

UNIQUENESS OF MEROMORPHIC SOLUTIONS OF A CERTAIN TYPE OF DIFFERENCE EQUATIONS

JUN-FAN CHEN AND SHU-QING LIN

ABSTRACT. In this paper, we study the uniqueness of two finite order transcendental meromorphic solutions $f(z)$ and $g(z)$ of the following complex difference equation

$$A_1(z)f(z+1) + A_0(z)f(z) = F(z)e^{\alpha(z)}$$

when they share $0, \infty$ CM, where $A_1(z), A_0(z), F(z)$ are non-zero polynomials, $\alpha(z)$ is a polynomial. Our result generalizes and complements some known results given recently by Cui and Chen, Li and Chen. Examples for the precision of our result are also supplied.

1. Introduction

Let \mathbb{C} denote the complex plane. In this paper, we assume that the reader is familiar with Nevanlinna theory and standard notations (see [9, 11]), such as

$$T(r, f), N(r, f), m(r, f), \bar{N}(r, f), \bar{N}(r, \frac{1}{f}), \dots$$

We denote by $S(r, f)$ any quantity satisfying

$$S(r, f) = o(T(r, f)), \quad r \rightarrow \infty,$$

outside possibly an exceptional set of finite logarithmic measure.

Let $f(z)$ and $g(z)$ be meromorphic functions. We say that $f(z)$ and $g(z)$ share a CM, provided that $f(z) - a$ and $g(z) - a$ have the same zeros counting multiplicities. And we say that $f(z)$ and $g(z)$ share ∞ CM, provided that they have the same poles counting multiplicities.

The uniqueness theory of meromorphic functions is an important part of Nevanlinna theory. The famous Nevanlinna 4 CM theorem will be given in detail below.

Received July 3, 2021; Revised September 24, 2021; Accepted November 4, 2021.

2010 *Mathematics Subject Classification.* 39B32, 30D35.

Key words and phrases. Difference equation, transcendental meromorphic solution, Nevanlinna theory, finite order.

Project supported by the Natural Science Foundation of Fujian Province, China (Grant No. 2021J01651).

Theorem 1 (see [15]). *Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions and let a_1, a_2, a_3, a_4 be four distinct values in the extended complex plane. Furthermore, suppose that $f(z)$ and $g(z)$ share a_1, a_2, a_3, a_4 CM. If $f(z) \not\equiv g(z)$, then $f(z) = T(g(z))$, where T is a Möbius transformation.*

Later, Yang and Yi [19] supplemented and perfected the uniqueness theory of meromorphic functions. One major problem in the study of complex difference equations has so far been the lack of efficient tools. Fortunately, till the 1970s and 1980s, Bank and Kaufman [1], Shimomura [17] and Yanagihara [18] initially obtained the existence of meromorphic solutions of certain difference equations in the complex plane. Recently, many scholars combined the uniqueness with the differences and the solutions of difference equations, and obtained lots of results [2–7, 10, 12–14, 16]. Specially, in 2013, Chen and Shon [3] studied the following non-homogeneous linear difference equation

$$(1.1) \quad A_n(z)f(z+n) + \cdots + A_1(z)f(z+1) + A_0(z)f(z) = F(z),$$

and proved the following theorem.

Theorem 2 (see [3]). *Let $A_0(z), \dots, A_n(z), F(z)$ be polynomials such that*

$$F(z)A_0(z)A_n(z) \not\equiv 0, \quad A_0 + \cdots + A_n \not\equiv 0.$$

Then every finite order transcendental meromorphic solution $f(z)$ of the equation (1.1) satisfies $\lambda(f(z)) = \sigma(f(z)) \geq 1$. Here, the order $\sigma(f(z))$ of a meromorphic solution $f(z)$ is defined to be

$$\sigma(f(z)) = \limsup_{r \rightarrow \infty} \frac{\log^+ T(r, f(z))}{\log r},$$

and its exponent of convergence of zeros is defined by

$$\lambda(f(z)) = \limsup_{r \rightarrow \infty} \frac{\log^+ N\left(r, \frac{1}{f(z)}\right)}{\log r}.$$

In 2016, Cui and Chen [6] considered the uniqueness of meromorphic solutions sharing three values with a meromorphic function to the homogeneous linear difference equation

$$(1.2) \quad A_1(z)f(z+1) + A_0(z)f(z) = 0$$

and proved the following theorem.

Theorem 3 (see [6]). *Let $f(z)$ be a transcendental meromorphic solution of (1.2) and assume that the order of $f(z)$ is finite. Suppose that $A_1(z)$ and $A_0(z)$ are non-zero polynomials such that $A_1(z) + A_0(z) \not\equiv 0$. If a meromorphic function $g(z)$ and $f(z)$ share $0, 1, \infty$ CM, then either $f(z) \equiv g(z)$ or $f(z)g(z) \equiv 1$.*

In 2017, Cui and Chen [7] extended the complex difference equation in Theorem 3 as

$$(1.3) \quad A_1(z)f(z+1) + A_0(z)f(z) = F(z)$$

and proved the following theorem.

Theorem 4 (see [7]). *Let $f(z)$ be a transcendental meromorphic solution of (1.3) and assume that the order of $f(z)$ is finite. Suppose that $A_1(z)$, $A_0(z)$, $F(z)$ are non-zero polynomials such that $A_1(z) + A_0(z) \not\equiv 0$. If a meromorphic function $g(z)$ and $f(z)$ share $0, 1, \infty$ CM, then one of the following cases holds:*

- (i) $f(z) \equiv g(z)$; (ii) $f(z) + g(z) = f(z)g(z)$;
- (iii) there exist a polynomial $\beta(z) = b_1z + b_0$ and a constant a_0 satisfying $e^{a_0} \neq e^{b_0}$ such that

$$f(z) = \frac{1 - e^{\beta(z)}}{e^{\beta(z)}(e^{a_0 - b_0} - 1)}, \quad g(z) = \frac{1 - e^{\beta(z)}}{1 - e^{b_0 - a_0}},$$

where $b_1 \neq 0$ and b_0 are constants.

In 2019, Li and Chen [12] considered the uniqueness of two meromorphic solutions of non-homogeneous linear difference equation

$$(1.4) \quad B_1(z)f(z+1) + B_2(z)f(z) = B_3(z)$$

when they share two values, and proved the following theorem.

