Bull. Korean Math. Soc. **59** (2022), No. 4, pp. 951–960
https://doi.org/10.4134/BKMS.b210562
pISSN: 1015-8634 / eISSN: 2234-3016

A GENERALIZATION OF $\mathcal{A}_2$-GROUPS

Junqiang Zhang

Abstract. In this paper, we determine the finite p-group such that the intersection of its any two distinct minimal nonabelian subgroups is a maximal subgroup of the two minimal nonabelian subgroups, and the finite p-group in which any two distinct $\mathcal{A}_1$-subgroups generate an $\mathcal{A}_2$-subgroup. As a byproduct, we answer a problem proposed by Berkovich and Janko.

1. Introduction

A finite group G is said to be minimal nonabelian if G is nonabelian but all its proper subgroups are abelian. Obviously, every finite nonabelian group contains a minimal nonabelian subgroup. In particular, every nonabelian p-group, by [2, Proposition 10.28], can be generated by its minimal nonabelian subgroups. Hence minimal nonabelian subgroups, in a sense, can be regarded as "basic elements" of a nonabelian p-group, which play a fundamental role in studying the structure of nonabelian p-groups.

Berkovich and Janko [3] introduced a more general concept than that of minimal nonabelian p-groups. A nonabelian p-group is said to be an $\mathcal{A}_t$-group, $t \in \mathbb{N}$, if it has a nonabelian subgroup of index p^{t-1} but all its subgroups of index p^t are abelian. Obviously, an $\mathcal{A}_1$-group is a minimal nonabelian p-group. Given a nonabelian p-group G, there is a $t \in \mathbb{N}$ such that G is an $\mathcal{A}_t$-group. Hence, in a sense, the study of nonabelian p-groups can be regarded as that of $\mathcal{A}_t$-groups for some $t \in \mathbb{N}$. For convenience, an abelian p-group is called an $\mathcal{A}_0$-group. We also use $G \in \mathcal{A}_t$ to denote G is an $\mathcal{A}_t$-group. $\mathcal{A}_t$-groups were classified up to isomorphism for $t \leqslant 3$ in [8, 15, 17].

In this paper, we continue the research about the structure of a nonabelian p-group by imposing hypothesis on its $\mathcal{A}_1$-subgroups. Motivated by [13], our interest is: what can be said about the p-groups all of whose two distinct $\mathcal{A}_1$-subgroups generate an $\mathcal{A}_2$-subgroup? Such p-groups with at lease two distinct $\mathcal{A}_1$-subgroups are called $\mathcal{P}_1$-groups. In addition, we observed that if $G \in \mathcal{A}_2$,

Received July 31, 2021; Accepted November 8, 2021.
2020 *Mathematics Subject Classification.* Primary 20D15.
Key words and phrases. Finite p-groups, minimal nonabelian subgroups, maximal subgroups.
This work was financially supported by NSFC (No. 11771258 & 11971280).

then G has property $\mathcal{P}_2$: $H_1 \cap H_2$ is maximal in both H_1 and H_2 for any two distinct $\mathcal{A}_1$-subgroups H_1 and H_2 of G. The p-groups with property $\mathcal{P}_2$ are called $\mathcal{P}_2$-groups. The p-groups all of whose nonabelian proper subgroups are generated by two elements are called $\mathcal{P}_3$-groups. We will prove that the class of $\mathcal{P}_1$-groups is exactly the class of $\mathcal{A}_2$-groups, and hence [4, Problem 1016] is solved. We also prove that the class of the $\mathcal{P}_2$-groups is a proper subclass of the $\mathcal{P}_3$-groups. Although $\mathcal{P}_3$-groups were classified by Xu et al. in [9], it is not easy to pick out $\mathcal{P}_2$-groups from the list of $\mathcal{P}_3$-groups by using the conditions of $\mathcal{P}_2$-groups. Here we give a self-contained proof to classify $\mathcal{P}_2$-groups. It turns out that

$$\{\mathcal{A}_2\text{-groups}\} = \{\mathcal{P}_1\text{-groups}\} \subset \{\mathcal{P}_2\text{-groups}\} \subset \{\mathcal{P}_3\text{-groups}\}.$$

It should be mentioned that the class of the $\mathcal{A}_2$-groups is also a proper subclass of the finite p-groups classified by Fang and An in [5].

