Bull. Korean Math. Soc. **0** (0), No. 0, pp. 1–0
https://doi.org/10.4134/BKMS.b200561
pISSN: 1015-8634 / eISSN: 2234-3016

SCHATTEN-HERZ CLASS TOEPLITZ OPERATORS ON WEIGHTED BERGMAN SPACES INDUCED BY DOUBLING WEIGHTS

Juntao Du and Songxiao Li

Abstract. Schatten-Herz class Toeplitz operators on weighted Bergman spaces induced by doubling weights are investigated in this paper.

1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane, and $H(\mathbb{D})$ the class of all functions analytic on $\mathbb{D}$. For any $z \in \mathbb{D}$ and $r > 0$, let $D(z,r) = \{w \in \mathbb{D} : \beta(z,w) < r\}$ be the Bergman disk. Here $\beta(\cdot,\cdot)$ is the Bergman metric on $\mathbb{D}$. If $\{a_j\}_{j=1}^{\infty} \subset \mathbb{D}$ satisfying $\inf_{i\neq j} \beta(a_i,a_j) \geq s > 0$, we say that $\{a_j\}_{j=1}^{\infty} \subset \mathbb{D}$ is s-separated.

A function $\omega : \mathbb{D} \to [0,\infty)$ is called a weight if it is positive and integrable. A weight ω is radial if $\omega(z) = \omega(|z|)$ for all $z \in \mathbb{D}$. Let ω be a radial weight and $\hat{\omega}(r) = \int_r^1 \omega(s)ds$ for $r \in [0,1)$. We say that ω is a doubling weight, denoted by $\omega \in \hat{\mathcal{D}}$, if there is a constant $C > 0$ such that $\hat{\omega}(r) < C\hat{\omega}(\frac{1+r}{2})$ when $0 \leq r < 1$. If there exist $K > 1$ and $C > 1$ such that $\hat{\omega}(r) \geq C\hat{\omega}(1 - \frac{1-r}{K})$ when $0 \leq r < 1$, we say that ω is a reverse doubling weight, denoted by $\omega \in \check{\mathcal{D}}$. We say that ω is a regular weight, denoted by $\omega \in \mathcal{R}$, if there exist $C > 1$ and $\delta \in (0,1)$, such that

$$\frac{1}{C} < \frac{\hat{\omega}(r)}{(1-r)\omega(r)} < C, \quad \text{when} \quad \delta < r < 1.$$

We denote by $\mathcal{D} = \hat{\mathcal{D}} \bigcap \check{\mathcal{D}}$. From [9], we see that $\mathcal{R} \subset \mathcal{D}$. More details about $\mathcal{R}$, $\mathcal{D}$ and $\hat{\mathcal{D}}$ can be found in [7–11].

When $0 < p < \infty$ and $\omega \in \hat{\mathcal{D}}$, the weighted Bergman space A^p_ω is the space of all $f \in H(\mathbb{D})$ such that

$$\|f\|^p_{A^p_\omega} = \int_{\mathbb{D}} |f(z)|^p \omega(z) dA(z) < \infty,$$

Received June 26, 2020; Accepted November 2, 2020.
2010 *Mathematics Subject Classification.* Primary 30H20, 47B35.
Key words and phrases. Weighted Bergman space, Toeplitz operator, doubling weight, Schatten-Herz class.
The project was supported by NNSF of China (No.11720101003).

where dA is the normalized Lebesgue area measure on $\mathbb{D}$. When $\omega(z) = (1-|z|^2)^\alpha (\alpha > -1)$, the space A^p_ω becomes the classical weighted Bergman space A^p_α. When $\alpha = 0$, we will write $A^p_\alpha = A^p$. Suppose $0 < p \le \infty$ and μ is a positive Borel measure on $\mathbb{D}$. Let L^p_μ denote the Lebesgue space defined in a standard way. Meanwhile, L^p_τ denotes L^p_μ for $d\mu(z) = (1-|z|^2)^\tau dA(z)$ for some real number τ.

Let $\omega_s = \int_0^1 r^s \omega(r)dr$ and $B^\omega_z(w) = \frac{1}{2}\sum_{k=0}^\infty \frac{(w\overline{z})^k}{\omega_{2k+1}}$. Then B^ω_z is the reproducing kernel for A^2_ω, which means that for any $f \in A^2_\omega$(see [10]),

$$f(z) = \langle f, B^\omega_z \rangle_{A^2_\omega} = \int_{\mathbb{D}} f(w)\overline{B^\omega_z(w)}\omega(w)dA(w).$$

Let μ be a positive Borel measure on $\mathbb{D}$. Let $f \in H(\mathbb{D})$ and $z \in \mathbb{D}$. The Toeplitz operator $\mathcal{T}_\mu$ and the Berezin transform $\widetilde{\mathcal{T}_\mu}$ of $\mathcal{T}_\mu$ are defined by

$$\mathcal{T}_\mu f(z) = \int_{\mathbb{D}} f(\xi)\overline{B^\omega_z(\xi)}d\mu(\xi), \qquad \widetilde{\mathcal{T}_\mu}(z) = \frac{\langle \mathcal{T}_\mu B^\omega_z, B^\omega_z\rangle_{A^2_\omega}}{\|B^\omega_z\|^2_{A^2_\omega}},$$

respectively.

For $k = 0, 1, 2, \ldots$, let $\Lambda_k = \{z \in \mathbb{D} : 1 - \frac{1}{2^k} \le |z| < 1 - \frac{1}{2^{k+1}}\}$ and χ_k the characteristic function of Λ_k. Let $d\lambda(z) = \frac{dA(z)}{(1-|z|^2)^2}$ be the Möbious invariant area measure on $\mathbb{D}$. For $0 < p \le \infty$ and $0 \le q \le \infty$, let $\mathrm{K}^p_q(\lambda)$ denote the Herz space, which consists of all measurable functions f such that

$$\|f\|_{\mathrm{K}^p_q(\lambda)} = \|\{\|f\|_{L^p_{\lambda\chi_k}}\}_{k=0}^\infty\|_{l^q} < \infty.$$

Note that $\mathrm{K}^p_p(\lambda) = L^p_\lambda$. Here, $l^q (0 \le q \le \infty)$ consists of all complex sequences $\{a_k\}_{k=1}^\infty$ such that $\|\{a_k\}\|_{l^q} < \infty$, where

$$\|\{a_k\}\|_{l^q} = \begin{cases} \left(\sum_{k=1}^\infty |a_k|^q\right)^{\frac{1}{q}}, & 0 < q < \infty; \\ \sup_{k\ge 1} |a_k|, & q = \infty; \\ \limsup_{k\to\infty} |a_k|, & q = 0. \end{cases}$$