Theorem 5 (see [12]). *Let $f(z)$ and $g(z)$ be two finite order transcendental meromorphic solutions of (1.4). Furthermore, suppose that $B_1(z) \not\equiv 0$, $B_3(z) \not\equiv 0$, $B_2(z)$ are rational functions. If $f(z)$ and $g(z)$ share $0, \infty$ CM, then either $f(z) \equiv g(z)$ or*

$$f(z) = \frac{B_3(z)}{2B_2(z)}(e^{b_1z+b_0} + 1)$$

and

$$g(z) = \frac{B_3(z)}{2B_2(z)}(e^{-b_1z-b_0} + 1),$$

where b_1 and b_0 are constants such that $e^{-b_1} = e^{b_1} = -1$, and the coefficients of (1.4) satisfy

$$B_1(z)B_3(z+1) \equiv B_3(z)B_2(z+1).$$

Here in this paper, if we generalize the equation (1.3) to the following more general form

$$(1.5) \quad A_1(z)f(z+1) + A_0(z)f(z) = F(z)e^{\alpha(z)},$$

then how can we guarantee the meromorphic solutions of the equation (1.5) are uniquely determined by their zeros and poles? Next we study this problem and prove the following main result.

Theorem 6. *Let $f(z)$ and $g(z)$ be two finite order transcendental meromorphic solutions of the equation (1.5). Furthermore, suppose that $A_1(z)$, $A_0(z)$, $F(z)$ are non-zero polynomials, $\alpha(z)$ is a polynomial. If $f(z)$ and $g(z)$ share $0, \infty$ CM, then one of the following cases holds:*

- (i) $f(z) \equiv g(z)$;
- (ii)

$$f(z) = \frac{c_1 F(z)}{2A_0(z)} (e^{b_1 z + b_0} + 1)$$

and

$$g(z) = \frac{c_1 F(z)}{2A_0(z)} (e^{-b_1 z - b_0} + 1),$$

where $e^{\alpha(z)} = c_1 (\neq 0)$, $e^{-b_1} = e^{b_1} = -1$, $b_1 = (2k + 1)\pi i$, k is an integer and the coefficients of (1.5) satisfy

$$A_1(z)F(z + 1) \equiv F(z)A_0(z + 1);$$

- (iii)

$$f(z) = \frac{F(z)e^{a_1 z + a_0} (e^{b_1 z + b_0} - e^{-b_1})}{A_0(z)(1 - e^{-b_1})}$$

and

$$g(z) = \frac{F(z)e^{a_1 z + a_0} (1 - e^{-b_1 z - b_1 - b_0})}{A_0(z)(1 - e^{-b_1})},$$

where $(1 - e^{-b_1})^{-1} \neq 0$, $a_1 \neq 0$, $b_1 \neq 0$, a_0, b_0 are constants and the coefficients of (1.5) satisfy

$$A_1(z)F(z + 1) \equiv \frac{e^{-b_1} (e^{-b_1 z - b_0} - 1)}{e^{a_1} (1 - e^{-b_1 z - 2b_1 - b_0})} F(z)A_0(z + 1).$$

By Theorem 6, we can also give the following corollaries.

Corollary 1. *Let $f(z)$ and $g(z)$ be two finite order transcendental meromorphic solutions of the equation (1.5), where $A_1(z)$, $A_0(z)$, $F(z)$ are non-zero polynomials such that $A_1(z)F(z + 1) \not\equiv qF(z)A_0(z + 1)$, where $q = 1$ or $q = \frac{e^{-b_1} (e^{-b_1 z - b_0} - 1)}{e^{a_1} (1 - e^{-b_1 z - 2b_1 - b_0})}$, a_1 and b_1 are non-zero constants, b_0 is a constant. If $f(z)$ and $g(z)$ share $0, \infty$ CM, then $f(z) \equiv g(z)$.*

Corollary 2. *Let $f(z)$ and $g(z)$ be two finite order transcendental meromorphic solutions of the equation (1.5), where*

$$F(z)e^{\alpha(z)} \not\equiv A_1(z) + A_0(z),$$

$$A_1(z)[c_1 F(z + 1) - A_1(z + 1)] \not\equiv [c_1 F(z) - A_0(z)]A_0(z + 1)$$

and

$$\frac{A_1(z)[F(z + 1)e^{a_1 z + a_1 + a_0} - A_1(z + 1) - A_0(z + 1)]}{A_0(z + 1)[F(z)e^{a_1 z + a_0} - A_1(z) - A_0(z)]} \not\equiv \frac{e^{-b_1 z - b_1 - b_0} - e^{-b_1}}{1 - e^{-b_1 z - 2b_1 - b_0}},$$

where a_1, b_1, c_1 are non-zero constants, a_0 and b_0 are constants. If $f(z)$ and $g(z)$ share $1, \infty$ CM, then $f(z) \equiv g(z)$.

Remark 1. Corollary 1 and Corollary 2 follow from Theorem 6 immediately.

Example 1. The function $f(z) = \frac{ze^{\pi iz} + z}{2}$ and another function $g(z) = \frac{ze^{-\pi iz} + z}{2}$ solve the equation

$$zf(z + 1) + (z + 1)f(z) = z^2 + z.$$

Here $f(z)$ and $g(z)$ share $0, \infty$ CM, $e^{-\pi i} = e^{\pi i} = -1$. Without loss of generality, we may suppose that $e^{\alpha(z)} \equiv 1, F(z) = z^2 + z$. Then the coefficients of (1.5) satisfy $A_1(z) = z, A_0(z) = z + 1$ and $A_1(z)F(z + 1) \equiv F(z)A_0(z + 1) = z^3 + 3z^2 + 2z$. The above statement shows that Case (ii) in Theorem 6 certainly exists.

Example 2. The function $f(z) = \frac{e^{3\pi iz} - e^{\pi i(2z-1)}}{2}$ and another function $g(z) = \frac{e^{2\pi iz} - e^{\pi i(z-1)}}{2}$ solve the equation

$$f(z + 1) + f(z) = e^{2\pi iz}.$$

Here $f(z)$ and $g(z)$ share $0, \infty$ CM, $(1 - e^{-b_1})^{-1} = \frac{1}{2} \neq 0$, and the coefficients of (1.5) satisfy $A_1(z) \equiv A_0(z + 1) \equiv F(z) \equiv 1, A_1(z)F(z + 1) \equiv 1 \equiv \frac{e^{-b_1}(e^{-b_1 z - b_0} - 1)}{e^{a_1}(1 - e^{-b_1 z - 2b_1 - b_0})} F(z)A_0(z + 1)$, where $a_1 = 2\pi i, b_1 = \pi i, a_0 = b_0 = 0$. The above statement shows that Case (iii) in Theorem 6 certainly exists.

