ldots, F_0$ are w-flat. It is clear that $w\text{-fd}(M) \le w\text{-sd}(M) \le \mathrm{pd}_R(M)$. Some equivalent characterizations of w-split dimension are given in [1]. For example, let M be an R-module and n be a nonnegative integer. Then $w\text{-sd}(M) \le n$ if and only if $\mathrm{Ext}_R^{n+1}(M,N)$ is GV-torsion for all R-modules N, if and only if K_n is w-split whenever there is an exact sequence $0 \to K_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0$ with $P_0, \ldots, P_{n-1}$ projective. A ring R is called a DW ring if every ideal of R is a w-ideal. Fanggui Wang and Hwankoo Kim in [3] show that if R is a DW ring, then the class of projective R-modules, the class of w-split R-modules, and the class of w-projective R-modules are equivalent. Almahdi and Assaad in [1, Proposition 3.4] built on this in the following way. A ring R is a DW ring if and only if $\mathrm{pd}_R(M) = w\text{-sd}(R)$ for all R-modules M, if and only if $\mathrm{fd}_R(M) = w\text{-fd}(M)$ for all R-modules M.

In 2021, Almahdi and Assaad used w-split modules to give a characterization of DW rings. For example, let R be a ring. Then R is a DW ring if and only if every w-projective R-module is projective, if and only if every w-split R-module is projective, if and only if every w-flat R-module is flat. Based on this use of w-split modules to characterize DW rings, the natural question arises whether iw-split modules can similarly characterize DW rings.

Theorem 2.17. *Let R be a ring. Then the following statements are equivalent.*

(1) *Every w-injective R-module is injective.*
(2) *Every iw-split R-module is injective.*
(3) *R is a DW ring.*

Proof. (1)⇒(2) By the definitions of w-injective modules and iw-split modules, we can get the conclusion.

(2)⇒(3) Let $J \in \mathrm{GV}(R)$ and $N = J/J^2$. Then $JN = 0$. According to Theorem 2.10, N is an iw-split module. Now let $f : J \to N$ be the natural homomorphism, so $f(a) = \bar{a}$, $a \in J$. By (2), N is an injective module. Thus there exists a homomorphism $g : R \to N$ such that $f(a) = g(a)$ for all $a \in J$. Denoted by $g(1) = \bar{b}$ for all $b \in J$, since $\bar{b} = f(b) = g(b) = bg(1) = \bar{b}^2$, so $g(1) = \bar{b} = \bar{0}$, therefore $g = 0$. Since f is an epimorphism, it follows that g is also an epimorphism. Hence $N = 0$, that is, $J = J^2$. By [4, Theorem 1.8.22], J can be generated by an idempotent element, thus J is a projective ideal. Therefore J is a w-module, so $J = J_w = R$. Hence R is a DW ring.

(3)⇒(1) Clear. □

Corollary 2.18. *Let R be a DW ring and N be an R-module. Then*

$$iw\text{-sd}_R(N) = w\text{-id}_R(N).$$

Proof. This is obtained directly from Corollary 2.5. □

3. Characterizations of semi-simple rings and Dedekind domains

In the following, we will discuss the relationships between w-split dimension of modules and iw-split dimension of modules.

Theorem 3.1. *The following statements are equivalent for a ring R and a nonnegative integer n.*

(1) $w\text{-sd}(M) \leq n$ *for all R-modules M.*
(2) $iw\text{-sd}(N) \leq n$ *for all R-modules N.*
(3) $\mathrm{Ext}_R^k(M, N)$ *is* GV*-torsion for all R-modules M, N and for all $k > n$.*
(4) $\mathrm{Ext}_R^{n+1}(M, N)$ *is* GV*-torsion for all R-modules M, N.*

Proof. (1)⇒(2) This follows from [1, Proposition 3.3] and Proposition 2.15.

(2)⇒(3) Since $iw\text{-sd}(N) \leq n$ for all R-modules N, then there is a resolution of an iw-split module N

$$0 \to N \to E_0 \to \cdots \to E_{n-1} \to E_n \to 0,$$

where $E_0, E_1, \ldots, E_n$ are iw-split modules. Thus when $k > n$, $\mathrm{Ext}_R^k(M, N) \cong \mathrm{Ext}_R^{k-n}(M, E_n)$ for all R-modules M. Hence $\mathrm{Ext}_R^k(M, N)$ is GV-torsion.

(3)⇒(4) This is obvious.

(4)⇒(1) This follows easily from [1, Proposition 3.3]. □

The following results give new equivalent characterizations of semi-simple rings and Dedekind domains by iw-split modules.

Theorem 3.2. *The following statements are equivalent for a ring R.*

(1) *Every R-module is a w-split module.*
(2) *Every R-module is an iw-split module.*
(3) *R is a semi-simple ring.*
(4) *Every cyclic R-module is an iw-split module.*

Proof. (1)⇒(2) Let M be an R-module. It follows the assumption that M is w-split, so $w\text{-sd}(M) = 0$. By Theorem 3.1, $iw\text{-sd}(N) = 0$ for all R-modules N. Thus every R-module is an iw-split module.

