

## REAL-VARIABLE CHARACTERIZATIONS OF VARIABLE HARDY SPACES ON LIPSCHITZ DOMAINS OF $\mathbb{R}^n$

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ABSTRACT. Let  $\Omega$  be a proper open subset of  $\mathbb{R}^n$  and  $p(\cdot) : \Omega \rightarrow (0, \infty)$  be a variable exponent function satisfying the globally log-Hölder continuous condition. In this article, the author introduces the “geometrical” variable Hardy spaces  $H_r^{p(\cdot)}(\Omega)$  and  $H_z^{p(\cdot)}(\Omega)$  on  $\Omega$ , and then obtains the grand maximal function characterizations of  $H_r^{p(\cdot)}(\Omega)$  and  $H_z^{p(\cdot)}(\Omega)$  when  $\Omega$  is a strongly Lipschitz domain of  $\mathbb{R}^n$ . Moreover, the author further introduces the “geometrical” variable local Hardy spaces  $h_r^{p(\cdot)}(\Omega)$ , and then establishes the atomic characterization of  $h_r^{p(\cdot)}(\Omega)$  when  $\Omega$  is a bounded Lipschitz domain of  $\mathbb{R}^n$ .

### 1. Introduction

The real-variable theory of Hardy spaces  $H^p(\Omega)$  with  $p \in (0, 1]$  on domains of  $\mathbb{R}^n$  and their duals are well studied (see, for example, [14]) and have been playing an important and fundamental role in the boundary value problems for the Laplace equation. In recent years, there has been a lot of attention paid to the study of Hardy spaces on domains of  $\mathbb{R}^n$ , which has become a very active research topic in harmonic analysis (see, for instance, [1, 3–7, 13, 16]).

Originally, Chang et al. [7] introduced the Hardy spaces  $H_r^p(\Omega)$  and  $H_z^p(\Omega)$  on domains  $\Omega$  of  $\mathbb{R}^n$ , respectively, by restricting arbitrary elements of  $H^p(\mathbb{R}^n)$  to  $\Omega$ , and restricting elements of  $H^p(\mathbb{R}^n)$  which are zero outside  $\bar{\Omega}$  to  $\Omega$ , where and in what follows,  $\bar{\Omega}$  denotes the closure of  $\Omega$  in  $\mathbb{R}^n$ . For the Hardy spaces  $H_r^p(\Omega)$  and  $H_z^p(\Omega)$ , atomic characterizations have been obtained in [7] when  $\Omega$  is a special Lipschitz domain or a bounded Lipschitz domain of  $\mathbb{R}^n$ , and grand maximal function characterizations have been established in [8] when  $\Omega$  is a strongly Lipschitz domain of  $\mathbb{R}^n$ . Moreover, Chang et al. [7] also introduced the local Hardy spaces  $h_r^p(\Omega)$  and  $h_z^p(\Omega)$  in a similar way and obtained atomic

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decompositions for these local Hardy spaces when  $\Omega$  is a bounded Lipschitz domain of  $\mathbb{R}^n$ .

On the other hand, as a natural generalization of classical Hardy spaces, Nakai and Sawano [15] introduced the variable Hardy space  $H^{p(\cdot)}(\mathbb{R}^n)$ , established their atomic characterizations and investigated their dual spaces. Independently, Cruz-Uribe and Wang [10] also investigated the variable Hardy space  $H^{p(\cdot)}(\mathbb{R}^n)$  with  $p(\cdot)$  satisfying some conditions slightly weaker than those used in [15]. In [10], equivalent characterizations of  $H^{p(\cdot)}(\mathbb{R}^n)$  by means of radial or non-tangential maximal functions or atoms were established. Moreover, Yang et al. [19, 22] established equivalent characterizations of variable Hardy space  $H^{p(\cdot)}(\mathbb{R}^n)$  via Riesz transforms and intrinsic square functions. Furthermore, Tan [17] introduced the variable local Hardy space  $h^{p(\cdot)}(\mathbb{R}^n)$  and established atomic characterizations for  $h^{p(\cdot)}(\mathbb{R}^n)$  by using the discrete Littlewood-Paley-Stein theory.

As a more general class of function spaces including both Hardy spaces on Euclidean spaces with variable exponents  $H^{p(\cdot)}(\mathbb{R}^n)$  and Hardy spaces on RD-spaces with constant exponents  $H^p(\mathcal{X})$ . Recently, Zhuo et al. [20] introduced the variable Hardy space  $H^{*,p(\cdot)}(\mathcal{X})$  on the so-called RD-space with infinite measures via the grand maximal function, and then obtained its several real-variable characterizations, respectively, in terms of atoms and Littlewood-Paley functions.

Motivated by the above results, especially by the theory of the classical Hardy space on domains in [5, 7, 8] and the variable Hardy space in [10, 15, 20], it is the main target of this article to establish a real-variable theory of the “geometrical” variable (local) Hardy spaces on a proper open subset  $\Omega$  in  $\mathbb{R}^n$ . Precisely, let  $\Omega$  be a proper open subset of  $\mathbb{R}^n$  and  $p(\cdot) : \Omega \rightarrow (0, \infty)$  be a variable exponent function satisfying the globally log-Hölder continuous condition. In this article, the author introduces the “geometrical” variable Hardy spaces  $H_r^{p(\cdot)}(\Omega)$  and  $H_z^{p(\cdot)}(\Omega)$  on  $\Omega$ , and then obtains the grand maximal function characterizations of  $H_r^{p(\cdot)}(\Omega)$  and  $H_z^{p(\cdot)}(\Omega)$  when  $\Omega$  is a strongly Lipschitz domain of  $\mathbb{R}^n$ . Moreover, the author further introduces the “geometrical” variable local Hardy spaces  $h_r^{p(\cdot)}(\Omega)$ , and then establishes the atomic characterization of  $h_r^{p(\cdot)}(\Omega)$  when  $\Omega$  is a bounded Lipschitz domain of  $\mathbb{R}^n$ .

To state the main results of this article, we begin with recall some notation and notions. Let  $\Omega$  be an open subset in  $\mathbb{R}^n$ . A measurable function  $p(\cdot) : \Omega \rightarrow (0, \infty)$  is called a *variable exponent*. Moreover, for any given variable exponent  $p(\cdot)$ , let

$$(1.1) \quad p_- := \operatorname{ess\,inf}_{x \in \Omega} p(x), \quad p_+ := \operatorname{ess\,sup}_{x \in \Omega} p(x) \quad \text{and} \quad \underline{p} := \min\{p_-, 1\}.$$

Denote by  $\mathcal{P}(\Omega)$  the set of all variable exponents  $p(\cdot)$  on  $\Omega$  satisfying  $0 < p_- \leq p_+ < \infty$ .

Let  $f$  be a measurable function on  $\Omega$  and  $p(\cdot) \in \mathcal{P}(\Omega)$ . Then the *modular function* (for simplicity, the *modular*)  $\varrho_{p(\cdot)}$ , associated with  $p(\cdot)$ , is defined by

setting

$$\varrho_{p(\cdot)}(f) := \int_{\Omega} |f(x)|^{p(x)} dx$$

and the *Luxemburg* (also called *Luxemburg–Nakano*) *quasi-norm*  $\|f\|_{L^{p(\cdot)}(\Omega)}$  of  $f$  is defined by

$$\|f\|_{L^{p(\cdot)}(\Omega)} := \inf \{ \lambda \in (0, \infty) : \varrho_{p(\cdot)}(f/\lambda) \leq 1 \}.$$

Furthermore, the *variable Lebesgue space*  $L^{p(\cdot)}(\Omega)$  is defined to the set of all measurable functions  $f$  on  $\Omega$  satisfying that  $\varrho_{p(\cdot)}(f) < \infty$ , equipped with the quasi-norm  $\|f\|_{L^{p(\cdot)}(\Omega)}$ .

A function  $p(\cdot) \in \mathcal{P}(\Omega)$  is said to satisfy the *globally log-Hölder continuous condition*, denoted by  $p(\cdot) \in C^{\log}(\Omega)$ , if there exist positive constants  $C_{\log}, C_{\infty} \in (0, \infty)$  and  $p_{\infty} \in \mathbb{R}$ , where  $p_{\infty} := \lim_{x \rightarrow \infty} p(x)$ , such that, for any  $x, y \in \Omega$ ,

$$|p(x) - p(y)| \leq \frac{C_{\log}}{\log(e + 1/|x - y|)} \quad \text{and} \quad |p(x) - p_{\infty}| \leq \frac{C_{\infty}}{\log(e + |x|)}.$$

In the whole article, we denote by  $\mathcal{S}(\mathbb{R}^n)$  the *space of all Schwartz functions* and by  $\mathcal{S}'(\mathbb{R}^n)$  its *topological dual space*. For  $N \in \mathbb{N} := \{1, 2, \dots\}$  and  $\mathbb{Z}_+ := \{0\} \cup \mathbb{N}$ , let

$$(1.2) \quad \mathcal{F}_N(\mathbb{R}^n) := \left\{ \psi \in \mathcal{S}(\mathbb{R}^n) : \sum_{\alpha \in \mathbb{Z}_+^n, |\alpha| \leq N} \sup_{x \in \mathbb{R}^n} (1 + |x|)^N |\partial^{\alpha} \psi(x)| \leq 1 \right\},$$

where, for any  $\alpha := (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n := (\mathbb{Z}_+)^n$ ,  $|\alpha| := \alpha_1 + \dots + \alpha_n$  and  $\partial^{\alpha} := (\frac{\partial}{\partial x_1})^{\alpha_1} \dots (\frac{\partial}{\partial x_n})^{\alpha_n}$ . Then, for all  $f \in \mathcal{S}'(\mathbb{R}^n)$ , the *grand maximal function*  $M_{\psi}(f)$  of  $f$  is defined by setting, for all  $x \in \mathbb{R}^n$ ,

$$(1.3) \quad M_{\psi}(f)(x) := \sup \{ |f * \psi_t(x)| : t \in (0, \infty) \text{ and } \psi \in \mathcal{F}_N(\mathbb{R}^n) \},$$

where, for all  $t \in (0, \infty)$  and  $x \in \mathbb{R}^n$ ,  $\psi_t(x) := t^{-n} \psi(x/t)$ .