Example 3. The function $f(z) = -e^{3\pi iz}$ and another function $g(z) = -e^{-3\pi iz}$ solve the equation

$$f(z + 1) + f(z) = 0.$$

Here $f(z)$ and $g(z)$ share $1, \infty$ CM, $e^{-3\pi i} = e^{3\pi i} = -1$, and the coefficients of (1.5) satisfy $A_0(z) \equiv A_1(z) \equiv 1; A_0(z) + A_1(z) \equiv 2 \neq F(z)e^{\alpha(z)} \equiv 0$. But

$$A_1(z)[c_1 F(z + 1) - A_1(z + 1)] \equiv -1 \equiv [c_1 F(z) - A_0(z)]A_0(z + 1)$$

and

$$\frac{A_1(z)[F(z + 1)e^{a_1 z + a_1 + a_0} - A_1(z + 1) - A_0(z + 1)]}{A_0(z + 1)[F(z)e^{a_1 z + a_0} - A_1(z) - A_0(z)]} \equiv 1 \equiv \frac{e^{-b_1 z - b_1 - b_0} - e^{-b_1}}{1 - e^{-b_1 z - 2b_1 - b_0}},$$

where $b_1 = 3\pi i, b_0 = 0, c_1 \neq 0, a_1 \neq 0, a_0$ are constants.

Remark 2. From Example 3, we find that the conditions

$$A_1(z)[c_1 F(z + 1) - A_1(z + 1)] \equiv [c_1 F(z) - A_0(z)]A_0(z + 1)$$

and

$$\frac{A_1(z)[F(z + 1)e^{a_1 z + a_1 + a_0} - A_1(z + 1) - A_0(z + 1)]}{A_0(z + 1)[F(z)e^{a_1 z + a_0} - A_1(z) - A_0(z)]} \equiv \frac{e^{-b_1 z - b_1 - b_0} - e^{-b_1}}{1 - e^{-b_1 z - 2b_1 - b_0}}$$

in Corollary 2 cannot be deleted.

Example 4. The functions $f(z) = \frac{e^z}{z}$ and $g(z) = \frac{e^z}{z}$ solve the equation

$$f(z + 1) + ef(z) = \frac{2z + 1}{z(z + 1)} e^{z+1}.$$

Here $f(z)$ and $g(z)$ share $0, \infty$ CM. The above statement shows that Case (i) in Theorem 6 certainly exists.

Example 5. The function $f(z) = \frac{z(e^{3\pi iz} + e^{2\pi iz})}{2}$ and another function $g(z) = \frac{z(e^{2\pi iz} + e^{\pi iz})}{2}$ solve the equation

$$\frac{z}{z+1}f(z+1) + f(z) = ze^{2\pi iz}.$$

Here $f(z)$ and $g(z)$ share $0, \infty$ CM, $(1 - e^{-b_1})^{-1} = \frac{1}{2} \neq 0$, and the coefficients of (1.5) satisfy $A_1(z) \equiv \frac{z}{z+1}$, $A_0(z) \equiv 1$, $F(z) \equiv z$, $A_1(z)F(z+1) \equiv z \equiv \frac{e^{-b_1}(e^{-b_1 z - b_0} - 1)}{e^{a_1(1 - e^{-b_1 z - 2b_1 - b_0})}}F(z)A_0(z+1)$, where $a_1 = 2\pi i$, $b_1 = \pi i$, $a_0 = b_0 = 0$. The above statement shows that when $F(z)$ and $\alpha(z)$ are non-constant polynomials, Case (iii) in Theorem 6 certainly exists.

2. Some lemmas

Lemma 1 (see [5, 8]). *Let ε be a positive constant, ξ_1 and ξ_2 be two distinct complex constants, and let $f(z)$ be a meromorphic function of finite order $\sigma = \sigma(f(z))$. Then we have*

$$(2.1) \quad m\left(r, \frac{f(z + \xi_1)}{f(z + \xi_2)}\right) = O(r^{\sigma-1+\varepsilon}) = o(T(r, f)).$$

Lemma 2 (see [5]). *Let $f(z)$ be a meromorphic function such that the order $\sigma = \sigma(f(z)) < +\infty$, and let η be a nonzero complex number. If $\varepsilon > 0$, then*

$$(2.2) \quad T(r, f(z + \eta)) = T(r, f(z)) + O(r^{\sigma-1+\varepsilon}) + O(\log r).$$

Lemma 3 (see [19]). *If meromorphic functions $f_j(z)$ ($j = 1, 2, \dots, n$) ($n \geq 2$) and entire functions $g_j(z)$ ($j = 1, 2, \dots, n$) ($n \geq 2$) satisfy the following conditions:*

- (1) $\sum_{j=1}^n f_j e^{g_j} \equiv 0$,
- (2) when $1 \leq j < l \leq n$, $g_j - g_l$ are not constants,
- (3) when $1 \leq j \leq n$, $1 \leq h < l \leq n$, $T(r, f_j) = o(T(r, e^{g_h - g_l}))$ ($r \rightarrow \infty$, $r \notin E$),

then we have $f_j \equiv 0$ ($j = 1, 2, \dots, n$).

3. Proof of Theorem 6

Since $f(z)$ and $g(z)$ are finite order transcendental meromorphic functions that share $0, \infty$ CM, we get

$$(3.1) \quad \frac{f(z)}{g(z)} = e^{\beta(z)},$$

where $\beta(z)$ is a polynomial such that $\deg \beta(z) \leq \max\{\sigma(f), \sigma(g)\}$.

From the equations (1.5) and (3.1), we have

$$(3.2) \quad A_1(z)g(z+1) + A_0(z)g(z) = F(z)e^{\alpha(z)}$$

and

$$(3.3) \quad A_1(z)g(z+1)e^{\beta(z+1)} + A_0(z)g(z)e^{\beta(z)} = F(z)e^{\alpha(z)}.$$

Combining (3.2) with (3.3), we have

$$(3.4) \quad g(z)A_0(z)[1 - e^{\beta(z)-\beta(z+1)}] = F(z)e^{\alpha(z)}[1 - e^{-\beta(z+1)}].$$

If $1 - e^{\beta(z)-\beta(z+1)} \equiv 0$, then from (3.4), we have $1 - e^{-\beta(z+1)} \equiv 0$, which means that $f(z) \equiv g(z)$.