Let $j = 0, 1, 2, \ldots$ and $\lambda_j = \inf\{\|\mathcal{T}_\mu - R\|_{A^2_\omega \to A^2_\omega} : \mathrm{rank}(R) \le j\}$. If $\{\lambda_j\}_{j=0}^\infty \in l^p$ for some $p \in (0, \infty)$, we say that $\mathcal{T}_\mu$ belongs to the Schatten p-class, denoted by $\mathcal{T}_\mu \in \mathcal{S}_p(A^2_\omega)$. We denote by $\mathcal{S}_\infty$ the class of all bounded linear operators on A^2_ω. For $0 < p \le \infty$ and $0 \le q \le \infty$, $\mathcal{T}_\mu$ is said to belong to the Schatten-Herz class, denoted by $\mathcal{S}_{p,q}$, if $\mathcal{T}_{\mu\chi_k} \in \mathcal{S}_p$ and

$$\|\mathcal{T}_\mu\|_{\mathcal{S}_{p,q}} = \|\{\|\mathcal{T}_{\mu\chi_k}\|_{\mathcal{S}_p}\}_{k=0}^\infty\|_{l^q} < \infty.$$

Here $\|\mathcal{T}_{\mu\chi_k}\|_{\mathcal{S}_p}$ means the Schatten p-class norm of $\mathcal{T}_{\mu\chi_k}$ on A^2_ω, see [12] for example. In [4], Loaiza, López-García and Pérez-Esteva considered Schatten-Herz class Toeplitz operators on A^2 for the first time. For more study on Herz spaces and Schatten-Herz class Toeplitz operators, see [1–6].

In [9, 11], Peláez, Rättyä and Sierra investigated the boundedness, compactness and Schatten class Toeplitz operators on Bergman spaces induced by doubling weights. In this paper, we investigate Schatten-Herz class Toeplitz operators on A^2_ω with $\omega \in \hat{\mathcal{D}}$. The main result of this paper is stated as follows.

Theorem 1.1. *Suppose $0 < p \leq \infty$, $0 \leq q \leq \infty$, $r > 0$, $\omega \in \hat{\mathcal{D}}$ and μ is a positive Borel measure. Let $\{a_j\} \subset \mathbb{D}$ such that $\mathbb{D} = \cup_{j=1}^{\infty} D(a_j, r)$ and $\{a_j\}$ are s-separated for some $0 < s < r < \infty$. Let*

$$\widehat{\mu_r}(z) = \frac{\mu(D(z,r))}{(1-|z|)\hat{\omega}(z)} \quad \textit{and} \quad \|\{\widehat{\mu_r}(a_j)\}\|_{l^p_q} = \|\{\|\{(\widehat{\mu_r}\chi_k)(a_j)\}_{j=1}^{\infty}\|_{l^p}\}_{k=0}^{\infty}\|_{l^q}.$$

(i) *The following statements are equivalent.*
   (a) $\mathcal{T}_\mu \in \mathcal{S}_{p,q}$;
   (b) $\widehat{\mu_r} \in \mathrm{K}^p_q(\lambda)$;
   (c) $\|\{\widehat{\mu_r}(a_j)\}\|_{l^p_q} < \infty$.

   Moreover,

$$\|\mathcal{T}_\mu\|_{\mathcal{S}_{p,q}} \approx \|\widehat{\mu_r}\|_{\mathrm{K}^p_q(\lambda)} \approx \|\{\widehat{\mu_r}(a_j)\}\|_{l^p_q}.$$

(ii) *If $\omega \in \mathcal{D}$ such that $\frac{(1-|\cdot|)^p \hat{\omega}(\cdot)^p}{(1-|\cdot|)^2} \in \mathcal{D}$, then $\mathcal{T}_\mu \in \mathcal{S}_{p,q}$ if and only if $\widetilde{\mathcal{T}_\mu} \in \mathrm{K}^p_q(\lambda)$.*

Moreover,

$$\|\mathcal{T}_\mu\|_{\mathcal{S}_{p,q}} \approx \|\widetilde{\mathcal{T}_\mu}\|_{\mathrm{K}^p_q(\lambda)}.$$

Throughout this paper, the letter C will denote constants and may differ from one occurrence to the other. The notation $A \lesssim B$ means that there is a positive constant C such that $A \leq CB$. The notation $A \approx B$ means $A \lesssim B$ and $B \lesssim A$.

2. The proof of main result

In this section, we prove the main result in this paper. For this purpose, let's recall some related definitions and state some lemmas.

For $j = 0, 1, 2, \ldots$ and $k = 0, 1, 2, \ldots, 2^j - 1$, let

$$I_{j,k} = \left\{ e^{\mathrm{i}\theta} : \frac{2\pi k}{2^j} \leq \theta < \frac{2\pi(k+1)}{2^j} \right\}$$

and

$$R_{j,k} = \left\{ z \in \mathbb{D} : \frac{z}{|z|} \in I_{j,k}, 1 - \frac{|I_{j,k}|}{2\pi} \leq |z| < 1 - \frac{|I_{j,k}|}{4\pi} \right\}.$$

Lemma 2.1. *Suppose $0 < s < r < \infty$ are given and $\{a_n\}_{n=1}^{\infty}$ is s-separated. Then the following statements hold.*

(i) *For any $z \in \mathbb{D}$, there exist at most $N_1 = N_1(r)$ elements of $\{R_{j,k}\}$ intersecting with $D(z, r)$;*
(ii) *For any $R_{j,k}$, there exist at most $N_2 = N_2(r, s)$ elements of $\{D(a_n, r)\}$ intersecting with $R_{k,j}$.*

Proof. (i) Suppose $z \in \Lambda_i$ and $w \in D(z,r)$. Then we have $1-|z| \approx 1-|w|$. So there exists $M = M(r)$ such that $w \in \cup_{m=i-M}^{i+M} \Lambda_m$. Here, let $\Lambda_m = \emptyset$ if $m < 0$. By Proposition 4.4 in [12], $D(z,r)$ is an Euclidean disk with center $C_0 = \frac{1-(\tanh r)^2}{1-(\tanh r)^2|z|^2} z$ and radius $R_0 = \frac{1-|z|^2}{1-(\tanh r)^2|z|^2} \tanh r$. So, for all $w \in D(z,r)$, we have

$$|\mathrm{Arg}(\overline{z}w)| \leq \arcsin \frac{R_0}{C_0} \approx 1-|z| \text{ as } |z| \to 1.$$

When $j = i-M, \ldots, i+M$ and $k = 1,2,\ldots,2^j-1$, $|I_{j,k}| \approx \frac{1}{2^i} \approx 1-|z|$. So, there exists $N_1 = N_1(r)$ such that there are at most N_1 elements of $\{R_{j,k}\}$ intersecting with $D(z,r)$.