We begin with recall the definition of the variable Hardy space  $H^{p(\cdot)}(\mathbb{R}^n)$ , which can be found in [15, Definition 1.1].

**Definition 1.1.** Let  $p(\cdot) \in C^{\log}(\mathbb{R}^n)$  and  $N \in (n/(p_-) + n + 1, \infty) \cap \mathbb{N}$ , where  $p_-$  is as in (1.1). The *variable Hardy space* denoted by  $H^{p(\cdot)}(\mathbb{R}^n)$ , is defined to be the set of all  $f \in \mathcal{S}'(\mathbb{R}^n)$  such that  $M_{\psi}(f) \in L^{p(\cdot)}(\mathbb{R}^n)$  with the quasi-norm

$$\|f\|_{H^{p(\cdot)}(\mathbb{R}^n)} := \|M_{\psi}(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)},$$

where  $M_{\psi}(f)$  is as in (1.3).

For an open subset  $\Omega \subset \mathbb{R}^n$ , let  $\mathcal{D}(\Omega)$  denote the *space of all infinitely differentiable functions with compact supports in  $\Omega$*  equipped with the inductive topology and  $\mathcal{D}'(\Omega)$  its *topological dual* equipped with the weak-\* topology, which is called the *space of distributions on  $\Omega$* . Then we introduce the “geometric” variable Hardy spaces  $H_z^{p(\cdot)}(\Omega)$  and  $H_r^{p(\cdot)}(\Omega)$  on proper open subset  $\Omega \subset \mathbb{R}^n$  following the way in [7, 8].

**Definition 1.2.** Let  $\Omega \subset \mathbb{R}^n$  be a proper open subset and  $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ . Then the *variable Hardy space*  $H_z^{p(\cdot)}(\Omega)$  is defined by setting

$$H_z^{p(\cdot)}(\Omega) := \left\{ f \in \mathcal{D}'(\mathbb{R}^n) : f \in H^{p(\cdot)}(\mathbb{R}^n), \text{supp}(f) \subset \overline{\Omega} \right\}$$

equipped with the *quasi-norm*  $\|f\|_{H_z^{p(\cdot)}(\Omega)} := \|f\|_{H^{p(\cdot)}(\mathbb{R}^n)}$ , where  $\overline{\Omega}$  denotes the closure of  $\Omega$  in  $\mathbb{R}^n$ .

A distribution  $f$  on  $\Omega$  is said to belong to the *variable Hardy space*  $H_r^{p(\cdot)}(\Omega)$  if  $f$  is the restriction to  $\Omega$  of a distribution  $F \in H^{p(\cdot)}(\mathbb{R}^n)$ , namely,

$$\begin{aligned} H_r^{p(\cdot)}(\Omega) &:= \left\{ f \in \mathcal{D}'(\Omega) : \text{there exists an } F \in H^{p(\cdot)}(\mathbb{R}^n) \text{ such that } F|_{\Omega} = f \right\} \\ &= H^{p(\cdot)}(\mathbb{R}^n) / \left\{ F \in H^{p(\cdot)}(\mathbb{R}^n) : F = 0 \text{ on } \Omega \right\}. \end{aligned}$$

Moreover, for any  $f \in H_r^{p(\cdot)}(\Omega)$ , the *quasi-norm*  $\|f\|_{H_r^{p(\cdot)}(\Omega)}$  of  $f$  in  $H_r^{p(\cdot)}(\Omega)$  is defined by setting

$$\|f\|_{H_r^{p(\cdot)}(\Omega)} := \inf \left\{ \|F\|_{H^{p(\cdot)}(\mathbb{R}^n)} : F \in H^{p(\cdot)}(\mathbb{R}^n) \text{ and } F|_{\Omega} = f \right\},$$

where the infimum is taken over all  $F \in H^{p(\cdot)}(\mathbb{R}^n)$  satisfying  $F = f$  on  $\Omega$ .

Our first main result is the grand maximal function characterizations of the variable Hardy spaces  $H_z^{p(\cdot)}(\Omega)$  and  $H_r^{p(\cdot)}(\Omega)$  on a strongly Lipschitz domain  $\Omega$  of  $\mathbb{R}^n$ . To this end, we recall the following definition of grand maximal functions (see, [8]). In what follows, for any  $q \in [1, \infty]$ , we denote by  $q'$  its *conjugate index*, namely,  $1/q + 1/q' = 1$ .

For any  $x \in \mathbb{R}^n$  and  $\frac{n}{n+1} < p_- \leq p_+ \leq 1$ , where  $p_-$  and  $p_+$  are as in (1.1), denote by  $F_x(\Omega)$  the collection of all  $\phi \in \mathcal{D}(\mathbb{R}^n)$ , for which there exists a cube  $Q$  such that  $\text{supp}(\phi) \subset Q$ ,  $x \in Q$ ,  $c_Q \in \Omega$  and

$$\|\phi\|_{L^\infty(\mathbb{R}^n)} + \ell(Q)\|\nabla\phi\|_{L^\infty(\mathbb{R}^n)} \leq |Q|^{-1},$$

here and hereafter,  $c_Q$  denotes the *center* of the cube  $Q$  and  $\ell(Q)$  its *sidelength*. For each  $x \in \Omega$ , let

$$G_x(\Omega) := \{\phi \in F_x(\Omega) : \phi = 0 \text{ on } \partial\Omega\}.$$

For each  $f \in \mathcal{D}'(\mathbb{R}^n)$  and any  $x \in \mathbb{R}^n$ , let

$$M_z(f)(x) := \sup_{\phi \in F_x(\Omega)} |\langle f, \phi \rangle|.$$

Let  $p^* := \frac{np_-}{n-p_-}$ , where  $p_-$  is as in (1.1),  $q := \frac{p^*}{p^*-1}$  and  $W_0^{1,q}(\Omega)$  denote the *Sobolev space with zero boundary values* on  $\Omega$ . For each bounded linear functional  $f$  on  $W_0^{1,q}(\Omega)$ , for any  $x \in \Omega$ , let

$$M_r(f)(x) := \sup_{\phi \in G_x(\Omega)} |\langle f, \phi \mathbf{1}_\Omega \rangle|,$$

where  $\mathbf{1}_\Omega$  denotes the *characteristic function* of  $\Omega$ . From the fact that  $\phi \in G_x(\Omega)$ , it follows that  $\phi \mathbf{1}_\Omega \in W_0^{1,q}(\Omega)$ , which implies that  $M_r(f)$  is well defined.

For any  $x \in \mathbb{R}^n$ ,  $\frac{n}{n+1} < p_- \leq p_+ \leq 1$ , where  $p_-$  and  $p_+$  are as in (1.1), and  $\frac{p_-^*}{p_-^*-1} < q < \infty$ , we denote by  $F_x^q(\Omega)$  the collection of all  $\phi \in \mathcal{D}(\mathbb{R}^n)$ , which satisfy  $\text{supp}(\phi) \subset Q$ ,  $x \in Q$ ,  $c_Q \in \Omega$  and

$$\|\phi\|_{L^q(\mathbb{R}^n)} + \ell(Q)\|\nabla\phi\|_{L^q(\mathbb{R}^n)} \leq |Q|^{-1/q'}.$$

Similarly, for each  $x \in \Omega$ , we let

$$G_x^q(\Omega) := \{\phi \in F_x^q(\Omega) : \phi = 0 \text{ on } \partial\Omega\}.$$

We then let, for  $f \in \mathcal{D}'(\mathbb{R}^n)$  and  $x \in \mathbb{R}^n$ ,

$$M_z^{(q)}(f)(x) := \sup_{\phi \in F_x^q(\Omega)} |\langle f, \phi \rangle|.$$

For each bounded linear functional  $f$  on  $W_0^{1,q}(\Omega)$  and for any  $x \in \Omega$ , let

$$M_r^{(q)}(f)(x) := \sup_{\phi \in G_x^q(\Omega)} |\langle f, \phi \mathbf{1}_\Omega \rangle|.$$

From the fact that  $\phi \in G_x^q(\Omega)$ , it follows that  $\phi \mathbf{1}_\Omega \in W_0^{1,q}(\Omega)$ , which implies that  $M_r^{(q)}(f)$  is well defined.