If $1 - e^{\beta(z)-\beta(z+1)} \not\equiv 0$, then $\beta(z)$ must be a non-constant polynomial. Thus from (3.4) we may solve out $g(z)$ of the form

$$(3.5) \quad g(z) = \frac{F(z)e^{\alpha(z)}(1 - e^{-\beta(z+1)})}{A_0(z)[1 - e^{\beta(z)-\beta(z+1)}]}.$$

By (3.2) and (3.5), we see that

$$\frac{A_1(z)F(z+1)e^{\alpha(z+1)}[1 - e^{-\beta(z+2)}]}{A_0(z+1)[1 - e^{\beta(z+1)-\beta(z+2)}]} + \frac{A_0(z)F(z)e^{\alpha(z)}[1 - e^{-\beta(z+1)}]}{A_0(z)[1 - e^{\beta(z)-\beta(z+1)}]} = F(z)e^{\alpha(z)}.$$

Equally,

$$(3.6) \quad A_1(z)q(z+1)[1 - e^{-\beta(z+2)}] + A_0(z)q(z)[1 - e^{-\beta(z+1)}] = F(z)e^{\alpha(z)},$$

where

$$q(z) = \frac{F(z)e^{\alpha(z)}}{A_0(z)[1 - e^{\beta(z)-\beta(z+1)}]}.$$

Now we set

$$(3.7) \quad \begin{aligned} \alpha(z) &= a_m z^m + a_{m-1} z^{m-1} + \cdots + a_0, \\ \beta(z) &= b_n z^n + b_{n-1} z^{n-1} + \cdots + b_0, \end{aligned}$$

where $a_m \neq 0$, $a_{m-1}, \dots, a_0, b_n \neq 0$, b_{n-1}, \dots, b_0 are constants, m and n are integers.

Note that $\beta(z)$ is a non-constant polynomial and $\alpha(z)$ is a polynomial. Next, we divide our argument into two cases respectively: Case 1, $\alpha(z)$ is a constant, $\beta(z)$ is not a constant; Case 2, $\alpha(z)$ and $\beta(z)$ are not constants.

Case 1. If $\alpha(z)$ is a constant, $\beta(z)$ is not a constant, then it is natural to have $\deg \beta(z) \geq 1$. Now we will prove that $\deg \beta(z) = 1$. Suppose on the contrary that $\deg \beta(z) = n \geq 2$. Then, it is obvious that

$$(3.8) \quad \deg[\beta(z+1) - \beta(z)] = \deg[\beta(z+2) - \beta(z+1)] = n - 1,$$

and $\sigma(e^{\beta(z+2)-\beta(z+1)}) = \sigma(e^{\beta(z+1)-\beta(z)}) = n - 1 \geq 1$.

Without loss of generality, we assume that $e^{\alpha(z)} = c_1$ ($\neq 0$). By the first main theorem of Nevanlinna theory, we have

$$\begin{aligned} T(r, q(z)) &= T\left(r, \frac{1}{q(z)}\right) + O(1) = T\left(r, \frac{A_0(z) - A_0(z)e^{\beta(z)-\beta(z+1)}}{c_1 F(z)}\right) + O(1) \\ &= T\left(r, e^{\beta(z)-\beta(z+1)}\right) + O(\log r). \end{aligned}$$

Therefore, $\sigma(q(z)) = n - 1$. From Lemma 1, we get

$$(3.9) \quad m\left(r, \frac{q(z+1)}{q(z)}\right) = O(r^{\sigma(q)-1+\varepsilon}) = O(r^{n-2+\varepsilon}) = o(r^{n-1})$$

for each $\varepsilon \in (0, 1)$.

The equation (3.6) can be changed to

$$(3.10) \quad \begin{aligned} & -A_1(z)q(z+1) - A_0(z)q(z)e^{\beta(z+2)-\beta(z+1)} \\ & = [c_1F(z) - A_1(z)q(z+1) - A_0(z)q(z)]e^{\beta(z+2)}. \end{aligned}$$

If $c_1F(z) - A_1(z)q(z+1) - A_0(z)q(z) \not\equiv 0$, then by (3.8), (3.10) and the fact $\sigma(q(z)) = n - 1$, we have

$$\begin{aligned} n & = \sigma\left((c_1F(z) - A_1(z)q(z+1) - A_0(z)q(z))e^{\beta(z+2)}\right) \\ & = \sigma\left(-A_1(z)q(z+1) - A_0(z)q(z)e^{\beta(z+2)-\beta(z+1)}\right) \leq n - 1. \end{aligned}$$

This is absurd. Thus $c_1F(z) - A_1(z)q(z+1) - A_0(z)q(z) \equiv 0$.

From the above discussion and (3.10), we see that

$$(3.11) \quad -A_1(z)q(z+1) - A_0(z)q(z)e^{\beta(z+2)-\beta(z+1)} = 0.$$

Since $A_1(z)$ and $A_0(z)$ are non-zero polynomials, by (3.9) and (3.11), we obtain

$$\begin{aligned} T\left(r, e^{\beta(z+2)-\beta(z+1)}\right) & = m\left(r, e^{\beta(z+2)-\beta(z+1)}\right) \\ & = m\left(r, \frac{-A_1(z)q(z+1)}{A_0(z)q(z)}\right) \leq o(r^{n-1}) + O(\log r), \end{aligned}$$

which contradicts $\sigma(e^{\beta(z+2)-\beta(z+1)}) = n - 1 \geq 1$. We thus have proved that $\deg \beta(z) = 1$. Next we derive from (3.7) that there exist two constants $b_1 (\neq 0)$, b_0 such that $\beta(z) = b_1z + b_0$.

Substituting $e^{\alpha(z)} = c_1$, $\beta(z) = b_1z + b_0$ into (3.5), we obtain

$$(3.12) \quad g(z) = \frac{c_1F(z)(1 - e^{-b_1z - b_1 - b_0})}{A_0(z)(1 - e^{-b_1})},$$

where $(1 - e^{-b_1})^{-1} \neq 0$.