(ii) Let $c_{t,j,k} = (1-\frac{1}{2^t})e^{\frac{2\pi i k}{2^j}}$. For any $w \in R_{j,k}$ as $j \to \infty$, we have

$$\beta(c_{j,j,k}, w) \leq \beta(c_{j,j,k}, c_{j+1,j,k}) + \beta(c_{j,j,k}, c_{j,j,k+1}) \lesssim 1.$$

So, there exists $R' > 0$, for all $j = 0,1,2,\ldots$, $k = 0,1,\ldots,2^j-1$ and $w \in R_{j,k}$, we have $\beta(c_{j,j,k}, w) < R'$.

Without loss of generality, assume $R_{j,k}$ intersect with $D(a_i,r)$ for $i = 1,2,\ldots,N_{j,k}$. We have

$$1-|c_{j,j,k}| \approx 1-|a_i|, \ i = 1,2,\ldots,N_{j,k}$$

and

$$\cup_{i=1}^{N_{j,k}} D(a_i,s) \subset \cup_{i=1}^{N_{j,k}} D(a_i,r) \subset D(c_{j,j,k}, R'+2r).$$

Thus

$$N_{j,k} \leq \frac{|D(c_{j,j,k}, R'+2r)|}{\inf_{1 \leq i \leq N_{j,k}} |D(a_i,s)|} \lesssim 1,$$

which implies the desired result. The proof is complete. □

The following lemma is a main result in [10] and plays an important role in the studying of Toeplitz operators on A^2_ω.

Lemma 2.2. *Let $0 < p < \infty$ and $\omega \in \hat{\mathcal{D}}$. Then the following assertions hold.*

(i) $M_p^p(r, B_z^\omega) \approx \int_0^{r|z|} \frac{1}{\hat{\omega}(t)^p(1-t)^p} dt$, $r|z| \to 1$.

(ii) *If* $\upsilon \in \hat{\mathcal{D}}$, $\|B_z^\omega\|_{A^p_\upsilon}^p \approx \int_0^{|z|} \frac{\hat{\upsilon}(t)}{\hat{\omega}(t)^p(1-t)^p} dt$, $|z| \to 1$.

Here and hence forth, $M_p^p(r, B_z^\omega) = \frac{1}{2\pi}\int_0^{2\pi} |B_z^\omega(re^{i\theta})|^p d\theta$.

For any measurable function f, let

$$B_\omega f(z) = \frac{1}{\|B_z^\omega\|_{A^2_\omega}^2} \int_{\mathbb{D}} f(w)|B_z^\omega(w)|^2 \frac{\hat{\omega}(w) dA(w)}{1-|w|}.$$

Then we have the following lemma.

Lemma 2.3. *Let $\omega \in \mathcal{D}$ and $1 \leq p < \infty$. Then there exists $\varepsilon > 0$ such that $B_\omega : L^p_\tau \to L^p_\tau$ is bounded when $\tau = -2 \pm \varepsilon$.*

Proof. Since $\omega \in \mathcal{D}$, by Lemmas A and B in [11], there are constants $0 < a < b < \infty$ such that

$$\frac{\hat{\omega}(t)}{(1-t)^b} \nearrow \infty \quad \text{and} \quad \frac{\hat{\omega}(t)}{(1-t)^a} \searrow 0, \quad \text{when} \quad 0 \le t < 1. \tag{1}$$

Suppose $0 < \varepsilon < a$. Then $(1-|\cdot|)^{-1\pm\varepsilon}\hat{\omega}(\cdot) \in \mathcal{R}$.

We only prove the case of $\tau = -2-\varepsilon$. The case of $\tau = -2+\varepsilon$ can be proved in the same way.

Suppose $p = 1$. By Lemma 2.2,

$$B_z^\omega(z) = \|B_z^\omega\|_{A_\omega^2}^2 \approx \frac{1}{(1-|z|)\hat{\omega}(z)}, \quad z \in \mathbb{D}. \tag{2}$$

For all $f \in L_\tau^p$, by Fubini's Theorem, (2) and Lemma 2.2(ii), we have

$$\begin{aligned}
\|B_\omega f\|_{L_\tau^1} &= \int_{\mathbb{D}} \frac{1}{\|B_z^\omega\|_{A_\omega^2}^2} \left| \int_{\mathbb{D}} f(w)|B_z^\omega(w)|^2 \frac{\hat{\omega}(w)dA(w)}{1-|w|} \right| (1-|z|^2)^{-2-\varepsilon} dA(z) \\
&\lesssim \int_{\mathbb{D}} |f(w)| \frac{\hat{\omega}(w)}{1-|w|} \left(\int_{\mathbb{D}} |B_w^\omega(z)|^2 (1-|z|)^{-1-\varepsilon} \hat{\omega}(z) dA(z) \right) dA(w) \\
&\lesssim \int_{\mathbb{D}} |f(w)| \frac{\hat{\omega}(w)}{1-|w|} \left(\int_0^{\frac{|w|+1}{2}} \frac{1}{(1-t)^{2+\varepsilon}\hat{\omega}(t)} dt \right) dA(w).
\end{aligned}$$

Since $0 < \varepsilon < a$,

$$\begin{aligned}
\int_0^{\frac{|w|+1}{2}} \frac{1}{(1-t)^{2+\varepsilon}\hat{\omega}(t)} dt &\lesssim \frac{(1-|w|)^a}{\hat{\omega}(w)} \int_0^{\frac{|w|+1}{2}} \frac{1}{(1-t)^{2+a+\varepsilon}} dt \\
&\lesssim \frac{1}{(1-|w|)^{1+\varepsilon}\hat{\omega}(w)}.
\end{aligned}$$

So, $\|B_\omega f\|_{L_\tau^p}^p \lesssim \|f\|_{L_\tau^p}^p$ when $p = 1$.