A domain  $\Omega$  of  $\mathbb{R}^n$  ( $n \geq 2$ ) is said to be *strongly Lipschitz* if it is a Lipschitz domain and its boundary  $\partial\Omega$  is a finite union of parts of rotated graphs of Lipschitz maps and, at most one of these parts possibly unbounded. Moreover, a domain  $\Omega \subset \mathbb{R}^n$  is said to be a *special Lipschitz domain*, i.e.,  $\Omega := \{(x', x_n) : x_n > \lambda(x')\}$ . Here  $\lambda : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$  is a function which satisfies the Lipschitz condition  $|\lambda(x') - \lambda(y')| \leq A|x' - y'|$  for all  $x', y' \in \mathbb{R}^{n-1}$ . It is well known that strongly Lipschitz domains include special Lipschitz domains, bounded Lipschitz domains and exterior domains (see, for example, [1, 2, 7, 11]). Then we have the following grand maximal function characterizations of  $H_z^{p(\cdot)}(\Omega)$  and  $H_r^{p(\cdot)}(\Omega)$  on a strongly Lipschitz domain  $\Omega$  of  $\mathbb{R}^n$ .

**Theorem 1.3.** *Let  $\Omega \subset \mathbb{R}^n$  be a strongly Lipschitz domain,  $p(\cdot) \in C^{\log}(\Omega)$  and  $\frac{n}{n+1} < p_- \leq p_+ \leq 1$ , where  $p_-$  and  $p_+$  are as in (1.1).*

- (i) *If  $\Omega$  is bounded, then  $f \in H_z^{p(\cdot)}(\Omega)$  if and only if  $f \in \mathcal{D}'(\mathbb{R}^n)$ ,  $\text{supp}(f) \subset \bar{\Omega}$ ,  $M_z(f) \in L^{p(\cdot)}(\Omega)$  and  $\langle f, \phi \rangle = 0$  for each  $\phi \in \mathcal{D}(\mathbb{R}^n)$  with  $\phi \equiv 1$  on  $\Omega$ . Moreover, there exists a positive constant  $C$  independent of  $f$ , such that*

$$C^{-1} \|M_z(f)\|_{L^{p(\cdot)}(\Omega)} \leq \|f\|_{H_z^{p(\cdot)}(\Omega)} \leq C \|M_z(f)\|_{L^{p(\cdot)}(\Omega)}.$$

- (ii) *If  $\Omega$  is unbounded, then  $f \in H_z^{p(\cdot)}(\Omega)$  if and only if  $f \in \mathcal{D}'(\mathbb{R}^n)$ ,  $\text{supp}(f) \subset \bar{\Omega}$  and  $M_z(f) \in L^{p(\cdot)}(\Omega)$ . Moreover, there exists a positive constant  $C$  independent of  $f$ , such that*

$$C^{-1} \|M_z(f)\|_{L^{p(\cdot)}(\Omega)} \leq \|f\|_{H_z^{p(\cdot)}(\Omega)} \leq C \|M_z(f)\|_{L^{p(\cdot)}(\Omega)}.$$

- (iii) Assume that  $\frac{p^*}{p^*-1} < q \leq \infty$ . If  $\Omega$  is bounded, then  $f \in H_z^{p(\cdot)}(\Omega)$  if and only if  $f \in \mathcal{D}'(\mathbb{R}^n)$ ,  $\text{supp}(f) \subset \overline{\Omega}$ ,  $M_z^{(q)}(f) \in L^{p(\cdot)}(\Omega)$  and  $\langle f, \phi \rangle = 0$  for each  $\phi \in \mathcal{D}(\mathbb{R}^n)$  with  $\phi \equiv 1$  on  $\Omega$ . Moreover, there exists a positive constant  $C$  independent of  $f$ , such that

$$C^{-1} \left\| M_z^{(q)}(f) \right\|_{L^{p(\cdot)}(\Omega)} \leq \|f\|_{H_z^{p(\cdot)}(\Omega)} \leq C \left\| M_z^{(q)}(f) \right\|_{L^{p(\cdot)}(\Omega)}.$$

- (iv) Assume that  $\frac{p^*}{p^*-1} < q \leq \infty$ . If  $\Omega$  is unbounded, then  $f \in H_z^{p(\cdot)}(\Omega)$  if and only if  $f \in \mathcal{D}'(\mathbb{R}^n)$ ,  $\text{supp}(f) \subset \overline{\Omega}$  and  $M_z^{(q)}(f) \in L^{p(\cdot)}(\Omega)$ . Moreover, there exists a positive constant  $C$  independent of  $f$ , such that

$$C^{-1} \left\| M_z^{(q)}(f) \right\|_{L^{p(\cdot)}(\Omega)} \leq \|f\|_{H_z^{p(\cdot)}(\Omega)} \leq C \left\| M_z^{(q)}(f) \right\|_{L^{p(\cdot)}(\Omega)}.$$

**Theorem 1.4.** Let  $\Omega \subset \mathbb{R}^n$  be a strongly Lipschitz domain,  $p(\cdot) \in C^{\log}(\Omega)$  and  $\frac{n}{n+1} < p_- \leq p_+ \leq 1$ , where  $p_-$  and  $p_+$  are as in (1.1).

- (i) Assume that  $q = (p^*)'$  and  $\Omega$  is bounded. Then  $f \in H_r^{p(\cdot)}(\Omega)$  if and only if  $f$  is a bounded linear functional on  $W_0^{1,q}(\Omega)$  and  $M_r(f) \in L^{p(\cdot)}(\Omega)$ . Moreover, there exists a positive constant  $C$  independent of  $f$ , such that

$$C^{-1} \|M_r(f)\|_{L^{p(\cdot)}(\Omega)} \leq \|f\|_{H_r^{p(\cdot)}(\Omega)} \leq C \|M_r(f)\|_{L^{p(\cdot)}(\Omega)}.$$

- (ii) Assume that  $\frac{p^*}{p^*-1} < q \leq \infty$ . Then  $f \in H_r^{p(\cdot)}(\Omega)$  if and only if  $f$  is a bounded linear functional on  $W_0^{1,q}(\Omega)$  and  $M_r^{(q)}(f) \in L^{p(\cdot)}(\Omega)$ . Moreover, there exists a positive constant  $C$  independent of  $f$ , such that

$$C^{-1} \left\| M_r^{(q)}(f) \right\|_{L^{p(\cdot)}(\Omega)} \leq \|f\|_{H_r^{p(\cdot)}(\Omega)} \leq C \left\| M_r^{(q)}(f) \right\|_{L^{p(\cdot)}(\Omega)}.$$

*Remark 1.5.* (i) Let  $p \in (\frac{n}{n+1}, 1]$  be a given constant. We point out that, if  $p(\cdot) := p$ , then Theorems 1.3 and 1.4 were established by Chen et al. in [8].

(ii) It is worth pointing out that in the process of the proofs of (i) and (ii) of Theorem 1.3 are composed by three steps: we first deal with the case  $\Omega$  being special Lipschitz, and then the case  $\Omega$  being bounded, finally  $\Omega$  being unbounded (see Section 2 below). Based on Theorems 1.3 and 1.4, [8, Corollaries 2.13 and 2.14] are also true under the setting of variable exponent function spaces  $H_r^{p(\cdot)}(\Omega)$  and  $H_z^{p(\cdot)}(\Omega)$ .

Our second main result concerning the atomic characterization of the variable local Hardy spaces  $h_r^{p(\cdot)}(\Omega)$ . We first introduce the notions of  $h^{p(\cdot)}(\mathbb{R}^n)$  and  $h_r^{p(\cdot)}(\Omega)$  as follows. The following definition was introduced by Tan in [17, Theorem 1.3].

**Definition 1.6.** Let  $f \in \mathcal{S}'(\mathbb{R}^n)$ ,  $p(\cdot) \in C^{\log}(\mathbb{R}^n)$  and  $x \in \mathbb{R}^n$ . Denote by  $M_{\text{loc}}(f)$  the grand maximal function given by

$$M_{\text{loc}}(f)(x) := \sup \{ |\phi_t * f(x)| : 0 < t < 1, \phi \in \mathcal{F}_N(\mathbb{R}^n) \}$$

for any fixed large integer  $N$ , where  $\mathcal{F}_N(\mathbb{R}^n)$  is as in (1.2). A distribution  $f$  on  $\mathbb{R}^n$  is in the variable local Hardy spaces  $h^{p(\cdot)}(\mathbb{R}^n)$  if and only if the grand maximal function  $M_{\text{loc}}(f)$  lies in  $L^{p(\cdot)}(\mathbb{R}^n)$ , i.e., for all  $f \in \mathcal{S}'(\mathbb{R}^n)$ ,

$$\|f\|_{h^{p(\cdot)}(\mathbb{R}^n)} \sim \|M_{\text{loc}}(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

**Definition 1.7.** Let  $\Omega \subset \mathbb{R}^n$  be a proper open subset and  $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ . A distribution  $f$  on  $\Omega$  is said to belong to the *variable local Hardy space*  $h_r^{p(\cdot)}(\Omega)$  if  $f$  is the restriction to  $\Omega$  of a distribution  $F \in h^{p(\cdot)}(\mathbb{R}^n)$ , namely,

$$\begin{aligned} h_r^{p(\cdot)}(\Omega) &:= \left\{ f \in \mathcal{D}'(\Omega) : \text{there exists an } F \in h^{p(\cdot)}(\mathbb{R}^n) \text{ such that } F|_{\Omega} = f \right\} \\ &= h^{p(\cdot)}(\mathbb{R}^n) / \left\{ F \in h^{p(\cdot)}(\mathbb{R}^n) : F = 0 \text{ on } \Omega \right\}. \end{aligned}$$

Moreover, for any  $f \in h_r^{p(\cdot)}(\Omega)$ , the *quasi-norm*  $\|f\|_{h_r^{p(\cdot)}(\Omega)}$  of  $f$  in  $h_r^{p(\cdot)}(\Omega)$  is defined by setting

$$\|f\|_{h_r^{p(\cdot)}(\Omega)} := \inf \left\{ \|F\|_{h^{p(\cdot)}(\mathbb{R}^n)} : F \in h^{p(\cdot)}(\mathbb{R}^n) \text{ and } F|_{\Omega} = f \right\},$$

where the infimum is taken over all  $F \in h^{p(\cdot)}(\mathbb{R}^n)$  satisfying  $F = f$  on  $\Omega$ .