For a finite p-group G, we use $M \lessdot G$ to denote M is a maximal subgroup of G and the nth term of the lower central series of G is denoted by G_n and $G' = G_2$. The other terminology and notations are standard, as in [6].

2. Preliminaries

In this section, we introduce the following lemmas which are used in this paper.

Lemma 2.1 ([9, Lemma 2.2]). *Suppose that G is a finite nonabelian p-group. Then the following conditions are equivalent*:

(1) *G is a minimal nonabelian group.*

(2) *$d(G) = 2$ and $|G'| = p$.*

(3) *$d(G) = 2$ and $\Phi(G) = Z(G)$.*

Lemma 2.2 ([2, Proposition 10.28]). *A nonabelian p-group is generated by its minimal nonabelian subgroups.*

Lemma 2.3 ([9, Lemma 3.1]). *Let G be a nonabelian two-generator p-group with an abelian maximal subgroup. Assume $|G/G'| = p^{m+1}$ and $c(G) = c$. Then*

(1) *$\Phi(G) = G'Z(G)$;*

(2) *$Z(M) = Z(G)$ and $M' = G_3, M_3 = G_4, \dots, M_{c-1} = G_c$ for any nonabelian maximal subgroup M of G;*

(3) *G has the lower central complexion $(m+1, \underbrace{1, 1, \dots, 1}_{c-1})$. Particularly, $|G_c| = p$.*

Lemma 2.4 ([4, Proposition 72.1]). *Let G be a metacyclic p-group. Then G is an $\mathcal{A}_t$-group if and only if $|G'| = p^t$.*

Lemma 2.5 ([2, §1, Exercise 6]). *Let G be a nonabelian p-group. Then the number of abelian subgroups of index p in G is $0, 1$ or $p+1$.*

Lemma 2.6. ([16, Theorem 3.2]) *Let G be a finite p-group. Then the following statements are equivalent*:

(1) *all nonabelian subgroups of G are generated by two elements.*
(2) *all subgroups of class 2 of G are generated by two elements.*
(3) *all $\mathcal{A}_2$-subgroups of G are generated by two elements.*

3. Determining the $\mathcal{P}_1$-groups

Due to the classification of $\mathcal{A}_3$-groups in [17], we found that $\mathcal{P}_1$-groups must be $\mathcal{A}_2$-groups. Although we can prove this by using the classification of $\mathcal{A}_3$-groups, it is a tedious work since $\mathcal{A}_3$-groups have a long list of groups. Here we give a short proof which is independent of the classification of $\mathcal{A}_3$-groups.

Lemma 3.1. *Let G be a finite p-group and $T = \bigcap_{i=1}^{m} M_i$, where $M_i \lessdot G$ and $M_i \neq M_j$ if $i \neq j$. If $m \geqslant 2 + p + \cdots + p^{k-2}$, then $|G : T| \geqslant p^k$.*

Proof. Since M_i is maximal in G, by Correspondence Theorem, M_i/T is maximal in G/T. Obviously, $M_i/T \neq M_j/T$ for $i \neq j$. Thus the number of maximal subgroups of G/T is at least m. Notice that the number of maximal subgroups of G/T is $\frac{p^{d(G/T)}-1}{p-1}$. Since $m \geqslant 2 + p + \cdots + p^{k-2}$, $d(G/T) \geqslant k$. Obviously, $\Phi(G) \leq T$. It follows that G/T is elementary abelian. Hence $|G : T| \geqslant p^k$. □

Corollary 3.2. *Let G be an $\mathcal{A}_t$-group, where $t \geqslant 2$. Then $\Phi(G)$ is the intersection of nonabelian maximal subgroups. In particular, the Frattini subgroup of $\mathcal{A}_2$-group is the intersection of all its $\mathcal{A}_1$-subgroups.*

Proof. Let T be the intersection of all nonabelian maximal subgroups of G. Then $\Phi(G) \leq T$. Following, we only need to show that $|G : T| \geqslant |G : \Phi(G)| = p^{d(G)}$.

If $d(G) = 2$, then the number of maximal subgroups of G is $1 + p$. Since $t \geqslant 2$, the number of abelian maximal subgroups of G is not equal $1 + p$. Thus the number of nonabelian maximal subgroups of G is at least p by Lemma 2.5. Let M_1 and M_2 be two distinct nonabelian maximal subgroups of G. Then $|G : T| \geqslant |G : M_1 \cap M_2| = p^2$.