(3)⇒(1) Let N be an R-module. Since R is a semi-simple ring, then N is injective. According to Corollary 2.4, we can get N is iw-split, thus $iw\text{-sd}(N) = 0$. By Theorem 3.1, $w\text{-sd}(N) = 0$. It follows from the definition of w-split dimension, N is a w-split module, so every R-module is a w-split module.

(2)⇒(4) This is trivial.

(4)⇒(3) Let M be a cyclic R-module and N be a cyclic torsion-free w-module. By the hypothesis, N is iw-split, so $\mathrm{Ext}_R^1(M, N)$ is GV-torsion. Thus M is w-projective. By [3, Theorem 3.15], we can get the conclusion. $\square$

Theorem 3.3. *The following statements are equivalent for a ring R.*

(1) *Every factor module of an iw-split module is iw-split.*
(2) *Every factor module of an injective module is iw-split.*
(3) *Every submodule of a w-split module is w-split.*
(4) *Every submodule of a projective module is w-split.*
(5) *R is hereditary.*

Proof. (1)⇒(2) This is trivial.

(2)⇒(3) Let P be a w-split module and A be a submodule of P. Let N be any R-module. Then there is an exact sequence $0 \to N \to E \to C \to 0$, where E is injective. By the hypothesis, $\mathrm{Ext}_R^1(A, N) \cong \mathrm{Ext}_R^2(P/A, N) \cong \mathrm{Ext}_R^1(P/A, C)$ is GV-torsion, and so A is a w-split module.

(3)⇒(4) This is clear.

(4)⇒(5) It suffices to prove that R is a DW ring. Let $J \in \mathrm{GV}(R)$ and $R_J = R$. Let $F = \bigoplus_{J\in \mathrm{GV}(R)} R_J$ and $M = \bigoplus_{J\in \mathrm{GV}(R)} J$. Then F is a free R-module, M is a submodule of F and $M_w = F$. It follows from the assumption that M is w-split. By [6, Proposition 2.8], there exists $I \in \mathrm{GV}(R)$ such that $IF \subseteq M$. Therefore $I \subseteq J$ for all $J \in \mathrm{GV}(R)$. Especially, $I \subseteq I^2$. So I is generated by an idempotent element. Thus I is a projective ideal, therefore $I = R$. Hence R has only one GV-ideal, that is R itself, By the definition of DW rings, R is a DW ring.

(5)⇒(1) By the hypothesis, every factor module of an injective module is injective. By Corollary 2.4, every factor module of an iw-split module is iw-split. $\square$

As a consequence of the statement of Theorem 3.3, we have the following corollary.

Corollary 3.4. *Let R be an integral domain. Then every factor module of an iw-split module is iw-split if and only if R is a Dedekind domain.*

Acknowledgements. The author would like to thank the referee for a careful reading and relevant comments which substantially improved the paper. This work was supported by the Foundation of Chengdu University of Information Technology (KYTZ2022147).

References

[1] F. A. A. Almahdi and R. A. K. Assaad, *A note on w-split modules*, Palest. J. Math. **10** (2021), no. 1, 160–168.

[2] F. Wang and H. Kim, *w-injective modules and w-semi-hereditary rings*, J. Korean Math. Soc. **51** (2014), no. 3, 509–525. `https://doi.org/10.4134/JKMS.2014.51.3.509`

[3] F. Wang and H. Kim, *Two generalizations of projective modules and their applications*, J. Pure Appl. Algebra **219** (2015), no. 6, 2099–2123. https://doi.org/10.1016/j.jpaa.2014.07.025

[4] F. Wang and H. Kim, *Foundations of Commutative Rings and Their Modules*, Algebra and Applications, 22, Springer, Singapore, 2016. https://doi.org/10.1007/978-981-10-3337-7

[5] F. Wang and L. Qiao, *The w-weak global dimension of commutative rings*, Bull. Korean Math. Soc. **52** (2015), no. 4, 1327–1338. https://doi.org/10.4134/BKMS.2015.52.4.1327

[6] F. Wang and L. Qiao, *A new version of a theorem of Kaplansky*, Comm. Algebra **48** (2020), no. 8, 3415–3428. https://doi.org/10.1080/00927872.2020.1739289

[7] F. Wang and L. Qiao, *Two applications of Nagata rings and modules*, J. Algebra Appl. **19** (2020), no. 6, 2050115, 15 pp. https://doi.org/10.1142/S0219498820501157

Xiaoying Wu
College of Applied Mathematics
Chengdu University of Information Technology
Chengdu, Sichuan 610225, P. R. China
Email address: mengwxy2017@163.com