In what follows, to establish the atomic characterization of the variable local Hardy space  $h_r^{p(\cdot)}(\Omega)$ , we introduce the notion of  $(p(\cdot), q)_{\Omega}$ -atoms.

**Definition 1.8.** Let  $\Omega$  be an open subset of  $\mathbb{R}^n$ ,  $p(\cdot) \in \mathcal{P}(\Omega)$  and  $q \in (1, \infty]$ . Assume that  $p_-$  and  $\underline{p}$  are as in (1.1).

- (i) A cube  $Q \subset \mathbb{R}^n$  is said to be of *type (a) cube* if  $4Q \subset \Omega$  with  $\ell(Q) < 1$ ; a cube  $\tilde{Q} \subset \mathbb{R}^n$  is said to be of *type (b) cube* if either  $\ell(Q) \geq 1$  or  $2\tilde{Q} \cap \Omega^c = \emptyset$  and  $4\tilde{Q} \cap \Omega^c \neq \emptyset$ .
- (ii) A measurable function  $a$  on  $\Omega$  is called a *type (a)  $(p(\cdot), q)_{\Omega}$ -atom* if
  - (ii)<sub>1</sub>  $\text{supp}(a) \subset Q$ , where  $\text{supp}(a) := \{x \in \mathbb{R}^n : a(x) \neq 0\}$  and  $Q$  is a type (a) cube;
  - (ii)<sub>2</sub>  $\|a\|_{L^q(\Omega)} \leq |Q|^{1/q} \|\mathbf{1}_Q\|_{L^{p(\cdot)}(\Omega)}^{-1}$ ;
  - (ii)<sub>3</sub> there exists an integer  $s \geq d_{p(\cdot)}$ , where  $d \geq d_{p(\cdot)} := \min\{d \in \mathbb{Z}_+ : p_-(n+d+1) > n\}$ , such that, for any  $\alpha \in \mathbb{Z}_+^n$  with  $|\alpha| \leq s$ ,  $\int_{\mathbb{R}^n} a(x)x^\alpha dx = 0$ .

Moreover, a measurable function  $b$  on  $\Omega$  is called a *type (b)  $(p(\cdot), q)_{\Omega}$ -atom* if  $\text{supp}(b) \subset \tilde{Q}$  with  $\tilde{Q}$  being a type (b) cube and  $\|b\|_{L^q(\Omega)} \leq |\tilde{Q}|^{1/q} \|\mathbf{1}_{\tilde{Q}}\|_{L^{p(\cdot)}(\Omega)}^{-1}$ . Furthermore, a measurable function  $a$  on  $\mathbb{R}^n$  is called a  *$(p(\cdot), q)$ -atom*, if it satisfies the conditions (ii)<sub>2</sub>, (ii)<sub>3</sub> above and  $\text{supp}(a) \subset Q \subset \mathbb{R}^n$ .

For a sequence  $\{\lambda_j\}_{j=1}^\infty \subset \mathbb{C}$  and a cubes sequence  $\{Q_j\}_{j=1}^\infty$  of the supports of atoms, define that

$$\mathcal{A}(\{\lambda_j\}_{j=1}^\infty, \{Q_j\}_{j=1}^\infty) := \left\| \left\{ \sum_{j=1}^\infty \left( \frac{|\lambda_j| \mathbf{1}_{Q_j}}{\|\mathbf{1}_{Q_j}\|_{L^{p(\cdot)}(\Omega)}} \right)^p \right\}^{1/p} \right\|_{L^{p(\cdot)}(\Omega)}.$$

Then we have the atomic characterization of the variable local Hardy space  $h_r^{p(\cdot)}(\Omega)$  as follows.

**Theorem 1.9.** *Assume that  $\Omega \subset \mathbb{R}^n$  is a bounded Lipschitz domain. Let  $p(\cdot) \in C^{\log}(\Omega)$  with  $\frac{n}{n+1} < p_- \leq p_+ \leq 1$  and  $q \in (1, \infty]$ , where  $p_-$  and  $p_+$  are as in (1.1). Then,  $f \in h_r^{p(\cdot)}(\Omega)$  if and only if there exist sequences  $\{\lambda_j\}_{j=1}^\infty$ ,  $\{\kappa_j\}_{j=1}^\infty \subset \mathbb{C}$ , type (a)  $(p(\cdot), q)_\Omega$ -atoms  $\{a_j\}_{j=1}^\infty$ , and type (b)  $(p(\cdot), q)_\Omega$ -atoms  $\{b_j\}_{j=1}^\infty$  such that  $f = \sum_{j=1}^\infty \lambda_j a_j + \sum_{j=1}^\infty \kappa_j b_j$  in  $\mathcal{D}'(\mathbb{R}^n)$ , and*

$$\mathcal{A}(\{\lambda_j\}_{j=1}^\infty, \{Q_j\}_{j=1}^\infty) + \mathcal{A}(\{\kappa_j\}_{j=1}^\infty, \{\tilde{Q}_j\}_{j=1}^\infty) < \infty,$$

where  $\{Q_j\}_{j=1}^\infty$  and  $\{\tilde{Q}_j\}_{j=1}^\infty$ , respectively, denote the supports of  $\{a_j\}_{j=1}^\infty$  and  $\{b_j\}_{j=1}^\infty$ . Moreover, for any given  $f \in h_r^{p(\cdot)}(\Omega)$ , there exists a positive constant  $C$  independent of  $f$ , such that

$$C^{-1} \|f\|_{h_r^{p(\cdot)}(\Omega)} \leq \mathcal{A}(\{\lambda_j\}_{j=1}^\infty, \{Q_j\}_{j=1}^\infty) + \mathcal{A}(\{\kappa_j\}_{j=1}^\infty, \{\tilde{Q}_j\}_{j=1}^\infty) \leq C \|f\|_{h_r^{p(\cdot)}(\Omega)}.$$

*Remark 1.10.* When  $p(\cdot) := p \in (\frac{n}{n+1}, 1]$ , then Theorem 1.9 is reduced to [7, Theorem 2.7].

The layout of this article is as follows. Section 2 is devoted to the proofs of Theorems 1.3 and 1.4. In Section 3, we give the proof of Theorem 1.9.

Finally, we make some conventions on notation. Let  $\mathbb{N} := \{1, 2, \dots\}$  and  $\mathbb{Z}_+ := \{0\} \cup \mathbb{N}$ . Throughout the whole article, we denote by  $C$  a positive constant which is independent of the main parameters, but it may vary from line to line. The symbol  $f \lesssim g$  means that  $f \leq Cg$ . If  $f \lesssim g$  and  $g \lesssim f$ , then we write  $f \sim g$ . Denote by  $Q(c_Q, \ell(Q))$  the cube in  $\mathbb{R}^n$  with center  $c_Q \in \mathbb{R}^n$  and sidelength  $\ell(Q) \in (0, \infty)$ , and  $\alpha \in (0, \infty)$ , let  $\alpha Q := Q(c_Q, \alpha \ell(Q))$ . For any measurable subset  $E$  of  $\mathbb{R}^n$ , we denote the set  $\mathbb{R}^n \setminus E$  by  $E^c$  and its characteristic function by  $\mathbf{1}_E$ . Moreover, denote by  $\mathcal{S}(\mathbb{R}^n)$  the space of all Schwartz functions and  $\mathcal{S}'(\mathbb{R}^n)$  its dual space (namely, the space of all tempered distributions). For any sets  $E, F \subset \mathbb{R}^n$  and  $z \in \mathbb{R}^n$ , let  $\text{dist}(E, F) := \inf\{|x - y| : x \in E, y \in F\}$  and  $\text{dist}(z, E) := \inf\{|z - x| : x \in E\}$ . Finally, we also use  $W_0^{1,q}(\Omega)$  to denote the collection of elements in Sobolev spaces  $W^{1,q}(\Omega)$  with zero boundary values. For  $q \in (0, n)$ , let  $q^*$  be its Sobolev conjugate index  $\frac{nq}{n-q}$ . For any given  $p \in [1, \infty]$ , we denote by  $p'$  its conjugate exponent, namely,  $1/p + 1/p' = 1$ .



## 2. Proofs of Theorems 1.3 and 1.4

In this section, we give the proofs of Theorems 1.3 and 1.4. We begin with recall some basic properties of variable Lebesgue space  $L^{p(\cdot)}(\mathbb{R}^n)$ . The following Lemmas 2.1 and 2.2 come from [15, Lemma 2.2] and [9, 18], respectively.