If $d(G) \geqslant 3$, then $p^{d(G)-1} \geqslant 2 + p$. Let m be the number of nonabelian maximal subgroups of G. By Lemma 2.5, $m \geqslant p^2 + \cdots + p^{d(G)-1}$. Now we have

$$m \geqslant p^2 + \cdots + p^{d(G)-2} + p^{d(G)-1} \geqslant 2 + p + p^2 + \cdots + p^{d(G)-2}.$$

By Lemma 3.1, we get $|G : T| \geqslant p^{d(G)}$. □

Theorem 3.3. *Let G be a finite p-group. If G has at least two distinct $\mathcal{A}_1$-subgroups, then G is a $\mathcal{P}_1$-group if and only if G is an $\mathcal{A}_2$-group.*

Proof. ($\Leftarrow$) If G is an $\mathcal{A}_2$-group, then all $\mathcal{A}_1$-subgroups of G are of index p. It follows that any two distinct $\mathcal{A}_1$-subgroups generate an $\mathcal{A}_2$-subgroup. Thus G is a $\mathcal{P}_1$-group.

($\Rightarrow$) Let G be a counterexample of minimal order. Then G is an $\mathcal{A}_t$-group, where $t \geqslant 3$. Thus G has an $\mathcal{A}_3$-subgroup H. Then H is also a counterexample. It follows by the minimality of $|G|$ that $H = G$. So we may assume G is both an $\mathcal{A}_3$-group and a $\mathcal{P}_1$-group.

Let M be an $\mathcal{A}_2$-subgroup of G and H an $\mathcal{A}_1$-subgroup of M. Notice that G is an $\mathcal{A}_3$-group. We have $H \lessdot M \lessdot G$. By Lemma 2.2, there exists $\mathcal{A}_1$-subgroup A of G such that $A \nleq M$. It follows that $G = AM$. Since G is a $\mathcal{P}_1$-group, $\langle H, A\rangle$ is an $\mathcal{A}_2$-subgroup. It follows that $H \lessdot \langle H, A\rangle$, $A \lessdot \langle H, A\rangle$ and $\langle H, A\rangle \lessdot G$. Thus $|H| = |A|$ and $\langle H, A\rangle = HA$. Now we have

$$\frac{|A|}{|A \cap H|} = \frac{|HA|}{|H|} = p.$$

Notice that $H \lessdot M$ and $A \nleq M$. We have $A \cap H \leq A \cap M < A$. Thus $A \cap M = A \cap H \lessdot A$. By the arbitrariness of H and Corollary 3.2, we get $A \cap M \leq \Phi(M)$. Now we have

$$p^2 \leqslant |M : \Phi(M)| \leqslant |M : A \cap M| = |AM : A| = |G : A| = p^2.$$

We get $A \cap M = \Phi(M)$ and $|M : \Phi(M)| = p^2$. So $d(M) = 2$. That is, all $\mathcal{A}_2$-subgroups of G are generated by two elements. It follows from Lemma 2.6 that $d(G) = 2$. Thus we have $\Phi(G) = \langle H, A\rangle \cap M = H$. Notice that the arbitrariness of H. By Lemma 2.2, we get that $M = \Phi(G)$. This is a contradiction. $\square$

A direct result of Theorem 3.3 is:

Corollary 3.4. *An $\mathcal{A}_3$-group can be generated by its two distinct $\mathcal{A}_1$-subgroups.*

Remark 3.5. For $t \geqslant 4$, there exists $\mathcal{A}_t$-group can not be generated by its two $\mathcal{A}_1$-subgroups. For example, $G = H \times K$, where H is an $\mathcal{A}_1$-group and K is an elementary abelian group of order p^{t-1}. In this case, G is an $\mathcal{A}_t$-group by [1, Corollary 2.4]. Obviously, $d(G) = t + 1 \geqslant 5$. Thus G can not be generated by its two $\mathcal{A}_1$-subgroups.