Suppose $1 < p < \infty$. Let $p' = \frac{p}{p-1}$, $h(z) = (1-|z|)^s$ with $0 < s < \min\{\frac{a}{p'}, \frac{a}{p}\}$. Set

$$H(z,w) = \frac{|B_z^\omega(w)|^2 \hat{\omega}(w)}{\|B_z^\omega\|_{A_\omega^2}^2 (1-|w|)(1-|w|^2)^{-2-\varepsilon}}.$$

Then $B_\omega f(z) = \int_{\mathbb{D}} f(w)H(z,w)dA_\tau(w)$. On one hand,

$$\begin{aligned}
\int_{\mathbb{D}} H(z,w)h(w)^{p'} dA_\tau(w) &= \int_{\mathbb{D}} \frac{|B_z^\omega(w)|^2 \hat{\omega}(w)(1-|w|)^{p's-1}}{\|B_z^\omega\|_{A_\omega^2}^2} dA(w) \\
&\lesssim (1-|z|)\hat{\omega}(z) \int_0^{\frac{|z|+1}{2}} \frac{1}{(1-t)^{2-p's}\hat{\omega}(t)} dt \\
&\lesssim \frac{(1-|z|)^{1+a}\hat{\omega}(z)}{\hat{\omega}(z)} \int_0^{\frac{|z|+1}{2}} \frac{1}{(1-t)^{2+a-p's}} dt \\
&\lesssim h(z)^{p'}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}\int_{\mathbb{D}} H(z,w)h(z)^p dA_\tau(z) &\approx \int_{\mathbb{D}} \frac{|B_z^\omega(w)|^2\hat{\omega}(w)(1-|z|)^{-2-\varepsilon+ps}}{\|B_z^\omega\|^2_{A^2_\omega}(1-|w|^2)^{-1-\varepsilon}} dA(z)\\ &\approx \frac{\hat{\omega}(w)}{(1-|w|)^{-1-\varepsilon}}\int_{\mathbb{D}} |B_w^\omega(z)|^2(1-|z|)^{-1-\varepsilon+ps}\hat{\omega}(z)dA(z)\\ &\lesssim \frac{\hat{\omega}(w)}{(1-|w|)^{-1-\varepsilon}}\int_0^{\frac{|w|+1}{2}} \frac{1}{(1-t)^{2+\varepsilon-ps}\hat{\omega}(t)}dt\\ &\lesssim h(w)^p.\end{aligned}$$

By Schur's test, see [12, Theorem 3.6] for example, we have that $B_\omega : L^p_\tau \to L^p_\tau$ is bounded when $1 < p < \infty$. The proof is complete. □

Lemma 2.4. *Suppose $\omega \in \mathcal{D}$, $1 \le p \le \infty$ and $0 \le q \le \infty$. Then B_ω is bounded on $\mathrm{K}^p_q(\lambda)$.*

Proof. When $1 \le p < \infty$, choose $\varepsilon > 0$ such that Lemma 2.3 holds. For $f \in \mathrm{K}^p_q(\lambda)$, by Lemma 2.3, when $\tau = -2 \pm \varepsilon$, for $j = 0, 1, 2, \ldots$, we have

$$\begin{aligned}\|B_\omega(f\chi_j)\|^p_{L^p_{\lambda\chi_k}} &= \int_{1-\frac{1}{2^k}\le|z|<1-\frac{1}{2^{k+1}}} |B_\omega(f\chi_j)(z)|^p \frac{dA(z)}{(1-|z|^2)^2}\\ &\approx 2^{k(\tau+2)}\int_{1-\frac{1}{2^k}\le|z|<1-\frac{1}{2^{k+1}}} |B_\omega(f\chi_j)(z)|^p dA_\tau(z)\\ &\lesssim 2^{k(\tau+2)}\int_{\mathbb{D}} |f(z)\chi_j(z)|^p dA_\tau(z) \approx 2^{(\tau+2)(k-j)}\|f\chi_j\|^p_{L^p_\lambda}.\end{aligned}$$

When $j \le k$, letting $\tau = -2-\varepsilon$, we get

$$\|B_\omega(f\chi_j)\|^p_{L^p_{\lambda\chi_k}} \lesssim 2^{-\varepsilon|j-k|}\|f\chi_j\|^p_{L^p_\lambda}.$$

When $j > k$, letting $\tau = -2+\varepsilon$, we obtain

$$\|B_\omega(f\chi_j)\|^p_{L^p_{\lambda\chi_k}} \lesssim 2^{-\varepsilon|j-k|}\|f\chi_j\|^p_{L^p_\lambda}.$$

Therefore, when $1 \le p < \infty$,

$$\begin{aligned}\|(B_\omega f)\chi_k\|_{L^p_\lambda} &= \left\| B_\omega\left(\sum_{j=0}^\infty f\chi_j\right)\right\|_{L^p_{\lambda\chi_k}}\\ &\le \sum_{j=0}^\infty \|B_\omega(f\chi_j)\|_{L^p_{\lambda\chi_k}} \lesssim \sum_{j=0}^\infty \frac{\|f\chi_j\|_{L^p_\lambda}}{2^{\frac{\varepsilon|j-k|}{p}}}.\end{aligned} \tag{3}$$

When $p = \infty$,

$$\|(B_\omega f)\chi_k\|_{L^\infty_\lambda} = \|B_\omega f\|_{L^\infty_{\lambda\chi_k}} \le \left(\sum_{j=0}^k + \sum_{j=k+1}^\infty\right)\|B_\omega(f\chi_j)\|_{L^\infty_{\lambda\chi_k}}. \tag{4}$$

After a calculation,

(5)
$$
\begin{aligned}
&\|B_\omega(f\chi_j)\|_{L^\infty_{\lambda\chi_k}} \\
&\le \sup_{1-\frac{1}{2^k}\le|z|<1-\frac{1}{2^{k+1}}} \frac{1}{\|B_z^\omega\|^2_{A^2_\omega}} \int_{1-\frac{1}{2^j}\le|w|<1-\frac{1}{2^{j+1}}} |f(w)||B_z^\omega(w)|^2 \frac{\hat\omega(w)dA(w)}{1-|w|} \\
&\lesssim \|f\|_{L^\infty_{\lambda\chi_j}} \frac{1}{2^k}\hat\omega(1-\frac{1}{2^k}) \int_{1-\frac{1}{2^j}}^{1-\frac{1}{2^{j+1}}} \frac{\hat\omega(r)dr}{1-r} \int_0^{(1-\frac{1}{2^{j+1}})(1-\frac{1}{2^{k+1}})} \frac{1}{(1-t)^2\hat\omega(t)^2}dt \\
&\approx \|f\|_{L^\infty_{\lambda\chi_j}} \frac{1}{2^k}\hat\omega(1-\frac{1}{2^{k+1}})\hat\omega(1-\frac{1}{2^{j+1}}) \int_0^{x_{j,k}} \frac{1}{(1-t)^2\hat\omega(t)^2}dt.
\end{aligned}
$$

Here $x_{j,k} = (1-\frac{1}{2^{j+1}})(1-\frac{1}{2^{k+1}})$.