**Lemma 2.1.** *Let  $p(\cdot) \in C^{\log}(\mathbb{R}^n)$  and  $0 < p_- \leq p_+ < \infty$ , where  $p_-$  and  $p_+$  are as in (1.1).*

- (i) *For all cubes  $Q = Q(c_Q, \ell(Q))$  with  $c_Q \in \mathbb{R}^n$  and  $\ell(Q) \leq 1$ , we have  $|Q|^{1/p_-(Q)} \lesssim |Q|^{1/p_+(Q)}$ . In particular, we have*

$$|Q|^{1/p_-(Q)} \sim |Q|^{1/p_+(Q)} \sim |Q|^{1/p(c_Q)} \sim \|\mathbf{1}_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)};$$

- (ii) *For all cubes  $Q = Q(c_Q, \ell(Q))$  with  $c_Q \in \mathbb{R}^n$  and  $\ell(Q) \geq 1$ , we have  $\|\mathbf{1}_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)} \sim |Q|^{1/p_\infty}$ .*

Here the implicit constants in  $\sim$  do not depend on  $c_Q$  and  $\ell(Q) > 0$ .

**Lemma 2.2.** *Let  $p(\cdot) \in \mathcal{P}(\Omega)$ . Then, for any  $s \in (0, \infty)$ ,  $\lambda \in \mathbb{C}$ , and  $f \in L^{p(\cdot)}(\Omega)$ ,*

$$\| |f|^s \|_{L^{p(\cdot)}(\Omega)} = \|f\|_{L^{sp(\cdot)}(\Omega)}^s \quad \text{and} \quad \|\lambda f\|_{L^{p(\cdot)}(\Omega)} = |\lambda| \|f\|_{L^{p(\cdot)}(\Omega)}.$$

Recall that, for any  $f \in L^1_{\text{loc}}(\Omega)$  and  $x \in \Omega$ , the *Hardy–Littlewood maximal function*  $M(f)$  is defined by setting

$$M(f)(x) := \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)| dy,$$

where the supremum is taken over all balls  $B \subset \Omega$  satisfying  $B \ni x$ .

**Lemma 2.3.** *Let  $\Omega$  be an open subset of  $\mathbb{R}^n$  and  $r \in (1, \infty]$ . Assume that  $p(\cdot) \in C^{\log}(\Omega)$  satisfies  $1 < p_- \leq p_+ < \infty$ , where  $p_-$  and  $p_+$  are as in (1.1). Then there exists a positive constant  $C$  such that, for any sequence  $\{f_k\}_{k \in \mathbb{N}}$  of measurable functions on  $\Omega$ ,*

$$\left\| \left\{ \sum_{k \in \mathbb{N}} [M(f_k)]^r \right\}^{1/r} \right\|_{L^{p(\cdot)}(\Omega)} \leq C \left\| \left( \sum_{k \in \mathbb{N}} |f_k|^r \right)^{1/r} \right\|_{L^{p(\cdot)}(\Omega)}$$

with the usual modification made when  $r = \infty$ .

We point out that in the case of metric measurable spaces of homogeneous type, Lemma 2.3 was established in [20, Theorem 2.7]. Moreover, the proof of [20, Theorem 2.7] is also valid in the case of Lemma 2.3 and we omit the details here. The following remark is just [21, Remark 2.8].

*Remark 2.4.* Let  $k \in \mathbb{N}$  and  $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ . Then, from Lemmas 2.2 and 2.3, and the fact that, for any cubes  $Q$  of  $\mathbb{R}^n$ ,  $r \in (0, \underline{p})$ ,  $\mathbf{1}_{2^k Q} \lesssim 2^{kn/r} [M(\mathbf{1}_Q)]^{1/r}$ , where  $\underline{p}$  is as in (1.1) and  $M$  denotes the Hardy–Littlewood maximal function,

we deduce that there exists a positive constant  $C$  such that, for any  $\{\lambda_j\}_{j=1}^\infty \subset \mathbb{C}$  and cubes  $\{Q_j\}_{j=1}^\infty$  of  $\mathbb{R}^n$ ,

$$\left\| \left\{ \sum_{j=1}^{\infty} \left[ \frac{|\lambda_j| \mathbf{1}_{2^k Q_j}}{\|\mathbf{1}_{Q_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \right]^p \right\}^{1/p} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C 2^{kn/r} \mathcal{A}(\{\lambda_j\}_{j=1}^\infty, \{Q_j\}_{j=1}^\infty).$$

From the proof of [12, Theorem 1.12(i)] or see the proof of [8, lemma 2.3] with regular modification, it follows that the variable Hardy space  $H_r^{p(\cdot)}(\Omega)$  admits the following atomic decomposition.

**Lemma 2.5.** *Let  $p(\cdot) \in C^{\log}(\Omega)$ ,  $\frac{n}{n+1} < p_- \leq p_+ \leq 1$  and  $q \in (1, \infty]$ , where  $p_-$  and  $p_+$  are as in (1.1). Assume that  $\Omega \subset \mathbb{R}^n$  is a strongly Lipschitz domain. Then, for each  $f \in H_r^{p(\cdot)}(\Omega)$ , there exist type (a)  $(p(\cdot), q)_\Omega$ -atoms  $\{a_j\}_{j=1}^\infty$  and type (b)  $(p(\cdot), q)_\Omega$ -atoms  $\{b_j\}_{j=1}^\infty$  such that  $f = \sum_{j=1}^\infty \lambda_j a_j + \sum_{j=1}^\infty \kappa_j b_j$  in  $\mathcal{D}'(\mathbb{R}^n)$ , and*

$$\mathcal{A}(\{\lambda_j\}_{j=1}^\infty, \{Q_j\}_{j=1}^\infty) + \mathcal{A}(\{\kappa_j\}_{j=1}^\infty, \{\tilde{Q}_j\}_{j=1}^\infty) \leq C \|f\|_{H_r^{p(\cdot)}(\Omega)},$$

where  $\{Q_j\}_{j=1}^\infty$  and  $\{\tilde{Q}_j\}_{j=1}^\infty$ , respectively, denote the supports of  $\{a_j\}_{j=1}^\infty$  and  $\{b_j\}_{j=1}^\infty$ , and the positive constant  $C$  is independent of  $f$ .

**Lemma 2.6.** *Let  $\Omega$  be a strongly Lipschitz domain,  $p(\cdot) \in C^{\log}(\Omega)$ ,  $\frac{n}{n+1} < p_- \leq p_+ \leq 1$  and  $q = \frac{p^*}{p^*-1}$ , where  $p_-$  and  $p_+$  are as in (1.1). Then each  $f \in H_r^{p(\cdot)}(\Omega)$  induces a bounded linear functional on  $W_0^{1,q}(\Omega)$  and there exists a positive constant  $C$  such that, for all  $f \in H_r^{p(\cdot)}(\Omega)$  and  $g \in W_0^{1,q}(\Omega)$ ,*

$$|\langle f, g \rangle| \leq C \|f\|_{H_r^{p(\cdot)}(\Omega)} \|g\|_{W_0^{1,q}(\Omega)}.$$

*Proof.* Since  $\mathcal{D}(\Omega)$  is dense in  $W_0^{1,q}(\Omega)$ , we only need to show that Lemma 2.6 holds true for each  $\phi \in \mathcal{D}(\Omega)$ .

If  $p_- = p_+ = 1$ , then  $q = (p^*)' = n$ . For each  $f \in H_r^{p(\cdot)}(\Omega)$ , there exists  $F \in H^1(\mathbb{R}^n)$  such that  $F|_\Omega = f$  and  $\|F\|_{H^1(\mathbb{R}^n)} \leq 2\|f\|_{H_r^1(\Omega)}$ . By the duality of  $H^1(\mathbb{R}^n)$  and  $BMO(\mathbb{R}^n)$ , the embedding of  $W^{1,n}(\mathbb{R}^n) \subset BMO(\mathbb{R}^n)$  and similar to that of (2.2) in the proof of [8, Lemma 2.4], we know that

$$(2.1) \quad \left| \int_{\Omega} f(x) \phi(x) dx \right| \lesssim \|f\|_{H_r^1(\Omega)} \|\phi\|_{W^{1,n}(\Omega)}.$$

If  $\frac{n}{n+1} < p_- \leq p_+ < 1$ , then  $q = (p^*)' > n$ . For each  $f \in H_r^{p(\cdot)}(\Omega)$ , from Lemma 2.5, we deduce that there exist two sequences  $\{\lambda_j\}_{j=1}^\infty \subset \mathbb{C}$  and  $\{\kappa_j\}_{j=1}^\infty \subset \mathbb{C}$ , a sequence  $\{a_j\}_{j=1}^\infty$  of type (a)  $(p(\cdot), p^*)_\Omega$ -atoms and a sequence  $\{b_j\}_{j=1}^\infty$  of type (b)  $(p(\cdot), p^*)_\Omega$ -atoms such that  $f = \sum_{j=1}^\infty \lambda_j a_j + \sum_{j=1}^\infty \kappa_j b_j$  in

$\mathcal{D}'(\mathbb{R}^n)$ . Moreover, by [20, Lemma 5.9], we know that

$$(2.2) \quad \sum_{j=1}^{\infty} |\lambda_j| + \sum_{j=1}^{\infty} |\kappa_j| \lesssim \mathcal{A}(\{\lambda_j\}_{j=1}^{\infty}, \{Q_j\}_{j=1}^{\infty}) + \mathcal{A}(\{\kappa_j\}_{j=1}^{\infty}, \{\tilde{Q}_j\}_{j=1}^{\infty}).$$