4. Determining the $\mathcal{P}_2$-groups

In this section, we establish a criterion for a nonabelian p-group to be a $\mathcal{P}_2$-group. Based on the criterion, the $\mathcal{P}_2$-groups are classified. It turns out that the class of the $\mathcal{P}_2$-groups is a proper subclass of the $\mathcal{P}_3$-groups. For convenience, we list the results of the classification of the $\mathcal{P}_3$-groups, which were obtained by Xu et al. in [9]. Following Xu et al. [9],

$\mathcal{B}_p$ denotes the class of p-groups whose non-abelian proper subgroups are two-generator,

$\mathcal{B}'_p = \{G \in \mathcal{B}_p \mid G$ is neither abelian nor minimal non-abelian$\}$,

$\mathcal{D}_p = \{G \in \mathcal{B}'_p \mid G$ has an abelian maximal subgroup$\}$,

$\mathcal{M}_p = \{G \in \mathcal{B}'_p \mid G$ has no abelian maximal subgroup$\}$,

$\mathcal{D}_p(2) = \{G \in \mathcal{D}_p \mid d(G) = 2\}$ and $\mathcal{D}_p(3) = \{G \in \mathcal{D}_p \mid d(G) = 3\}$,

$\mathcal{D}'_p(2) = \{G \in \mathcal{D}_p(2) \mid G$ is not of maximal class$\}$ and

$\mathcal{M}'_p = \{G \in \mathcal{M}_p \mid G$ is neither metacyclic nor 3-group of maximal class$\}$.

In terms of $\mathcal{A}_t$-groups and notations mentioned above, the [9, Main Theorem] can be stated as follows.

Theorem 4.1. *Assume G is a $\mathcal{P}_3$-group. Then G is one of the following groups*:

(1) *$\mathcal{A}_t$-groups, where $t \leqslant 2$*;
(2) *metacyclic groups*;
(3) *p-groups of maximal class with an abelian maximal subgroup*;
(4) *3-groups of maximal class*;
(5) *$\mathcal{D}'_p(2)$-groups with $p \geqslant 3$*;
(6) *$\mathcal{M}'_3$-groups with a unique minimal non-abelian maximal subgroup*;
(7) *$\mathcal{M}'_p$-groups having no minimal non-abelian maximal subgroup, where $p \geqslant 3$.*

Remark 4.2. From the argument in [9] or a simple check, it is not difficult to get the converse of Theorem 4.1 is also true.

Lemma 4.3. *Let G be an $\mathcal{A}_t$-group with $t \geqslant 1$. Then the following statements are equivalent*:

(1) *the index of all $\mathcal{A}_1$-subgroups of G are equal.*
(2) *the index of all $\mathcal{A}_k$-subgroups of G are p^{t-k} for any $k \in \{1, 2, \dots, t\}$.*
(3) *all nonabelian subgroups of index p^{t-k} are $\mathcal{A}_k$-subgroups for any $k \in \{1, 2, \dots, t\}$.*

Proof. (3) $\Rightarrow$ (1): It is obvious.

(1) $\Rightarrow$ (2): Let K be an $\mathcal{A}_k$-subgroup of G, where $k \in \{1, 2, \dots, t\}$. Then K has an $\mathcal{A}_1$-subgroup T of index p^{k-1}. Since G is an $\mathcal{A}_t$-group, G has $\mathcal{A}_1$-subgroups of index p^{t-1}. It follows from (1) that $|G : T| = p^{t-1}$. So

$$|G : K| = \frac{|G : T|}{|K : T|} = \frac{p^{t-1}}{p^{k-1}} = p^{t-k}.$$

(2) $\Rightarrow$ (3): Let K_1 be a nonabelian subgroup of index p^{t-k}, where $k \in \{1, 2, \dots, t\}$. Assume K_1 is an $\mathcal{A}_s$-group. Then K_1 has an $\mathcal{A}_1$-subgroup H of index p^{s-1}. It follows that

$$|G : H| = |G : K_1||K_1 : H| = p^{t-k}p^{s-1} = p^{t+s-k-1}.$$

Since G is an $\mathcal{A}_t$-group, G has $\mathcal{A}_1$-subgroups of index p^{t-1}. By (2), we get $|G : H| = p^{t-1}$. It follows that $p^{t+s-k-1} = p^{t-1}$ and so $s = k$. Thus K_1 is an $\mathcal{A}_k$-group. □

Following the notation of [10], the intersection of all $\mathcal{A}_1$-subgroups of a p-group G is denoted by $\mathrm{I}_{\mathcal{A}_1}(G)$. We use $\mathcal{A}_k(G)$ to denote the set consisting of the $\mathcal{A}_k$-subgroups of a p-group G.