Then, combining (3.2) with (3.12), we have

$$\begin{aligned} & \left[\frac{-c_1A_1(z)F(z+1)}{A_0(z+1)(1 - e^{-b_1})}e^{-b_1} - \frac{c_1F(z)}{(1 - e^{-b_1})} \right] e^{-b_1z - b_1 - b_0} \\ & = c_1F(z) - \frac{c_1A_1(z)F(z+1)}{A_0(z+1)(1 - e^{-b_1})} - \frac{c_1F(z)}{(1 - e^{-b_1})}. \end{aligned}$$

From Lemma 3, we get

$$(3.13) \quad \frac{-c_1A_1(z)F(z+1)}{A_0(z+1)(1 - e^{-b_1})}e^{-b_1} - \frac{c_1F(z)}{(1 - e^{-b_1})} \equiv 0$$

and

$$(3.14) \quad c_1 F(z) - \frac{c_1 A_1(z) F(z+1)}{A_0(z+1)(1-e^{-b_1})} - \frac{c_1 F(z)}{(1-e^{-b_1})} \equiv 0.$$

By using the above equations, we have

$$e^{-b_1} = \frac{-A_0(z+1)F(z)}{A_1(z)F(z+1)} = e^{b_1},$$

where $e^{-b_1} = e^{b_1} = -1$.

Further, from (3.1) and (3.12), we obtain

$$f(z) = \frac{c_1 F(z)}{2A_0(z)} (1 + e^{b_1 z + b_0})$$

and

$$g(z) = \frac{c_1 F(z)}{2A_0(z)} (1 + e^{-b_1 z - b_0}),$$

where $e^{-b_1} = e^{b_1} = -1$, $b_1 = (2k+1)\pi i$, k is an integer. Finally, we obtain from (3.13) or (3.14) that

$$A_1(z)F(z+1) \equiv F(z)A_0(z+1)$$

holds for this case.

Case 2. If $\alpha(z)$ and $\beta(z)$ are not constants with $\deg \alpha(z) = m$, $\deg \beta(z) = n$, then (3.6) can be expressed as

$$(3.15) \quad \begin{aligned} & A_1(z)F(z+1)e^{\alpha(z+1)} - A_1(z)F(z+1)e^{\alpha(z+1)-\beta(z+2)} - A_0(z+1) \\ & F(z)e^{\alpha(z)-\beta(z+1)} + A_0(z+1)F(z)e^{\alpha(z)-\beta(z+2)} - A_1(z)F(z+1) \\ & e^{\beta(z)-\beta(z+1)+\alpha(z+1)} + A_0(z+1)F(z)e^{\alpha(z)+\beta(z)-\beta(z+1)} \\ & - A_0(z+1)F(z)e^{\alpha(z)+\beta(z)-\beta(z+2)} + A_1(z)F(z+1) \\ & e^{\alpha(z+1)+\beta(z)-\beta(z+1)-\beta(z+2)} = 0. \end{aligned}$$

We will discuss three subcases.

Subcase 2.1: If $\deg \alpha(z) > \deg \beta(z) \geq 1$, then (3.15) can be expressed as

$$J_{11}(z)e^{\alpha(z+1)} = 0,$$

so that

$$(3.16) \quad J_{11}(z) = 0,$$

where

$$\begin{aligned} J_{11}(z) &= A_1(z)F(z+1) - A_1(z)F(z+1)e^{-\beta(z+2)} \\ & - A_0(z+1)F(z)e^{\alpha(z)-\alpha(z+1)-\beta(z+1)} \\ & + A_0(z+1)F(z)e^{\alpha(z)-\beta(z+2)-\alpha(z+1)} - A_1(z)F(z+1)e^{\beta(z)-\beta(z+1)} \\ & + A_0(z+1)F(z)e^{\alpha(z)+\beta(z)-\beta(z+1)-\alpha(z+1)} \\ & - A_0(z+1)F(z)e^{\alpha(z)+\beta(z)-\beta(z+2)-\alpha(z+1)} \end{aligned}$$

$$+ A_1(z)F(z+1)e^{\beta(z)-\beta(z+1)-\beta(z+2)}.$$

Now we distinguish $m - 1 > n \geq 1$, $m - 1 = n \geq 1$ two subcases to derive the contradictions.

Subcase 2.1.1: If $m - 1 > n \geq 1$, then (3.16) can be expressed as

$$(3.17) \quad \begin{aligned} & J_{24}(z)e^{-\beta(z+2)} + J_{23}(z)e^{\alpha(z)-\alpha(z+1)-\beta(z+1)} + J_{22}(z) \\ & e^{\beta(z)-\beta(z+1)+\alpha(z)-\alpha(z+1)} + J_{21}(z)e^{h_0(z)} = 0, \end{aligned}$$

where $h_0(z) \equiv 0$ and

$$\begin{cases} J_{24}(z) = -A_1(z)F(z+1) + A_1(z)F(z+1)e^{\beta(z)-\beta(z+1)}, \\ J_{23}(z) = -A_0(z+1)F(z) + A_0(z+1)F(z)e^{\beta(z+1)-\beta(z+2)}, \\ J_{22}(z) = A_0(z+1)F(z) - A_0(z+1)F(z)e^{\beta(z+1)-\beta(z+2)}, \\ J_{21}(z) = A_1(z)F(z+1) - A_1(z)F(z+1)e^{\beta(z)-\beta(z+1)}. \end{cases}$$

Now by $m - 1 > n \geq 1$ it follows that $\deg(\beta(z+i) - \beta(z+j)) = \deg \beta(z) - 1 = n - 1 \geq 0$ ($i = 0, 1; j = 1, 2$), $\deg(\alpha(z+1) - \alpha(z) + \beta(z+1) - \beta(z+2)) = m - 1$, $\deg(\alpha(z+1) - \alpha(z) + \beta(z+1) - \beta(z+2) - \beta(z)) = m - 1$, $\deg(\alpha(z) - \alpha(z+1) - \beta(z+1)) = m - 1$, $\deg(\beta(z) - \beta(z+1) + \alpha(z) - \alpha(z+1)) = m - 1$. From Lemma 2, for $j = 1, 2, 3, 4$, we get

$$\begin{cases} T(r, J_{2j}(z)) = o\{T(r, e^{\beta(z)})\}, \\ T(r, J_{2j}(z)) = o\{T(r, e^{\alpha(z)-\alpha(z+1)-\beta(z+1)})\}, \\ T(r, J_{2j}(z)) = o\{T(r, e^{\alpha(z)-\alpha(z+1)+\beta(z)-\beta(z+1)})\}, \\ T(r, J_{2j}(z)) = o\{T(r, e^{\alpha(z+1)-\alpha(z)+\beta(z+1)-\beta(z+2)})\}, \\ T(r, J_{2j}(z)) = o\{T(r, e^{\alpha(z+1)-\alpha(z)+\beta(z+1)-\beta(z+2)-\beta(z)})\}. \end{cases}$$

Applying Lemma 3 to (3.17), we have $J_{2j}(z) \equiv 0$ ($j = 1, 2, 3, 4$). Then, by $J_{24}(z) \equiv 0$ and the assumption that $A_1(z)$ and $F(z)$ are non-zero polynomials, we have $e^{\beta(z)-\beta(z+1)} \equiv 1$, which contradicts that $e^{\beta(z)-\beta(z+1)} \neq 1$.