By (1),

$$
\begin{aligned}
\int_0^{x_{j,k}} \frac{1}{(1-t)^2\hat\omega(t)^2}dt &\lesssim \frac{(1-x_{j,k})^{2a}}{\hat\omega(x_{j,k})^2} \int_0^{x_{j,k}} \frac{1}{(1-t)^{2+2a}}dt \\
&\lesssim \frac{(1-x_{j,k})^{-1}}{\hat\omega(x_{j,k})^2} = \frac{\left(1-(1-\frac{1}{2^{k+1}})(1-\frac{1}{2^{j+1}})\right)^{-1}}{\left(\hat\omega((1-\frac{1}{2^{k+1}})(1-\frac{1}{2^{j+1}}))\right)^2}. \qquad (6)
\end{aligned}
$$

Let $r_j = \frac{1}{2^{j+1}}$. Since $\frac{1-\left(1-\frac{1}{2^{j+1}}-\frac{1}{2^{k+1}}\right)}{1-(1-\frac{1}{2^{k+1}})(1-\frac{1}{2^{j+1}})} \approx 1$, by (5) and (6) we have

$$
\begin{aligned}
\|B_\omega(f\chi_j)\|_{L^\infty_{\lambda\chi_k}} &\lesssim \|f\|_{L^\infty_{\lambda\chi_j}} \frac{\hat\omega(1-\frac{1}{2^{k+1}})\hat\omega(1-\frac{1}{2^{j+1}})}{(1+2^{k-j})\left(\hat\omega(1-\frac{1}{2^{k+1}}-\frac{1}{2^{j+1}})\right)^2} \\
&= \|f\|_{L^\infty_{\lambda\chi_j}} \frac{\hat\omega(1-r_k)\hat\omega(1-r_j)}{(1+2^{k-j})\,\hat\omega(1-r_k-r_j)^2}.
\end{aligned}
$$

Therefore, when $j \ge k$, using the monotonicity of $\hat\omega$, we have

$$
\begin{aligned}
\|B_\omega(f\chi_j)\|_{L^\infty_{\lambda\chi_k}} &\lesssim \|f\|_{L^\infty_{\lambda\chi_j}} \frac{\hat\omega(1-r_j)}{\hat\omega(1-r_k-r_j)} \\
&= \|f\|_{L^\infty_{\lambda\chi_j}} \frac{\frac{\hat\omega(1-r_j)}{(1-(1-r_j))^a}}{\frac{\hat\omega(1-r_k-r_j)}{(1-(1-r_k-r_j))^a}} \frac{r_j^a}{(r_k+r_j)^a} \lesssim \frac{\|f\|_{L^\infty_{\lambda\chi_j}}}{2^{a(j-k)}}. \qquad (7)
\end{aligned}
$$

When $j < k$, it is obvious that

$$
\|B_\omega(f\chi_j)\|_{L^\infty_{\lambda\chi_k}} \lesssim \frac{\|f\|_{L^\infty_{\lambda\chi_j}}}{2^{k-j}}. \qquad (8)
$$

From (4), (7) and (8), we have

$$
\|(B_\omega f)\chi_k\|_{L^\infty_\lambda} \le \sum_{j=0}^\infty \frac{\|f\chi_j\|_{L^\infty_\lambda}}{2^{\min\{1,a\}|j-k|}}. \qquad (9)
$$

By (3) and (9), there exists $\varepsilon_0 > 0$ such that

$$\|(B_\omega f)\chi_k\|_{L^p_\lambda} \lesssim \sum_{j=0}^{\infty} \frac{\|f\chi_j\|_{L^p_\lambda}}{2^{\varepsilon_0|j-k|}},$$

when $1 \le p \le \infty$.

When $1 \le q \le \infty$, we consider the sequences $X = \{x_k\}$, $Y = \{y_k\}$, where $x_k = 2^{-\varepsilon_0|k|}$ and

$$y_k = \begin{cases} \|f\chi_k\|_{L^p_\lambda}, & k \ge 0, \\ 0, & k < 0. \end{cases}$$

Then $\|(B_\omega f)\chi_k\|_{L^p_\lambda} \lesssim X * Y(k)$. Here, $X * Y$ means the convolution of X and Y. By Young's inequality,

$$\begin{aligned}\|B_\omega f\|_{\mathrm{K}^p_q(\lambda)} &= \|\{\|(B_\omega f)\chi_k\|_{L^p_\lambda}\}_{k=0}^{\infty}\|_{l^q} \\ &\lesssim \left\| \left\{ \sum_{j=0}^{\infty} \frac{\|f\chi_j\|_{L^p_\lambda}}{2^{\varepsilon_0|j-k|}} \right\}_{k=0}^{\infty} \right\|_{l^q} = \|X * Y\|_{l^q} \lesssim \|X\|_{l^1}\|Y\|_{l^q} \approx \|f\|_{\mathrm{K}^p_q(\lambda)}.\end{aligned}$$

When $0 < q < 1$,

$$\begin{aligned}\|B_\omega f\|^q_{K^p_q(\lambda)} &= \sum_{k=0}^{\infty} \left(\|(B_\omega f)\chi_k\|_{L^p_\lambda}\right)^q \lesssim \sum_{k=0}^{\infty} \left(\sum_{j=0}^{\infty} \frac{\|f\chi_j\|_{L^p_\lambda}}{2^{\varepsilon_0|j-k|}} \right)^q \\ &\le \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \frac{\|f\chi_j\|^q_{L^p_\lambda}}{2^{q\varepsilon_0|j-k|}} \le \sum_{j=0}^{\infty} \|f\chi_j\|^q_{L^p_\lambda} \sum_{k=-\infty}^{\infty} 2^{-q\varepsilon_0|k|} \approx \|f\|^q_{\mathrm{K}^p_q(\lambda)}.\end{aligned}$$

If $q = 0$, for any given $M \in \mathbb{N}$,

$$\begin{aligned}\|B_\omega f\|_{K^p_q(\lambda)} &= \limsup_{k\to\infty} \|(B_\omega f)\chi_k\|_{L^p_\lambda} \lesssim \limsup_{k\to\infty} \sum_{j=0}^{\infty} \frac{\|f\chi_j\|_{L^p_\lambda}}{2^{\varepsilon_0|j-k|}} \\ &= \limsup_{k\to\infty} \left(\sum_{j=0}^{M} \frac{\|f\chi_j\|_{L^p_\lambda}}{2^{\varepsilon_0|j-k|}} + \sum_{j=M+1}^{\infty} \frac{\|f\chi_j\|_{L^p_\lambda}}{2^{\varepsilon_0|j-k|}} \right) \lesssim \sup_{j\ge M+1} \|f\chi_j\|_{L^p_\lambda}.\end{aligned}$$