Theorem 4.4. *Let G be an $\mathcal{A}_t$-group with $t \geqslant 3$. Then the following statements are equivalent*:

(1) *G is a $\mathcal{P}_2$-group.*
(2) $\mathrm{I}_{\mathcal{A}_1}(G) \lessdot K$ *for any $\mathcal{A}_1$-subgroup K of G.*
(3) *The orders of all $\mathcal{A}_1$-subgroups of G are equal, and all $\mathcal{A}_2$-subgroups of G are generated by two elements and have a same Frattini subgroup.*

Proof. (2) $\Rightarrow$ (1): It is obvious.

(1) $\Rightarrow$ (3): Let H_1 and H_2 be two distinct $\mathcal{A}_1$-subgroups of G. Since G is $\mathcal{P}_2$-group, $H_1 \cap H_2 \lessdot H_i$ for $i = 1, 2$. Thus $|H_1| = |H_2|$. By the arbitrariness of H_i, we get the orders of all $\mathcal{A}_1$-subgroups of G are equal.

Let $M \in \mathcal{A}_2(G)$ and $H \in \mathcal{A}_1(M)$. Then $|M : H| = p$. Since G is an $\mathcal{A}_t$-group with $t \geqslant 3$, there exists $A \in \mathcal{A}_1(G) \backslash \mathcal{A}_1(M)$ by Lemma 2.2. Thus $A \cap H \leq A \cap M < A$. Since G is a $\mathcal{P}_2$-group, $A \cap H \lessdot A$. It follows that $A \cap H = A \cap M \lessdot A$. Notice that $|H| = |A|$. We get $A \cap M = A \cap H \lessdot H$. By the arbitrariness of H, we get $A \cap M \leq \mathrm{I}_{\mathcal{A}_1}(M)$. Notice that $M \in \mathcal{A}_2$. By Lemma 3.2, we get $\mathrm{I}_{\mathcal{A}_1}(M) = \Phi(M)$. Now, we have

$$p^2 \leqslant |M : \Phi(M)| = |M : \mathrm{I}_{\mathcal{A}_1}(M)| \leqslant |M : A \cap M| = |M : H| \cdot |H : A \cap M| = p^2.$$

It follows that $A \cap M = \Phi(M)$ and $d(M) = 2$. Particularly, $\Phi(M) \leq A$. Notice that $\Phi(M) \leq H$. Then $\Phi(M) \leq \mathrm{I}_{\mathcal{A}_1}(G)$. It follows that

$$\mathrm{I}_{\mathcal{A}_1}(G) \leq \mathrm{I}_{\mathcal{A}_1}(M) = \Phi(M) \leq \mathrm{I}_{\mathcal{A}_1}(G).$$

Thus $\Phi(M) = \mathrm{I}_{\mathcal{A}_1}(G)$. By the arbitrariness of M, we get all $\mathcal{A}_2$-subgroups of G have a same Frattini subgroup.

(3) $\Rightarrow$ (2): Let K be an $\mathcal{A}_1$-subgroup of G and M a subgroup of G such that $K \lessdot M$. Since the orders of all $\mathcal{A}_1$-subgroups of G are equal, M is an $\mathcal{A}_2$-group by Lemma 4.3. Since all $\mathcal{A}_2$-subgroups of G have a same Frattini subgroup, we may let T be the same Frattini subgroup of $\mathcal{A}_2$-subgroups of G. It follows that $T = \Phi(M)$ and so $T \leq K$. By the arbitrariness of K, we get $T \leq \mathrm{I}_{\mathcal{A}_1}(G)$. Now, we have

$$\Phi(M) = T \leq \mathrm{I}_{\mathcal{A}_1}(G) < K \lessdot M.$$

Since M is an $\mathcal{A}_2$-group, by the hypothesis, $d(M) = 2$ and so $|M : \Phi(M)| = p^2$. It follows that $\Phi(M) = \mathrm{I}_{\mathcal{A}_1}(G)$ and $\mathrm{I}_{\mathcal{A}_1}(G) \lessdot K$. $\square$

Combining with Lemma 2.6 and Theorem 4.4, we have:

Corollary 4.5. *Assume $G \in \mathcal{A}_t$, where $t \geqslant 3$. If $G \in \mathcal{P}_2$, then $G \in \mathcal{P}_3$.*

In following, we will classify the $\mathcal{P}_2$-groups. Since $\mathcal{A}_0$-, $\mathcal{A}_1$- and $\mathcal{A}_2$-groups are $\mathcal{P}_2$-groups, we assume G is an $\mathcal{A}_t$-group with $t \geqslant 3$ in Theorem 4.6 and Theorem 4.9.