Then for each type (a)  $(p(\cdot), p^*)_{\Omega}$ -atom  $a_j$ , from the moment condition of  $a_j$ ,  $\text{supp}(a_j) \subset Q_j$ ,  $\|a_j\|_{L^{p^*}(\Omega)} \leq |Q_j|^{1/p^*} \|\mathbf{1}_{Q_j}\|_{L^{p(\cdot)}(\Omega)}^{-1}$ , Lemma 2.1(i),  $p^* = \frac{np_-}{n-p_-}$ , and repeating the proof of [8, (2.3)], we deduce that

$$(2.3) \quad \begin{aligned} \left| \int_{\Omega} a_j(x) \phi(x) dx \right| &\lesssim \ell(Q_j) \int_0^1 \|a_j\|_{L^{p^*}(\Omega)} \left[ \int_{\Omega} |\nabla \phi(x)|^q dx \right]^{1/q} t^{-n/q} dt \\ &\lesssim \ell(Q_j) |Q_j|^{1/p^*} \|\mathbf{1}_{Q_j}\|_{L^{p(\cdot)}(\Omega)}^{-1} \int_0^1 t^{-n/q} dt \|\nabla \phi\|_{L^q(\Omega)} \\ &\lesssim [\ell(Q_j)]^{1+n/p^*} |Q_j|^{-1/p_-} \|\phi\|_{W^{1,q}(\Omega)} \sim \|\phi\|_{W^{1,q}(\Omega)}. \end{aligned}$$

For each type (b)  $(p(\cdot), p^*)_{\Omega}$ -atom  $b_j$ , by  $\text{supp}(b_j) \subset \tilde{Q}_j$ ,

$$\|b_j\|_{L^{p^*}(\Omega)} \leq |\tilde{Q}_j|^{1/p^*} \|\mathbf{1}_{\tilde{Q}_j}\|_{L^{p(\cdot)}(\Omega)}^{-1},$$

Lemma 2.1(ii),  $p^* = \frac{np_-}{n-p_-}$ , and a proof similar to those of [8, (2.4)], we know that

$$(2.4) \quad \begin{aligned} \left| \int_{\Omega} b_j(x) \phi(x) dx \right| &\lesssim \ell(\tilde{Q}_j) \int_0^1 \|b_j\|_{L^{p^*}(\Omega)} \|\nabla \phi\|_{L^q(\Omega)} t^{-n/q} dt \\ &\lesssim \ell(\tilde{Q}_j) |\tilde{Q}_j|^{1/p^*} \|\mathbf{1}_{\tilde{Q}_j}\|_{L^{p(\cdot)}(\Omega)}^{-1} \int_0^1 t^{-n/q} dt \|\nabla \phi\|_{L^q(\Omega)} \\ &\lesssim [\ell(\tilde{Q}_j)]^{1+n/p^*} |Q_j|^{-1/p_{\infty}} \|\phi\|_{W^{1,q}(\Omega)} \\ &\lesssim [\ell(\tilde{Q}_j)]^{n/p_- - n/p_{\infty}} \|\phi\|_{W^{1,q}(\Omega)} \lesssim \|\phi\|_{W^{1,q}(\Omega)}. \end{aligned}$$

Combining (2.2), (2.3), (2.4), and Lemma 2.5, we obtain

$$\begin{aligned} |\langle f, \phi \rangle| &\leq \sum_{j=1}^{\infty} |\langle \lambda_j a_j, \phi \rangle| + \sum_{j=1}^{\infty} |\langle \kappa_j b_j, \phi \rangle| \\ &\lesssim \left( \sum_{j=1}^{\infty} |\lambda_j| + \sum_{j=1}^{\infty} |\kappa_j| \right) \|\phi\|_{W^{1,q}(\Omega)} \lesssim \|f\|_{H_r^{p(\cdot)}(\Omega)} \|\phi\|_{W^{1,q}(\Omega)}, \end{aligned}$$

which, together with (2.1) and the density of  $\mathcal{D}(\Omega)$  in  $W_0^{1,q}(\Omega)$ , finishes the proof of Lemma 2.6.  $\square$

We first prove a weaker version of Theorem 1.3(i) as follows.

**Lemma 2.7.** *Let  $\Omega$  be a strongly Lipschitz domain,  $p(\cdot) \in C^{\log}(\Omega)$  and  $\frac{n}{n+1} < p_- \leq p_+ \leq 1$ , where  $p_-$  and  $p_+$  are as in (1.1). Then  $f \in H_z^{p(\cdot)}(\Omega)$  if and only*

if  $f \in \mathcal{D}'(\mathbb{R}^n)$ ,  $\text{supp}(f) \subset \bar{\Omega}$  and  $M_z(f) \in L^{p(\cdot)}(\mathbb{R}^n)$ . Moreover, there exists a positive constant  $C$  independent of  $f$ , such that

$$C^{-1} \|M_z(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq \|f\|_{H_z^{p(\cdot)}(\Omega)} \leq C \|M_z(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

*Proof.* Let  $f \in \mathcal{D}'(\mathbb{R}^n)$ ,  $\text{supp}(f) \subset \bar{\Omega}$  and  $M_z(f) \in L^{p(\cdot)}(\mathbb{R}^n)$ . Suppose that  $\psi \in \mathcal{D}(\mathbb{R}^n)$  is a radial function such that  $\text{supp}(\psi) \subset Q := Q(\bar{0}_n, 1)$  and  $\int_{\mathbb{R}^n} \psi(x) dx = 1$ , where and in what follows,  $\bar{0}_n$  denotes the origin of  $\mathbb{R}^n$ . Proceeding as in the proof of [8, Proposition 2.6], we know that

$$\sup_{t \in (0, \infty)} |f * \psi_t(x)| \lesssim M_z(f)(x),$$

which, together with (1.3) and Definition 1.1, implies that  $f \in H^{p(\cdot)}(\mathbb{R}^n)$ . Notice that  $\text{supp}(f) \subset \bar{\Omega}$ , this implies that  $f \in H_z^{p(\cdot)}(\Omega)$  and  $\|f\|_{H_z^{p(\cdot)}(\Omega)} \lesssim \|M_z(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)}$ .

On the contrary, let  $f \in H_z^{p(\cdot)}(\Omega)$ . By Definition 1.2 and [15, Theorem 4.6 and Definition 1.5], we conclude that  $f = \sum_{j=1}^{\infty} \lambda_j a_j$  is an atomic decomposition in  $H^{p(\cdot)}(\mathbb{R}^n)$ , where  $\{a_j\}_{j=1}^{\infty}$  are  $(p(\cdot), \infty)$ -atoms and  $\{\lambda_j\}_{j=1}^{\infty} \subset \mathbb{C}$  satisfy

$$\|f\|_{H_z^{p(\cdot)}(\Omega)} \sim \|f\|_{H^{p(\cdot)}(\mathbb{R}^n)} \sim \mathcal{A}(\{\lambda_j\}_{j=1}^{\infty}, \{Q_j\}_{j=1}^{\infty}).$$

Thus, we have

$$\begin{aligned} & \|M_z(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ & \leq \left\| \sum_{j=1}^{\infty} |\lambda_j| M_z(a_j) \mathbf{1}_{4Q_j} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} + \left\| \sum_{j=1}^{\infty} |\lambda_j| M_z(a_j) \mathbf{1}_{(4Q_j)^c} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ (2.5) \quad & =: \text{I} + \text{II}. \end{aligned}$$

For the term I, by  $x \in 4Q_j$  and a  $(p(\cdot), \infty)$ -atom  $a_j$  with  $\text{supp}(a_j) \subset Q_j$ , we have

$$\begin{aligned} M_z(a_j)(x) &= \sup_{\phi \in F_x(\Omega)} \left| \int_{\mathbb{R}^n} a_j(y) \phi(y) dy \right| \leq \sup_{\phi \in F_x(\Omega)} \|\phi\|_{L^\infty(\mathbb{R}^n)} \int_{\Omega \cap Q} |a_j(y)| dy \\ &\leq \sup_{\phi \in F_x(\Omega)} \frac{1}{|Q|} \int_{\Omega \cap Q} |a_j(y)| dy \leq \|a_j\|_{L^\infty(\mathbb{R}^n)} \leq \|\mathbf{1}_{Q_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{-1}, \end{aligned}$$

which, together with Remark 2.4, implies that

$$\begin{aligned} \text{I} &\lesssim \left\| \left\{ \sum_{j=1}^{\infty} \left[ \frac{|\lambda_j| \mathbf{1}_{4Q_j}}{\|\mathbf{1}_{Q_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \right]^p \right\}^{1/p} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ (2.6) \quad &\lesssim \mathcal{A}(\{\lambda_j\}_{j=1}^{\infty}, \{Q_j\}_{j=1}^{\infty}) \sim \|f\|_{H_z^{p(\cdot)}(\Omega)}. \end{aligned}$$

For II, from  $x \notin 4Q_j$ ,  $\phi \in F_x(\Omega)$  with  $\text{supp}(\phi) \subset Q$ , and the fact that  $\int_{\mathbb{R}^n} a_j(y)dy = 0$ , we deduce that

$$(2.7) \quad \begin{aligned} \left| \int_{\mathbb{R}^n} a_j(y)\phi(y)dy \right| &= \left| \int_{Q_j} a_j(y) [\phi(y) - \phi(c_{Q_j})] dy \right| \\ &\lesssim \|a_j\|_{L^\infty(\mathbb{R}^n)} [\ell(Q_j)]^{n+1} \|\nabla\phi\|_{L^\infty(\mathbb{R}^n)} \\ &\lesssim \|\nabla\phi\|_{L^\infty(\mathbb{R}^n)} [\ell(Q_j)]^{n+1} \|\mathbf{1}_{Q_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{-1}. \end{aligned}$$