Letting $M \to \infty$, we have $\|B_\omega f\|_{K^p_q(\lambda)} \lesssim \|f\|_{K^p_q(\lambda)}$. The proof is complete. □

Proof of Theorem 1.1. (i) By Lemma 2.1, for any $z \in \Lambda_k$, there exists $N = N(r)$ such that $D(z,r) \subset \cup_{j=k-N}^{k+N} \Lambda_j$. Here $\Lambda_j = \emptyset$ if $j < 0$. Thus,

$$(\widehat{\mu_r}\chi_k)(z) \le \sum_{j=k-N}^{k+N} \frac{\mu(\Lambda_j \cap D(z,r))}{(1-|z|)\hat{\omega}(z)} = \sum_{j=k-N}^{k+N} \widehat{(\mu\chi_j)}_r(z)$$

and

$$\widehat{(\mu\chi_j)}_r(z) = \frac{\mu(\Lambda_j \cap D(z,r))}{(1-|z|)\hat{\omega}(z)} \le \frac{\mu(D(z,r))}{(1-|z|)\hat{\omega}(z)} \sum_{k=j-N}^{j+N} \chi_k(z) = \sum_{k=j-N}^{j+N} (\widehat{\mu_r}\chi_k)(z).$$

Therefore, by Theorem 3 in [11],

$$\|\mathcal{T}_\mu\|_{\mathcal{S}_{p,q}} = \|\{\|\mathcal{T}_{\mu\chi_k}\|_{\mathcal{S}_p}\}_{k=0}^{\infty}\|_{l^q} \approx \|\{\|\widehat{(\mu\chi_k)_r}\|_{L^p_\lambda}\}_{k=0}^{\infty}\|_{l^q}$$
$$\approx \|\{\|\widehat{\mu_r}\chi_k\|_{L^p_\lambda}\}_{k=0}^{\infty}\|_{l^q} = \|\widehat{\mu_r}\|_{\mathrm{K}^p_q(\lambda)}$$

and

$$\|\mathcal{T}_\mu\|_{\mathcal{S}_{p,q}} = \|\{\|\mathcal{T}_{\mu\chi_k}\|_{\mathcal{S}_p}\}_{k=0}^{\infty}\|_{l^q} \approx \|\{\|\{\widehat{(\mu\chi_k)}_r(a_j)\}_{j=1}^{\infty}\|_{l^p}\}_{k=0}^{\infty}\|_{l^q}$$
$$\approx \|\{\|\{(\widehat{\mu_r}\chi_k)(a_j)\}_{j=1}^{\infty}\|_{l^p}\}_{k=0}^{\infty}\|_{l^q} = \|\{\widehat{\mu_r}(a_j)\}\|_{l^p_q}.$$

(ii) Suppose $r > 0$ such that Lemma 8 in [11] holds. That is, for all $z \in \mathbb{D}$ and $w \in D(z, r)$, $|B^\omega_z(w)| \approx B^\omega_z(z)$. By Lemma 11 in [11],

$$\langle \mathcal{T}_\mu B^\omega_z, B^\omega_z\rangle_{A^2_\omega} = \langle B^\omega_z, B^\omega_z\rangle_{L^2_\mu}.$$

Using (2), we get

$$\widehat{\mu_r}(z) = \frac{\mu(D(z,r))}{(1-|z|)\hat{\omega}(z)} \approx \frac{1}{B^\omega_z(z)}\int_{D(z,r)} |B^\omega_z(w)|^2 d\mu(w) \le \widetilde{\mathcal{T}_\mu}(z).$$

So,

$$\|\mathcal{T}_\mu\|_{\mathcal{S}_{p,q}} \approx \|\widehat{\mu_r}\|_{\mathrm{K}^p_q(\lambda)} \lesssim \|\widetilde{\mathcal{T}_\mu}\|_{\mathrm{K}^p_q(\lambda)}.$$

On the other hand, since $|B^\omega_z(w)|^2$ is subharmonic and $\omega \in \mathcal{D}$, by Fubini's theorem,

$$\widetilde{\mathcal{T}_\mu}(z) = \frac{1}{B^\omega_z(z)}\int_{\mathbb{D}} |B^\omega_z(w)|^2 d\mu(w) \lesssim \frac{1}{B^\omega_z(z)}\int_{\mathbb{D}}\int_{D(w,r)} \frac{|B^\omega_z(\eta)|^2}{(1-|\eta|)^2} dA(\eta)d\mu(w)$$
$$= \frac{1}{B^\omega_z(z)}\int_{\mathbb{D}} \widehat{\mu_r}(\eta)|B^\omega_z(\eta)|^2 \frac{\hat{\omega}(\eta)dA(\eta)}{1-|\eta|} = B_\omega(\widehat{\mu_r})(z).$$

By Lemma 2.4 we have

$$\|\widetilde{\mathcal{T}_\mu}\|_{\mathrm{K}^p_q(\lambda)} \lesssim \|B_\omega(\widehat{\mu_r})\|_{\mathrm{K}^p_q(\lambda)} \lesssim \|\widehat{\mu_r}\|_{\mathrm{K}^p_q(\lambda)} \approx \|\mathcal{T}_\mu\|_{\mathcal{S}_{p,q}},$$

when $1 \le p \le \infty$.

Next we prove the case of $p < 1$. For $z \in \mathbb{D}$,

$$\widetilde{\mathcal{T}_\mu}(z) = \frac{1}{B^\omega_z(z)}\int_{\mathbb{D}} |B^\omega_z(w)|^2 d\mu(w) \le \frac{1}{B^\omega_z(z)}\sum_{j=1}^{\infty}\int_{D(a_j,r)} |B^\omega_z(w)|^2 d\mu(w)$$
$$\le \frac{1}{B^\omega_z(z)}\sum_{j=1}^{\infty}\mu(D(a_j,r)) \sup_{w\in\overline{D(a_j,r)}} |B^\omega_z(w)|^2.$$

Using the subharmonicity of $|B^\omega_z|^{2p}$,

$$\left(\widetilde{\mathcal{T}_\mu}(z)\right)^p \le \frac{1}{(B^\omega_z(z))^p}\sum_{j=1}^{\infty}(\mu(D(a_j,r)))^p \sup_{w\in\overline{D(a_j,r)}} |B^\omega_z(w)|^{2p}$$
$$\lesssim \frac{1}{(B^\omega_z(z))^p}\sum_{j=1}^{\infty}(\widehat{\mu_r}(a_j))^p\, \frac{(1-|a_j|)^p\hat{\omega}(a_j)^p}{(1-|a_j|)^2}\int_{D(a_j,2r)} |B^\omega_z(w)|^{2p} dA(w).$$