Subcase 2.1.2: If $m - 1 = n \geq 1$, then from (3.7), we see that $\alpha(z) - \alpha(z+1) = -ma_m z^{m-1} - (C_m^2 a_m + C_{m-1}^1 a_{m-1})z^{m-2} - \dots - (a_m + a_{m-1} + \dots + a_1)$ is non-constant polynomial with $\deg(\alpha(z) - \alpha(z+1)) = m - 1 \geq 1$. Similarly, we can get $\deg(\beta(z) - \beta(z+1)) = n - 1 \geq 0$. The following discussion is divided into three subcases according to whether $b_n = \pm ma_m$.

Subcase 2.1.2.1: If $b_n \neq ma_m, b_n \neq -ma_m$, then (3.16) can be rewritten as (3.17). Therefore, using the same method as in the proof of Subcase 2.1.1, we can get a contradiction.

Subcase 2.1.2.2: If $b_n = -ma_m$, then (3.16) can be expressed as

$$(3.18) \quad J_{33}(z)e^{-\beta(z+2)} + J_{32}(z)e^{\beta(z)-\beta(z+1)+\alpha(z)-\alpha(z+1)} + J_{31}(z)e^{h_0(z)} = 0,$$

where $h_0(z) \equiv 0$ and

$$\begin{cases} J_{33}(z) = -A_1(z)F(z+1) + A_1(z)F(z+1)e^{\beta(z)-\beta(z+1)}, \\ J_{32}(z) = A_0(z+1)F(z) - A_0(z+1)F(z)e^{\beta(z+1)-\beta(z+2)}, \\ J_{31}(z) = A_1(z)F(z+1) - A_1(z)F(z+1)e^{\beta(z)-\beta(z+1)} \\ \quad + A_0(z+1)F(z)e^{\alpha(z)-\beta(z+2)-\alpha(z+1)} \\ \quad - A_0(z+1)F(z)e^{\alpha(z)-\alpha(z+1)-\beta(z+1)}. \end{cases}$$

By $b_n = -ma_m$, $m-1 = n \geq 1$, then $\deg(-\beta(z+2) - \beta(z) + \beta(z+1) - \alpha(z) + \alpha(z+1)) = \deg(\beta(z) - \beta(z+1) + \alpha(z) - \alpha(z+1)) = n$, $\deg(\beta(z) - \beta(z+1)) = \deg(\beta(z+1) - \beta(z+2)) = n-1$, $\deg(\alpha(z) - \alpha(z+1) - \beta(z+1)) = \deg(\alpha(z) - \alpha(z+1) - \beta(z+2)) \leq n-1$. From Lemma 2, for $j = 1, 2, 3$, we get

$$\begin{cases} T(r, J_{3j}(z)) = o\{T(r, e^{-\beta(z+2)})\}, \\ T(r, J_{3j}(z)) = o\{T(r, e^{\beta(z)-\beta(z+1)+\alpha(z)-\alpha(z+1)})\}, \\ T(r, J_{3j}(z)) = o\{T(r, e^{-\beta(z+2)-\beta(z)+\beta(z+1)-\alpha(z)+\alpha(z+1)})\}. \end{cases}$$

Applying Lemma 3 to (3.18), we have $J_{3j}(z) \equiv 0$ ($j = 1, 2, 3$). Then, by $J_{33}(z) \equiv 0$ and the assumption that $A_1(z)$ and $F(z)$ are non-zero polynomials, we obtain a contradiction again.

Subcase 2.1.2.3: If $b_n = ma_m$, then (3.16) can be expressed as

$$(3.19) \quad J_{43}(z)e^{-\beta(z+2)} + J_{42}(z)e^{\alpha(z)-\alpha(z+1)-\beta(z+1)} + J_{41}(z)e^{h_0(z)} = 0,$$

where $h_0(z) \equiv 0$ and

$$\begin{cases} J_{43}(z) = -A_1(z)F(z+1) + A_1(z)F(z+1)e^{\beta(z)-\beta(z+1)} \\ \quad - A_0(z+1)F(z)e^{\beta(z)+\alpha(z)-\alpha(z+1)} \\ \quad + A_0(z+1)F(z)e^{\beta(z)-\beta(z+1)+\beta(z+2)+\alpha(z)-\alpha(z+1)}, \\ J_{42}(z) = -A_0(z+1)F(z) + A_0(z+1)F(z)e^{\beta(z+1)-\beta(z+2)}, \\ J_{41}(z) = A_1(z)F(z+1) - A_1(z)F(z+1)e^{\beta(z)-\beta(z+1)}. \end{cases}$$

By $b_n = ma_m$, $m-1 = n \geq 1$, then $\deg(-\beta(z+2) - \alpha(z) + \alpha(z+1) + \beta(z+1)) = \deg(\alpha(z) - \alpha(z+1) - \beta(z+1)) = n$, $\deg(\beta(z) - \beta(z+1)) = \deg(\beta(z+1) - \beta(z+2)) = n-1$, $\deg(\beta(z) - \beta(z+1) + \beta(z+2) + \alpha(z) - \alpha(z+1)) = \deg(\beta(z) + \alpha(z) - \alpha(z+1)) \leq n-1$. From Lemma 2, for $j = 1, 2, 3$, we get

$$\begin{cases} T(r, J_{4j}(z)) = o\{T(r, e^{-\beta(z+2)})\}, \\ T(r, J_{4j}(z)) = o\{T(r, e^{\alpha(z)-\alpha(z+1)-\beta(z+1)})\}, \\ T(r, J_{4j}(z)) = o\{T(r, e^{-\beta(z+2)-\alpha(z)+\alpha(z+1)+\beta(z+1)})\}. \end{cases}$$

Applying Lemma 3 to (3.19), we have $J_{4j}(z) \equiv 0$ ($j = 1, 2, 3$). Then by $J_{41}(z) \equiv 0$, we can get the same contradiction as in Subcase 2.1.2.2.