Theorem 4.6. *Let G be an $\mathcal{A}_t$-group with an abelian subgroup of index p, where $t \geqslant 3$. Then $G \in \mathcal{P}_2$ if and only if $G \in \mathcal{P}_3$.*

Proof. ($\Rightarrow$) The conclusion follows by Corollary 4.5.

($\Leftarrow$) Since $G \in \mathcal{A}_t$ with $t \geqslant 3$, all $\mathcal{A}_2$-subgroups of G are proper subgroups. Thus all $\mathcal{A}_2$-subgroups of G are generated by two elements. It follows by Lemma 2.6 that all nonabelian subgroups of G are generated by two elements.

By Theorem 4.4, it is enough to show that the orders of all $\mathcal{A}_1$-subgroups of G are equal and all $\mathcal{A}_2$-subgroups have a same Frattini subgroup.

Let K be a nonabelian subgroup of index p^k of G. We assert that $K' = G_{k+1}$. Take a maximal subgroup M of G such that $K \leq M$. Then $|M : K| = p^{k-1}$. Assume $c(G) = c$. Then, by Lemma 2.3(2), we get

$$M_2 = G_3, M_3 = G_4, \ldots, M_{c-1} = G_c.$$

By induction on k, and by using Lemma 2.3(2), we get $K' = M_k = G_{k+1}$.

Let A be an $\mathcal{A}_1$-subgroup of G. Then $A' = G_i$ for some $i \leq c$. By Lemma 2.1, we have $|A'| = p$. By Lemma 2.3(3), we get $|G_c| = p$. It follows that $A' = G_c$ and so $|G : A| = p^{c-1}$. Thus the order of all $\mathcal{A}_1$-subgroups of G are equal.

Let H_1 and H_2 be two distinct $\mathcal{A}_2$-subgroups of G. Then $|G : H_1| = |G : H_2| = p^{c-2}$ by Lemma 4.3. It follows that $H_1' = H_2' = G_{c-1}$. By Lemma 2.3(2), we get $Z(H_1) = Z(H_2) = Z(G)$. It follows by Lemma 2.3(1) that

$$\Phi(H_1) = H_1' Z(H_1) = H_1' Z(G) = H_2' Z(G) = \Phi(H_2).$$

That is, all $\mathcal{A}_2$-subgroups have a same Frattini subgroup. □

Remark 4.7. By using Theorem 4.6 to check the groups in Theorem 4.1, we have $\mathcal{P}_2$-groups with an abelian subgroup of index p are the groups (3) and (5).

Lemma 4.8 ([9, Lemma 5.3 and Theorem 5.4])**.** *Let G be one of the groups* (7) *of Theorem* 4.1*, i.e., G is a $\mathcal{M}_p'$-group having no minimal non-abelian maximal subgroup, where $p \geqslant 3$. Then*

(1) $|G| = p^6$ *and* $|G_4| = p$;
(2) $K \in \mathcal{D}_p'(2)$, $\Phi(K) = G_3$ *and* $K_3 = G_4$ *for any maximal subgroup K of G.*

Theorem 4.9. *Let G be an $\mathcal{A}_t$-group without any abelian subgroup of index p, where $t \geqslant 3$. Then G is a $\mathcal{P}_2$-group if and only if G is one of the groups* (7) *in Theorem* 4.1.

Proof. Assume that G is a $\mathcal{P}_2$-group. Then, by Corollary 4.5, $G \in \mathcal{P}_3$. Thus G is one of the groups listed in Theorem 4.1. If G has an $\mathcal{A}_1$-subgroup of index p, by Theorem 4.4, we get all $\mathcal{A}_1$-subgroups are of index p. Thus G is an $\mathcal{A}_2$-group. This contradicts $t \geqslant 3$. Thus G has no $\mathcal{A}_1$-subgroup of index p. By hypothesis, G has no abelian subgroup of index p. By a simple check to those groups in Theorem 4.1, we get G is a metacyclic p-group or one of the groups (7) in Theorem 4.1.

Assume G is a metacyclic p-group. Let $G = \langle a, b \rangle$ and $G' < \langle a \rangle$. Then $M_1 = \langle a^p, b \rangle$ and $M_2 = \langle a, b^p \rangle$ are two distinct maximal subgroups of G. Since G has no abelian subgroup of index p, M_1 and M_2 are nonabeian maximal subgroups of G. Since G is a $\mathcal{P}_2$-group, the orders of all $\mathcal{A}_1$-subgroups of G are equal by Theorem 4.4. It follows by Lemma 4.3 that M_1 and M_2 are

$\mathcal{A}_{t-1}$-subgroups of G. From Lemma 2.4 we get $|M_1'| = |M_2'| = p^{t-1}$ and so $o([a^p, b]) = o([a, b^p]) = p^{t-1}$.