If  $\text{supp}(a_j) \cap \text{supp}(\phi) = \emptyset$ , then  $\int_{\mathbb{R}^n} a_j(y)\phi(y)dy = 0$ . If  $\text{supp}(a_j) \cap \text{supp}(\phi) \neq \emptyset$ , then  $Q_j \cap Q \neq \emptyset$ . Notice that  $x \notin 4Q_j$  and  $x \in Q$ , and hence  $\frac{3}{2}\ell(Q_j) \leq |x - y| < \ell(Q)$  for each  $y \in Q_j \cap Q$ , which implies that

$$|x - c_{Q_j}| \leq |x - y| + |y - c_{Q_j}| \lesssim \ell(Q).$$

By this and (2.7), we find that

$$\left| \int_{\mathbb{R}^n} a_j(y)\phi(y)dy \right| \lesssim \frac{[\ell(Q_j)]^{n+1}}{[\ell(Q)]^{n+1}} \|\mathbf{1}_{Q_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{-1} \lesssim \frac{[\ell(Q_j)]^{n+1}}{|x - c_{Q_j}|^{n+1}} \|\mathbf{1}_{Q_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{-1},$$

which implies that, for any  $x \notin 4Q_j$ ,

$$\begin{aligned} M_z(a_j)(x) &\lesssim \frac{[\ell(Q_j)]^{n+1}}{|x - c_{Q_j}|^{n+1}} \|\mathbf{1}_{Q_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{-1} \\ &\lesssim [M(\mathbf{1}_{Q_j})(x)]^{(n+1)/n} \|\mathbf{1}_{Q_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{-1}. \end{aligned}$$

From this,  $\theta := (n+1)/n$ , Lemmas 2.2 and 2.3, and Remark 2.4, we deduce that

$$(2.8) \quad \begin{aligned} \text{II} &\lesssim \left\| \sum_{j=1}^{\infty} \frac{|\lambda_j|}{\|\mathbf{1}_{Q_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} [M(\mathbf{1}_{Q_j})]^\theta \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ &\sim \left\| \left\{ \sum_{j=1}^{\infty} \frac{|\lambda_j|}{\|\mathbf{1}_{Q_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} [M(\mathbf{1}_{Q_j})]^\theta \right\}^{1/\theta} \right\|_{L^{\theta p(\cdot)}(\mathbb{R}^n)}^\theta \\ &\lesssim \left\| \left[ \sum_{j=1}^{\infty} \frac{|\lambda_j|}{\|\mathbf{1}_{Q_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} (\mathbf{1}_{Q_j})^\theta \right]^{1/\theta} \right\|_{L^{\theta p(\cdot)}(\mathbb{R}^n)}^\theta \\ &\sim \left\| \sum_{j=1}^{\infty} \frac{|\lambda_j| \mathbf{1}_{Q_j}}{\|\mathbf{1}_{Q_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \|f\|_{H_z^{p(\cdot)}(\Omega)}. \end{aligned}$$

Combining (2.5) (2.6), and (2.8), we obtain the desired result. This finishes the proof of Lemma 2.7.  $\square$

In consideration of Lemma 2.7, in order to prove (i) and (ii) of Theorem 1.3, it remains to prove that  $\|M_z(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \sim \|M_z(f)\|_{L^{p(\cdot)}(\Omega)}$ . To do this, we need the following lemma, which is a slight modification of [8, Lemma 2.7], with  $L^p(\Omega)$  norm therein replaced by  $L^{p(\cdot)}(\Omega)$  norm here, the details being omitted.

**Lemma 2.8.** *Let  $\Omega$  be a special Lipschitz domain,  $p(\cdot) \in C^{\log}(\Omega)$  and  $\frac{n}{n+1} < p_- \leq p_+ \leq 1$ , where  $p_-$  and  $p_+$  are as in (1.1). Then  $f \in H_z^{p(\cdot)}(\Omega)$  if and only if  $f \in \mathcal{D}'(\mathbb{R}^n)$ ,  $\text{supp}(f) \subset \bar{\Omega}$  and  $M_z(f) \in L^{p(\cdot)}(\Omega)$ . Moreover, there exists a positive constant  $C$  independent of  $f$ , such that*

$$C^{-1} \|M_z(f)\|_{L^{p(\cdot)}(\Omega)} \leq \|f\|_{H_z^{p(\cdot)}(\Omega)} \leq C \|M_z(f)\|_{L^{p(\cdot)}(\Omega)}.$$

Now let us deal with the general case of  $\Omega$ . In what follows, for each strongly Lipschitz domain  $\Omega$  and  $\epsilon \in (0, \infty)$ , we assume  $\Omega_\epsilon := \{x \in \mathbb{R}^n : \text{dist}(x, \Omega) < \epsilon\}$ . The following lemma is an analogue of [8, Lemma 2.8], since the proof is regular, its proof is omitted.

**Lemma 2.9.** *Let  $\Omega$  be a strongly Lipschitz domain,  $p(\cdot) \in C^{\log}(\Omega)$  and  $\frac{n}{n+1} < p_- \leq p_+ \leq 1$ , where  $p_-$  and  $p_+$  are as in (1.1). If  $f \in \mathcal{D}'(\mathbb{R}^n)$ ,  $\text{supp}(f) \subset \bar{\Omega}$  and  $M_z(f) \in L^{p(\cdot)}(\Omega)$ . Then there exist positive constants  $C$  and  $\epsilon$  independent of  $f$ , such that*

$$\|M_z(f)\|_{L^{p(\cdot)}(\Omega_\epsilon \setminus \bar{\Omega})} \leq C \|M_z(f)\|_{L^{p(\cdot)}(\Omega)}.$$

By [9, Corollary 2.27], we have the following embedding relationship between the variable and classical Lebesgue spaces.

**Lemma 2.10.** *Given  $\Omega$  and  $p(\cdot) : \Omega \rightarrow [1, \infty)$ , let  $|\Omega| < \infty$ . Then there exist constants  $c_1, c_2 > 0$  such that*

$$c_1 \|f\|_{L^{p_-}(\Omega)} \leq \|f\|_{L^{p(\cdot)}(\Omega)} \leq c_2 \|f\|_{L^{p_+}(\Omega)},$$

where  $1 \leq p_- \leq p_+ < \infty$ . In particular, given any  $\Omega$ , if  $f \in L^{p(\cdot)}(\Omega)$ , then  $f$  is locally integrable.

The following Lemma 2.11 extends [8, Lemma 2.9] from constant exponent case to the variable exponent case.

**Lemma 2.11.** *Let  $\Omega$  be a bounded Lipschitz domain,  $p(\cdot) \in C^{\log}(\Omega)$  and  $\frac{n}{n+1} < p_- \leq p_+ \leq 1$ , where  $p_-$  and  $p_+$  are as in (1.1). Then  $f \in H_z^{p(\cdot)}(\Omega)$  if and only if  $f \in \mathcal{D}'(\mathbb{R}^n)$ ,  $\text{supp}(f) \subset \bar{\Omega}$ ,  $M_z(f) \in L^{p(\cdot)}(\Omega)$  and  $\langle f, \phi \rangle = 0$  for each  $\phi \in \mathcal{D}(\mathbb{R}^n)$  with  $\phi \equiv 1$  on  $\Omega$ . Moreover, there exists a positive constant  $C$  independent of  $f$ , such that*

$$C^{-1} \|M_z(f)\|_{L^{p(\cdot)}(\Omega)} \leq \|f\|_{H_z^{p(\cdot)}(\Omega)} \leq C \|M_z(f)\|_{L^{p(\cdot)}(\Omega)}.$$

*Proof.* Let  $f \in H_z^{p(\cdot)}(\Omega)$ . From the fact that  $f \in H^{p(\cdot)}(\mathbb{R}^n)$  with  $\text{supp}(f) \subset \bar{\Omega}$ ,  $\langle f, \phi \rangle = 0$  for each  $\phi \in \mathcal{D}(\mathbb{R}^n)$  with  $\phi \equiv 1$  on  $\Omega$  and Lemma 2.7, it follows that  $\|M_z(f)\|_{L^{p(\cdot)}(\Omega)} \lesssim \|f\|_{H_z^{p(\cdot)}(\Omega)}$ .

On the contrary, by Lemma 2.7, we know that  $\|f\|_{H_x^{p(\cdot)}(\Omega)} \lesssim \|M_z(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)}$ , which implies that, to prove  $\|f\|_{H_x^{p(\cdot)}(\Omega)} \lesssim \|M_z(f)\|_{L^{p(\cdot)}(\Omega)}$ , it suffices to show  $\|M_z(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \|M_z(f)\|_{L^{p(\cdot)}(\Omega)}$ . This, further implies that, it suffices to show  $\|M_z(f)\|_{L^{p(\cdot)}(\Omega^c)} \lesssim \|M_z(f)\|_{L^{p(\cdot)}(\Omega)}$ .