Subcase 2.2: If $\deg \beta(z) > \deg \alpha(z) \geq 1$, then (3.15) can be expressed as

$$(3.20) \quad J_{52}(z)e^{\alpha(z+1)} + J_{51}(z)e^{\alpha(z+1)-\beta(z+2)} = 0,$$

where

$$\begin{cases} J_{52}(z) = A_1(z)F(z+1) - A_1(z)F(z+1)e^{\beta(z)-\beta(z+1)} \\ \quad + A_0(z+1)F(z)e^{\alpha(z)-\alpha(z+1)+\beta(z)-\beta(z+1)} \\ \quad - A_0(z+1)F(z)e^{\alpha(z)-\alpha(z+1)+\beta(z)-\beta(z+2)}, \\ J_{51}(z) = -A_1(z)F(z+1) - A_0(z+1)F(z)e^{\alpha(z)-\alpha(z+1)-\beta(z+1)+\beta(z+2)} \\ \quad + A_0(z+1)F(z)e^{\alpha(z)-\alpha(z+1)} + A_1(z)F(z+1)e^{\beta(z)-\beta(z+1)}. \end{cases}$$

By $\deg \beta(z) > \deg \alpha(z) \geq 1$, then $\deg(\beta(z+2)) = n$, $\deg(\beta(z)-\beta(z+1)) = n-1$, $\deg(\alpha(z)-\alpha(z+1)) = m-1 < n$, $\deg(\alpha(z)-\alpha(z+1)+\beta(z)-\beta(z+1)) = n-1$, $\deg(\alpha(z)-\alpha(z+1)+\beta(z)-\beta(z+2)) = n-1$, $\deg(\alpha(z)-\alpha(z+1)-\beta(z+1)+\beta(z+2)) = n-1$. From Lemma 2, for $j = 1, 2$, we get $T(r, J_{5j}(z)) = o\{T(r, e^{\beta(z+2)})\}$. Applying Lemma 3 to (3.20), we have $J_{5j}(z) \equiv 0$ ($j = 1, 2$). Then by $J_{52}(z) \equiv 0$, we have

$$(3.21) \quad \begin{aligned} & A_1(z)F(z+1) - A_1(z)F(z+1)e^{\beta(z)-\beta(z+1)} + A_0(z+1)F(z) \\ & e^{\alpha(z)-\alpha(z+1)+\beta(z)-\beta(z+1)} - A_0(z+1)F(z)e^{\alpha(z)-\alpha(z+1)+\beta(z)-\beta(z+2)} \equiv 0. \end{aligned}$$

We will discuss the degree of $\alpha(z)$ and obtain the contradiction.

Firstly, if $\deg \alpha(z) = m \geq 2$, by $\deg \beta(z) > \deg \alpha(z) \geq 1$, then we obtain $\deg \beta(z) = n \geq 3$, thus $\deg(\beta(z)-\beta(z+1)) = n-1 \geq 2$, $\deg(\beta(z+2)-\beta(z+1)) = n-1 \geq 2$, $\deg(\alpha(z)-\alpha(z+1)+\beta(z)-\beta(z+1)) = \deg(\alpha(z)-\alpha(z+1)+\beta(z)-\beta(z+2)) = \deg(\alpha(z+1)-\alpha(z)+\beta(z+2)-\beta(z+1)) = n-1 \geq 2$, $\deg(\alpha(z+1)-\alpha(z)) = m-1 \geq 1$. Applying Lemma 3 to (3.21), we have $A_1(z)F(z+1) \equiv A_0(z+1)F(z) \equiv 0$. This contradicts the assumption that $A_1(z), A_0(z), F(z)$ are non-zero polynomials. Hence, $\deg \alpha(z) = 1$. Setting $\alpha(z) = a_1z + a_0$, where $a_1 \neq 0$ and a_0 are constants. Then (3.21) can be expressed as

$$(3.22) \quad \begin{aligned} & A_1(z)F(z+1) + [A_0(z+1)F(z)e^{-a_1} - A_1(z)F(z+1)]e^{\beta(z)-\beta(z+1)} \\ & - A_0(z+1)F(z)e^{-a_1+\beta(z)-\beta(z+2)} \equiv 0. \end{aligned}$$

By $\deg \alpha(z) = 1$, then $\deg \beta(z) \geq 2$, thus $\deg(\beta(z+2) - \beta(z+1)) \geq 1$, $\deg(\beta(z) - \beta(z+i)) \geq 1$ ($i = 1, 2$). Applying Lemma 3 to (3.22), we have $A_1(z)F(z+1) \equiv A_0(z+1)F(z) \equiv 0$, which is a contradiction.

Subcase 2.3: If $\deg \beta(z) = \deg \alpha(z) = n \geq 1$, then (3.15) can be expressed as (3.20). By $\deg \beta(z) = \deg \alpha(z) = n \geq 1$, then $\deg(\beta(z)-\beta(z+1)) = \deg \beta(z) - 1 = \deg(\alpha(z)-\alpha(z+1)) = n-1$, $\deg(\alpha(z)-\alpha(z+1)+\beta(z)-\beta(z+1)) \leq n-1$, $\deg(\alpha(z)-\alpha(z+1)+\beta(z)-\beta(z+2)) \leq n-1$, $\deg(\alpha(z)-\alpha(z+1)-\beta(z+1)+\beta(z+2)) \leq n-1$. From Lemma 2, for $j = 1, 2$, we get $T(r, J_{5j}(z)) = o\{T(r, e^{\beta(z+2)})\}$. Applying Lemma 3 to (3.20), we have $J_{5j}(z) \equiv 0$ ($j = 1, 2$). Then by $J_{51}(z) \equiv 0$, we have

$$(3.23) \quad \begin{aligned} & -A_1(z)F(z+1) - A_0(z+1)F(z)e^{\alpha(z)-\alpha(z+1)+\beta(z+2)-\beta(z+1)} \\ & + A_0(z+1)F(z)e^{\alpha(z)-\alpha(z+1)} + A_1(z)F(z+1)e^{\beta(z)-\beta(z+1)} \equiv 0. \end{aligned}$$

Next we will prove that $\deg \alpha(z) = \deg \beta(z) = 1$.