By Fubini's theorem and Lemma 2.2(i),

$$\begin{aligned}&\int_{\Lambda_k}\int_{D(a_j,2r)}|B_z^\omega(w)|^{2p}dA(w)d\lambda(z)\\ \approx\ & 2^{2k}\int_{D(a_j,2r)}\int_{\Lambda_k}|B_z^\omega(w)|^{2p}dA(z)dA(w)\\ \lesssim\ & 2^k\int_{D(a_j,2r)}\int_0^{(1-\frac{1}{2^{k+1}})\frac{|w|+1}{2}}\frac{dtdA(w)}{\hat{\omega}(t)^{2p}(1-t)^{2p}}.\end{aligned}$$

Let $\upsilon(z)=\frac{(1-|z|)^p\hat{\omega}(z)^p}{(1-|z|)^2}$. Since $\upsilon\in\mathcal{D}$, there exist $C_1,C_2>1$ and $K>1$, such that

$$\hat{\upsilon}(t)=\int_t^{\frac{1+t}{2}}\upsilon(s)ds+\hat{\upsilon}(\frac{1+t}{2})>\int_t^{\frac{1+t}{2}}\upsilon(s)ds+\frac{1}{C_1}\hat{\upsilon}(t)$$

and

$$\hat{\upsilon}(t)=\int_t^{1-\frac{1-t}{K}}\upsilon(s)ds+\hat{\upsilon}(1-\frac{1-t}{K})<\int_t^{1-\frac{1-t}{K}}\upsilon(s)ds+\frac{1}{C_2}\hat{\upsilon}(t).$$

So,

$$(1-t)\upsilon(t)\approx\int_t^{\frac{1+t}{2}}\upsilon(s)ds\lesssim\hat{\upsilon}(t)\lesssim\int_t^{1-\frac{1-t}{K}}\upsilon(s)ds\approx(1-t)\upsilon(t),$$

which implies that $\upsilon\in\mathcal{R}$. Therefore,

$$\begin{aligned}&\int_0^{(1-\frac{1}{2^{k+1}})\frac{|w|+1}{2}}\frac{1}{\hat{\omega}(t)^{2p}(1-t)^{2p}}dt\\ \approx\ &\int_0^{(1-\frac{1}{2^{k+1}})\frac{|w|+1}{2}}\frac{1}{\hat{\upsilon}(t)^2(1-t)^2}dt\\ \lesssim\ &\frac{1}{\left(\hat{\upsilon}((1-\frac{1}{2^{k+1}})\frac{|w|+1}{2})\right)^2\left(1-(1-\frac{1}{2^{k+1}})\frac{|w|+1}{2}\right)}.\end{aligned}$$

Since $w\in D(a_j,r)$, we get

$$1-(1-\frac{1}{2^{k+1}})\frac{|a_j|+1}{2}\approx1-(1-\frac{1}{2^{k+1}})\frac{|w|+1}{2}.$$

Hence,

$$\begin{aligned}&\int_{\Lambda_k}\int_{D(a_j,2r)}|B_z^\omega(w)|^{2p}dA(w)d\lambda(z)\\ \lesssim\ &\frac{2^k(1-|a_j|)^2}{\left(\hat{\upsilon}((1-\frac{1}{2^{k+1}})\frac{|a_j|+1}{2})\right)^2\left(1-(1-\frac{1}{2^{k+1}})\frac{|a_j|+1}{2}\right)}.\end{aligned}$$

Let $\xi_m = \left(\sum_{a_j\in\Lambda_m}(\widehat{\mu_r}(a_j))^p\right)^{\frac{1}{p}}$. Then

$$\begin{aligned}
&\int_{\Lambda_k}(\widetilde{\mathcal{T}_\mu}(z))^p d\lambda(z)\\
\lesssim\ & \frac{1}{2^{(k+1)p}}\hat{\omega}(1-\frac{1}{2^{k+1}})^p\\
&\times\sum_{m=0}^{\infty}\xi_m^p\frac{\frac{1}{2^{(m+1)p}}\hat{\omega}(1-\frac{1}{2^{m+1}})^p}{\frac{1}{2^{2(m+1)}}}\frac{\frac{2^k}{2^{2(m+1)}}}{\left(\hat{\upsilon}(1-\frac{1}{2^{k+1}}-\frac{1}{2^{m+1}})\right)^2(\frac{1}{2^{k+1}}+\frac{1}{2^{m+1}})}\\
\approx\ & \sum_{m=0}^{\infty}\xi_m^p\frac{\hat{\upsilon}(1-\frac{1}{2^{m+1}})\hat{\upsilon}(1-\frac{1}{2^{k+1}})}{\left(\hat{\upsilon}(1-\frac{1}{2^{m+1}}-\frac{1}{2^{k+1}})\right)^2}\frac{1}{1+2^{m-k}}.
\end{aligned}$$

Similarly to get (7) and (8), there exists $\varepsilon>0$ such that

$$\int_{\Lambda_k}\left(\widetilde{\mathcal{T}_\mu}(z)\right)^p d\lambda(z)\lesssim\sum_{m=0}^{\infty}\xi_m^p\cdot\frac{1}{2^{\varepsilon|k-m|}}.$$

Note that

$$\|\{\xi_m\}\|_{l^q}=\left\|\left\{\left(\sum_{j=1}^{\infty}((\widehat{\mu_r}\chi_m)(a_j))^p\right)^{\frac{1}{p}}\right\}_{m=0}^{\infty}\right\|_{l^q}=\|\{\widehat{\mu_r}(a_j)\}\|_{l^p_q}\approx\|\mathcal{T}_\mu\|_{\mathcal{S}_{p,q}}.$$

If $0<p<q=\infty$, then

$$\|\widetilde{\mathcal{T}_\mu}\|_{\mathrm{K}^p_\infty(\lambda)}=\sup_{k\ge 0}\left(\int_{\Lambda_k}\left(\widetilde{\mathcal{T}_\mu}(z)\right)^p d\lambda(z)\right)^{\frac{1}{p}}\lesssim\|\{\xi_m\}\|_{l^\infty}\approx\|\mathcal{T}_\mu\|_{\mathcal{S}_{p,\infty}}.$$