Let $H_1 = \langle a^{p^{t-2}}, b\rangle$ and $H_2 = \langle a^{p^{t-3}}, b^p\rangle$. Notice that $t \geqslant 3$. We get

$$[a^{p^{t-2}}, b] = [a^p, b]^{p^{t-3}} \text{ and } [a^{p^{t-3}}, b^p] = [a, b^p]^{p^{t-3}}.$$

It follows that

$$|H_1'| = o([a^{p^{t-2}}, b]) = p^2 \text{ and } |H_2'| = o([a^{p^{t-3}}, b^p]) = p^2.$$

From Lemma 2.4 we get H_1 and H_2 are $\mathcal{A}_2$-subgroups of G. Now, it is obvious that $b^p \in \Phi(H_1)$ and $b^p \notin \Phi(H_2)$. This implies that $\Phi(H_1) \neq \Phi(H_2)$, which contradicts G is a $\mathcal{P}_2$-group by Theorem 4.4. Hence G is one of the groups (7) in Theorem 4.1.

Conversely, if G is one of the groups (7) in Theorem 4.1, by Lemma 4.8, $K \in \mathcal{D}_p'(2)$ and $|K_3| = |G_4| = p$ for any maximal subgroup K of G. Let H be a nonabelian subgroup of index p of K. Since $K \in \mathcal{D}_p'(2)$, $H' = K_3$ by Lemma 2.3(2). It follows that $|H'| = p$. Thus, by Lemma 2.1, H is an $\mathcal{A}_1$-subgroup. By the arbitrariness of H, we get K is an $\mathcal{A}_2$-subgroup. By Lemma 4.8(2), $\Phi(K) = G_3$. It follows by Theorem 4.4 that G is a $\mathcal{P}_2$-group. $\square$

Notice that $\mathcal{A}_t$-groups with $t \leqslant 2$ are $\mathcal{P}_2$-groups. Now, combining Theorem 4.9 and Remark 4.7, we have:

Theorem 4.10. *Let G be a finite p-group. Then G is a $\mathcal{P}_2$-group if and only if G is one of the groups* (1), (3), (5) and (7) *listed in Theorem* 4.1.

An $\mathcal{A}_t$-group G satisfies a *chain condition* if every $\mathcal{A}_i$-subgroup of G is contained in an $\mathcal{A}_{i+1}$-subgroup for all $i \in \{0, 1, 2, \ldots, t-1\}$. The concept was introduced by Zhang and Qu in [14]. Zhang in [11] proved that for $t \geqslant 3$, an $\mathcal{A}_t$-group G satisfies a *chain condition* if and only if G is an ordinary metcyclic p-group. We call an $\mathcal{A}_t$-group G satisfies a *weakly chain condition* if the included relations hold for $i \in \{1, 2, \ldots, t-1\}$. It is easy to see that G satisfies a weakly chain condition is equivalent to (3) in Lemma 4.3. In other words, G satisfies a weakly chain condition is equivalent to the orders of all $\mathcal{A}_1$-subgroups of G are equal. By Theorem 4.4(3) we get $\mathcal{P}_2$-groups satisfy (1) in Lemma 4.3, That is, $\mathcal{P}_2$-groups satisfies a weakly chain condition. Conversely, it is not true in general. We propose the following.

Problem. *Classify the p-groups satisfying a weak chain condition. Equivalently, classify the p-groups all of whose $\mathcal{A}_1$-subgroups have the same order.*

Remark 4.11. If the p-groups all of whose $\mathcal{A}_1$-subgroups are of order p^3, then for $p = 2$, such p-groups were classified by Janko in [7]. For p odd prime, the p-groups all of whose $\mathcal{A}_1$-subgroups are nonmetacyclic of order p^3 were classified by Zhang in [12].

Acknowledgements. We owe our sincere gratitude to the referee. She/He read our paper very carefully and put forward a lot of suggestions. The author also thanks Professor Qinhai Zhang for his value suggestions. Their suggestions are quite valuable and helpful for improving our paper.