Subcase 2.3.1: If $\deg \alpha(z) = \deg \beta(z) = n \geq 2$, $a_m \neq b_n$, then (3.23) can be expressed as

$$(3.24) \quad \begin{aligned} & J_{64}(z)e^{\alpha(z)-\alpha(z+1)} + J_{63}(z)e^{\beta(z)-\beta(z+1)} + J_{62}(z) \\ & e^{\alpha(z)-\alpha(z+1)+\beta(z+2)-\beta(z+1)} + J_{61}(z)e^{h_0(z)} \equiv 0, \end{aligned}$$

where $h_0(z) \equiv 0$ and

$$\begin{cases} J_{64}(z) = A_0(z+1), \\ J_{63}(z) = A_1(z) \frac{F(z+1)}{F(z)}, \\ J_{62}(z) = -A_0(z+1), \\ J_{61}(z) = -A_1(z) \frac{F(z+1)}{F(z)}. \end{cases}$$

By $\deg \alpha(z) = \deg \beta(z) = n \geq 2$, $a_m \neq b_n$, then $\deg(\alpha(z) - \alpha(z+1) - \beta(z) + \beta(z+1)) = n-1 \geq 1$, $\deg(\beta(z+1) - \beta(z+2)) = n-1 \geq 1$, $\deg(\alpha(z) - \alpha(z+1)) = n-1 \geq 1$, $\deg(\beta(z) - \alpha(z) + \alpha(z+1) - \beta(z+2)) = n-1 \geq 1$, $\deg(\beta(z) - \beta(z+1)) = n-1 \geq 1$, $\deg(\alpha(z) - \alpha(z+1) - \beta(z+1) + \beta(z+2)) = n-1 \geq 1$. From Lemma 2, for $j = 1, 2, 3, 4$, we get

$$\begin{cases} T(r, J_{6j}(z)) = o\{T(r, e^{\alpha(z)-\alpha(z+1)})\}, \\ T(r, J_{6j}(z)) = o\{T(r, e^{\beta(z)-\beta(z+1)})\}, \\ T(r, J_{6j}(z)) = o\{T(r, e^{\beta(z+1)-\beta(z+2)})\}, \\ T(r, J_{6j}(z)) = o\{T(r, e^{\beta(z)-\alpha(z)+\alpha(z+1)-\beta(z+2)})\}, \\ T(r, J_{6j}(z)) = o\{T(r, e^{\alpha(z)-\alpha(z+1)-\beta(z)+\beta(z+1)})\}, \\ T(r, J_{6j}(z)) = o\{T(r, e^{\alpha(z)-\alpha(z+1)-\beta(z+1)+\beta(z+2)})\}. \end{cases}$$

Applying Lemma 3 to (3.24), we have $J_{6j}(z) \equiv 0$ ($j = 1, 2$). Then by $J_{64}(z) \equiv 0$, we have $A_0(z+1) \equiv 0$. This contradicts the assumption that $A_0(z)$ is a non-zero polynomial.

Subcase 2.3.2: If $\deg \alpha(z) = \deg \beta(z) = n \geq 2$, $a_m = b_n$, then (3.23) can be expressed as

$$(3.25) \quad J_{72}(z)e^{\alpha(z)-\alpha(z+1)} + J_{71}(z)e^{\alpha(z)-\alpha(z+1)+\beta(z+2)-\beta(z+1)} \equiv 0,$$

where

$$\begin{cases} J_{72}(z) = A_0(z+1), \\ J_{71}(z) = -A_0(z+1) - A_1(z) \frac{F(z+1)}{F(z)} e^{\alpha(z+1)-\alpha(z)+\beta(z+1)-\beta(z+2)} \\ \quad + A_1(z) \frac{F(z+1)}{F(z)} e^{\alpha(z+1)-\alpha(z)+\beta(z)-\beta(z+2)}. \end{cases}$$

By $\deg \alpha(z) = \deg \beta(z) = n \geq 2$, $a_m = b_n$, then $\deg(\beta(z+1) - \beta(z+2)) = n-1 \geq 1$, $\deg(\alpha(z+1) - \alpha(z) + \beta(z) - \beta(z+2)) = \deg(\alpha(z+1) - \alpha(z) - \beta(z+2) + \beta(z+1)) \leq n-2$. From Lemma 2, for $j = 1, 2$, we get $T(r, J_{7j}(z)) = o\{T(r, e^{\beta(z+1)-\beta(z+2)})\}$. Applying Lemma 3 to (3.25), we have $J_{7j}(z) \equiv 0$ ($j =$

1, 2). Then by $J_{72}(z) \equiv 0$, we have $A_0(z+1) \equiv 0$, a contradiction. Hence, $\deg \alpha(z) = \deg \beta(z) = 1$.

Now we set $\alpha(z) = a_1z + a_0$, $\beta(z) = b_1z + b_0$. Then, substituting it into (3.1) and (3.5), we obtain

$$f(z) = \frac{F(z)e^{a_1z+a_0}(e^{b_1z+b_0} - e^{-b_1})}{A_0(z)(1 - e^{-b_1})}$$

and

$$g(z) = \frac{F(z)e^{a_1z+a_0}(1 - e^{-b_1z-b_1-b_0})}{A_0(z)(1 - e^{-b_1})},$$

where $(1 - e^{-b_1})^{-1} \neq 0$, $a_1 \neq 0$, $b_1 \neq 0$, a_0, b_0 are constants.

What is more, from (3.6), we see that the coefficients of (1.5) satisfy

$$A_1(z)F(z+1) \equiv \frac{e^{-b_1}(e^{-b_1z-b_0} - 1)}{e^{a_1}(1 - e^{-b_1z-2b_1-b_0})}F(z)A_0(z+1).$$

This completes the proof of Theorem 6.

Acknowledgments. The authors would like to thank the referees for their thorough comments and helpful suggestions.

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JUN-FAN CHEN
SCHOOL OF MATHEMATICS AND STATISTICS
FUJIAN NORMAL UNIVERSITY
FUZHOU 350117, P. R. CHINA
Email address: junfanchen@163.com

SHU-QING LIN
SCHOOL OF MATHEMATICS AND STATISTICS
FUJIAN NORMAL UNIVERSITY
FUZHOU 350117, P. R. CHINA
Email address: shuqinglin1996@163.com