When $0<p<q<\infty$, let $X=\{x_k\}$ and $Y=\{y_k\}$, where $x_k=\frac{1}{2^{\varepsilon|k|}}$ and

$$y_k=\begin{cases}\xi_k^p, & k\ge 0,\\ 0, & k<0.\end{cases}$$

Then,

$$\begin{aligned}
\|\widetilde{\mathcal{T}_\mu}\|_{\mathrm{K}^p_q(\lambda)}&=\left(\sum_{k=0}^{\infty}\left(\int_{\Lambda_k}\left(\widetilde{\mu}(z)\right)^p d\lambda(z)\right)^{\frac{q}{p}}\right)^{\frac{1}{q}}\lesssim\left(\sum_{k=0}^{\infty}\left(\sum_{m=0}^{\infty}\xi_m^p\cdot\frac{1}{2^{\varepsilon|k-m|}}\right)^{\frac{q}{p}}\right)^{\frac{1}{q}}\\
&=\|X*Y\|_{l^{\frac{q}{p}}}^{\frac{1}{p}}\le\left(\|X\|_{l^1}\|Y\|_{l^{\frac{q}{p}}}\right)^{\frac{1}{p}}\lesssim\|\xi_m\|_{l^q}\approx\|\mathcal{T}_\mu\|_{\mathcal{S}_{p,q}}.
\end{aligned}$$

If $0<q\le p<1$, then

$$\|\widetilde{\mathcal{T}_\mu}\|_{\mathrm{K}^p_q(\lambda)}=\left(\sum_{k=0}^{\infty}\left(\int_{\Lambda_k}\left(\widetilde{\mathcal{T}_\mu}(z)\right)^p d\lambda(z)\right)^{\frac{q}{p}}\right)^{\frac{1}{q}}$$

$$\lesssim \left(\sum_{k=0}^{\infty}\left(\sum_{m=0}^{\infty}\xi_m^p\cdot\frac{1}{2^{\varepsilon|k-m|}}\right)^{\frac{q}{p}}\right)^{\frac{1}{q}}$$

$$\lesssim \left(\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\xi_m^q\cdot\frac{1}{2^{\frac{q\varepsilon|k-m|}{p}}}\right)^{\frac{1}{q}}$$

$$\lesssim \left(\sum_{m=0}^{\infty}\xi_m^q\sum_{k=-\infty}^{\infty}\cdot\frac{1}{2^{\frac{q\varepsilon|k|}{p}}}\right)^{\frac{1}{q}} \lesssim \|\xi_m\|_{l^q} \approx \|\mathcal{T}_\mu\|_{\mathcal{S}_{p,q}}.$$

When $0 = q < p < 1$, for any given $M \in \mathbb{N}$,

$$\limsup_{k\to\infty}\int_{\Lambda_k}\left(\widetilde{\mathcal{T}_\mu}(z)\right)^p d\lambda(z) \lesssim \limsup_{k\to\infty}\left(\sum_{m=0}^{M}+\sum_{m=M+1}^{\infty}\right)\xi_m^p\cdot\frac{1}{2^{\varepsilon|k-m|}} \leq \sup_{m>M}\xi_m^p.$$

Letting $M \to \infty$, we obtain

$$\limsup_{k\to\infty}\left(\int_{\Lambda_k}\left(\widetilde{\mathcal{T}_\mu}(z)\right)^p d\lambda(z)\right)^{\frac{1}{p}} \lesssim \limsup_{m\to\infty}\xi_m.$$

So,

$$\|\widetilde{\mathcal{T}_\mu}\|_{\mathrm{K}_0^p(\lambda)} \lesssim \|\xi_m\|_{l^0} \approx \|\mathcal{T}_\mu\|_{\mathcal{S}_{p,q}}.$$

The proof is complete. □

References

[1] B. Choe, H. Koo, and K. Na, *Positive Toeplitz operators of Schatten-Herz type*, Nagoya Math. J. **185** (2007), 31–62. https://doi.org/10.1017/S0027763000025733

[2] E. S. Choi and K. Na, *Schatten-Herz type positive Toeplitz operators on pluriharmonic Bergman spaces*, J. Math. Anal. Appl. **327** (2007), no. 1, 679–694. https://doi.org/10.1016/j.jmaa.2006.04.032

[3] Z. Hu and X. Tang, *Schatten(-Herz) class extended Cesàro operators on Bergman spaces in the unit ball of* $\mathbf{C}^n$, Proc. Amer. Math. Soc. **138** (2010), no. 8, 2803–2814. https://doi.org/10.1090/S0002-9939-10-10365-7

[4] M. Loaiza, M. López-García, and S. Pérez-Esteva, *Herz classes and Toeplitz operators in the disk*, Integral Equations Operator Theory **53** (2005), no. 2, 287–296. https://doi.org/10.1007/s00020-004-1320-x

[5] X. F. Lü and Z. J. Hu, *Schatten(-Herz) classes of Toeplitz operators on weighted Bergman spaces*, Acta Math. Sinica (Chin. Ser.) **53** (2010), no. 4, 699–716.

[6] ———, *Schatten-Herz class Toeplitz operators on harmonic Bergman spaces of smooth domains in* $\mathbb{R}^n$, Math. Nachr. **286** (2013), no. 2-3, 248–259. https://doi.org/10.1002/mana.201100062

[7] J. Peláez, *Small weighted Bergman spaces*, in Proceedings of the Summer School in Complex and Harmonic Analysis, and Related Topics, 29–98, Publ. Univ. East. Finl. Rep. Stud. For. Nat. Sci., 22, Univ. East. Finl., Fac. Sci. For., Joensuu, 2016.

[8] J. Peláez and J. Rättyä, *Weighted Bergman spaces induced by rapidly increasing weights*, Mem. Amer. Math. Soc. **227** (2014), no. 1066, vi+124 pp.

[9] ———, *Trace class criteria for Toeplitz and composition operators on small Bergman spaces*, Adv. Math. **293** (2016), 606–643. https://doi.org/10.1016/j.aim.2016.02.017

[10] ———, *Two weight inequality for Bergman projection*, J. Math. Pures Appl. (9) **105** (2016), no. 1, 102–130. https://doi.org/10.1016/j.matpur.2015.10.001
[11] J. Peláez, J. Rättyä, and K. Sierra, *Berezin transform and Toeplitz operators on Bergman spaces induced by regular weights*, J. Geom. Anal. **28** (2018), no. 1, 656–687. https://doi.org/10.1007/s12220-017-9837-9
[12] K. Zhu, *Operator theory in function spaces*, second edition, Mathematical Surveys and Monographs, 138, American Mathematical Society, Providence, RI, 2007. https://doi.org/10.1090/surv/138

Juntao Du
Department of Mathematics
Shantou University
Shantou, Guangdong 515063, P. R. China
Email address: jtdu007@163.com

Songxiao Li
Institute of Fundamental and Frontier Sciences
University of Electronic Science and Technology of China
610054, Chengdu, Sichuan, P. R. China
Email address: jyulsx@163.com