References

[1] L. An, L. Li, H. Qu, and Q. Zhang, *Finite p-groups with a minimal non-abelian subgroup of index p (II)*, Sci. China Math. **57** (2014), no. 4, 737–753. https://doi.org/10.1007/s11425-013-4735-5

[2] Y. Berkovich, *Groups of prime power order. Vol. 1*, De Gruyter Expositions in Mathematics, 46, Walter de Gruyter GmbH & Co. KG, Berlin, 2008. https://doi.org/10.1515/9783110208238.512

[3] Y. Berkovich and Z. Janko, *Structure of finite p-groups with given subgroups*, in Ischia group theory 2004, 13–93, Contemp. Math., 402, Israel Math. Conf. Proc, Amer. Math. Soc., Providence, RI, 2006. https://doi.org/10.1090/conm/402/07570

[4] Y. Berkovich and Z. Janko, *Groups of prime power order, Vol. 2*, Walter de Gruyter. Berlin, New York, 2008.

[5] X. Fang and L. An, *A classification of finite metahamiltonian p-groups*, Commun. Math. Stat. **9** (2021), no. 2, 239–260. https://doi.org/10.1007/s40304-020-00229-0

[6] B. Huppert, *Endliche Gruppen. I*, Die Grundlehren der mathematischen Wissenschaften, Band 134, Springer-Verlag, Berlin, 1967.

[7] Z. Janko, *On finite nonabelian* 2*-groups all of whose minimal nonabelian subgroups are of exponent* 4, J. Algebra **315** (2007), no. 2, 801–808. https://doi.org/10.1016/j.jalgebra.2007.02.010

[8] L. Rédei, *Das "schiefe Produkt" in der Gruppentheorie mit Anwendung auf die endlichen nichtkommutativen Gruppen mit lauter kommutativen echten Untergruppen und die Ordnungszahlen, zu denen nur kommutative Gruppen gehören*, Comment. Math. Helv. **20** (1947), 225–264. https://doi.org/10.1007/BF02568131

[9] M. Xu, L. An, and Q. Zhang, *Finite p-groups all of whose non-abelian proper subgroups are generated by two elements*, J. Algebra **319** (2008), no. 9, 3603–3620. https://doi.org/10.1016/j.jalgebra.2008.01.045

[10] L. Zhang, *The intersection of nonabelian subgroups of finite p-groups*, J. Algebra Appl. **16** (2017), no. 1, 1750020, 9 pp. https://doi.org/10.1142/S0219498817500207

[11] Q. Zhang, *Finite p-groups whose subgroups of given order are isomorphic and minimal non-abelian*, Algebra Colloq. **26** (2019), no. 1, 1–8. https://doi.org/10.1142/S1005386719000026

[12] Q. Zhang, *Finite p-groups all of whose minimal nonabelian subgroups are nonmetacyclic of order p^3*, Acta Math. Sin. (Engl. Ser.) **35** (2019), no. 7, 1179–1189. https://doi.org/10.1007/s10114-019-7308-x

[13] Q. Zhang, L. J. Deng, and M. Y. Xu, *Finite p-groups in which any two noncommutative elements generate a subgroup of order p^3*, Acta Math. Sci. Ser. A (Chin. Ed.) **29** (2009), no. 3, 737–740.

[14] L. Zhang and H. Qu, *$\mathcal{A}_t$-groups satisfying a chain condition*, J. Algebra Appl. **13** (2014), no. 4, 1350137, 5 pp. https://doi.org/10.1142/S0219498813501375

[15] Q. Zhang, X. Sun, L. An, and M. Xu, *Finite p-groups all of whose subgroups of index p^2 are abelian*, Algebra Colloq. **15** (2008), no. 1, 167–180. https://doi.org/10.1142/S1005386708000163

[16] L. Zhang and J. Zhang, *Finite p-groups all of whose $\mathcal{A}_2$-subgroups are generated by two elements*, J. Group Theory **24** (2021), no. 1, 177–193. https://doi.org/10.1515/jgth-2019-0159

[17] Q. Zhang, L. Zhao, M. Li, and Y. Shen, *Finite p-groups all of whose subgroups of index p^3 are abelian*, Commun. Math. Stat. **3** (2015), no. 1, 69–162. https://doi.org/10.1007/s40304-015-0053-2

Junqiang Zhang
Department of Mathematics
Shanxi Normal University
Linfen Shanxi 041004, P. R. China
Email address: junqiangchang@163.com