For  $x \in \Omega_\epsilon \setminus \bar{\Omega}$ , from Lemma 2.9, we deduce that there exists an  $\epsilon \in (0, \infty)$  such that

$$(2.9) \quad \|M_z(f)\|_{L^{p(\cdot)}(\Omega_\epsilon \setminus \bar{\Omega})} \lesssim \|M_z(f)\|_{L^{p(\cdot)}(\Omega)}.$$

For  $x \in 2B \setminus \bar{\Omega}_\epsilon$ , we choose a ball  $B := B(\vec{0}_n, r_B)$  with  $r_B$  large enough such that  $\Omega_\epsilon \subset B$ . For any  $x \in 2B \setminus \bar{\Omega}_\epsilon$  and any  $\tilde{x} \in \Omega$ , by the proof of [8, Lemma 2.9], we know that,  $M_z(f)(x) \lesssim M_z(f)(\tilde{x})$  for each  $x \in 2B \setminus \bar{\Omega}_\epsilon$ , and hence

$$\begin{aligned} & \frac{1}{|2B \setminus \bar{\Omega}_\epsilon|} \int_{2B \setminus \bar{\Omega}_\epsilon} \left[ \frac{|M_z(f)(x)|}{\lambda} \right]^{p(x)} dx \\ & \leq \sup_{x \in 2B \setminus \bar{\Omega}_\epsilon} \left[ \frac{|M_z(f)(x)|}{\lambda} \right]^{p(x)} \lesssim \inf_{x \in \Omega} \left[ \frac{|M_z(f)(x)|}{\lambda} \right]^{p(x)} \\ & \lesssim \frac{1}{|\Omega|} \int_{\Omega} \left[ \frac{|M_z(f)(x)|}{\lambda} \right]^{p(x)} dx, \end{aligned}$$

which further implies that, for any  $x \in 2B \setminus \bar{\Omega}_\epsilon$ ,

$$\begin{aligned} \|M_z(f)\|_{L^{p(\cdot)}(2B \setminus \bar{\Omega}_\epsilon)} &= \inf \left\{ \lambda \in (0, \infty) : \int_{2B \setminus \bar{\Omega}_\epsilon} \left[ \frac{|M_z(f)(x)|}{\lambda} \right]^{p(x)} dx \leq 1 \right\} \\ &\lesssim \inf \left\{ \lambda \in (0, \infty) : \int_{\Omega} \left[ \frac{|M_z(f)(x)|}{\lambda} \right]^{p(x)} dx \leq 1 \right\} \\ (2.10) \quad &\sim \|M_z(f)\|_{L^{p(\cdot)}(\Omega)}. \end{aligned}$$

For  $x \in (2B)^c$ , let  $\phi \in F_x(\Omega)$  with  $\text{supp}(\phi) \subset Q$  and  $x \in Q$ . If  $Q \cap \Omega = \emptyset$ , then  $\langle f, \phi \rangle = 0$ . Otherwise, we have  $|x| \lesssim \ell(Q)$  and hence  $[\ell(Q)]^{-n-1} \lesssim |x|^{-n-1}$ . Thus, by choosing  $I$  to be a cube centered at origin such that  $2B \subset I$  and  $\ell(I) \sim \text{diam}(\Omega)$  (see, [8, Lemma 2.9]), where and in what follows,  $\text{diam}(\Omega)$  denotes the *diameter* of  $\Omega$ , namely,  $\text{diam}(\Omega) := \sup\{|x - y| : x, y \in \Omega\}$  (see, for example, [1]), and similarly to the proof of [8, Lemma 2.9], we see that

$$\begin{aligned} M_z(f)(x) &\lesssim \frac{|I| \text{diam}(\Omega)}{|x|^{n+1}} \inf_{y \in \Omega} M_z(f)(y) \\ &\lesssim \frac{|I| \text{diam}(\Omega)}{|x|^{n+1}} \frac{1}{\|\mathbf{1}_\Omega\|_{L^{p(\cdot)}(\Omega)}} \|M_z(f)\|_{L^{p(\cdot)}(\Omega)}, \end{aligned}$$

which, together with Lemmas 2.2 and 2.10,  $\frac{n}{n+1} < p_+ \leq 1$ , further implies that

$$\|M_z(f)\|_{L^{p(\cdot)}((2B)^c)} \lesssim \frac{|I| \text{diam}(\Omega)}{\|\mathbf{1}_\Omega\|_{L^{p(\cdot)}(\Omega)}} \left\| \frac{1}{|x|^{n+1}} \right\|_{L^{p(\cdot)}((2B)^c)} \|M_z(f)\|_{L^{p(\cdot)}(\Omega)}$$

$$\begin{aligned}
&\lesssim [\text{diam}(\Omega)]^{n+1} \|(\mathbf{1}_\Omega)^{p_-}\|_{L^{p(\cdot)/p_-}(\Omega)}^{-1/p_-} \\
&\quad \times \left\| |x|^{(-n-1)p_-} \right\|_{L^{p(\cdot)/p_-}((2B)^c)}^{1/p_-} \|M_z(f)\|_{L^{p(\cdot)}(\Omega)} \\
&\lesssim \|(\mathbf{1}_\Omega)^{p_-}\|_{L^{p_+/p_-}(\Omega)}^{-1/p_-} \left\| |x|^{(-n-1)p_-} \right\|_{L^{p_+/p_-}((2B)^c)}^{1/p_-} \\
&\quad \times \|M_z(f)\|_{L^{p(\cdot)}(\Omega)} \\
&\lesssim \left[ \int_{2r_B}^\infty \int_{|x|=\lambda} \left[ |x|^{(-n-1)p_+} \right]^{p_+} \lambda^{n-1} dx d\lambda \right]^{1/p_+} \\
&\quad \times \|M_z(f)\|_{L^{p(\cdot)}(\Omega)} \\
(2.11) \quad &\lesssim \|M_z(f)\|_{L^{p(\cdot)}(\Omega)}.
\end{aligned}$$

Combining the estimates of (2.9), (2.10) and (2.11), we obtain the desired inequality. This finishes the proof of Lemma 2.11.  $\square$

To prove Theorem 1.3(ii), we need the following Lemma 2.12, its proof is a repetition of the argument in [8, Lemma 2.10] except that [8, Lemma 2.7] is modified to be Lemma 2.8 and we omit the details here.

**Lemma 2.12.** *Let  $\Omega$  be an unbounded strongly Lipschitz domain,  $p(\cdot) \in C^{\log}(\Omega)$  and  $\frac{n}{n+1} < p_- \leq p_+ \leq 1$ , where  $p_-$  and  $p_+$  are as in (1.1). Then  $f \in H_z^{p(\cdot)}(\Omega)$  if and only if  $f \in \mathcal{D}'(\mathbb{R}^n)$ ,  $\text{supp}(f) \subset \bar{\Omega}$  and  $M_z(f) \in L^{p(\cdot)}(\Omega)$ . Moreover, there exists a positive constant  $C$  independent of  $f$ , such that*

$$C^{-1} \|M_z(f)\|_{L^{p(\cdot)}(\Omega)} \leq \|f\|_{H_z^{p(\cdot)}(\Omega)} \leq C \|M_z(f)\|_{L^{p(\cdot)}(\Omega)}.$$

*Proofs of (i) and (ii) of Theorem 1.3.* By Lemmas 2.11 and 2.12, we obtain that the desired of (i) and (ii) of Theorem 1.3, respectively. This finishes the proofs of (i) and (ii) of Theorem 1.3.  $\square$

*Proofs of (iii) and (iv) of Theorem 1.3.* We only prove for Theorem 1.3(iii), since the proof of Theorem 1.3(iv) is analogous to that of Theorem 1.3(iii) and we omit the details here. From the fact that  $F_x(\Omega) \subset F_x^q(\Omega)$ , it follows that  $M_z(f)(x) \leq M_z^{(q)}(f)(x)$  for all  $x \in \mathbb{R}^n$ , which, combined with Theorem 1.3(i), further implies that

$$\|f\|_{H_z^{p(\cdot)}(\Omega)} \lesssim \|M_z(f)\|_{L^{p(\cdot)}(\Omega)} \lesssim \left\| M_z^{(q)}(f) \right\|_{L^{p(\cdot)}(\Omega)}.$$

On the contrary, let  $\Omega$  be bounded and  $f \in H_z^{p(\cdot)}(\Omega)$ . Then  $f \in H^{p(\cdot)}(\mathbb{R}^n)$  with  $\text{supp}(f) \subset \bar{\Omega}$ , for each  $\phi \in \mathcal{D}(\mathbb{R}^n)$  with  $\phi \equiv 1$  on  $\Omega$ , it follows that  $\langle f, \phi \rangle = 0$ . Suppose that  $f = \sum_{j=1}^\infty \lambda_j a_j$  is an atomic decomposition in  $H^{p(\cdot)}(\mathbb{R}^n)$ , where  $\{a_j\}_{j=1}^\infty$  are  $(p(\cdot), \infty)$ -atoms and  $\{\lambda_j\}_{j=1}^\infty \subset \mathbb{C}$  satisfy

$$\|f\|_{H_z^{p(\cdot)}(\Omega)} \sim \|f\|_{H^{p(\cdot)}(\mathbb{R}^n)} \sim \mathcal{A}(\{\lambda_j\}_{j=1}^\infty, \{Q_j\}_{j=1}^\infty).$$



Now let us to prove that  $M_z^{(q)}(f) \in L^{p(\cdot)}(\Omega)$ . For  $x \in 4Q_j$  and a  $(p(\cdot), \infty)$ -atom  $a_j$  with  $\text{supp}(a_j) \subset Q_j$ , by the proof of [8, (2.18)] and  $\|a\|_{L^\infty(\mathbb{R}^n)} \leq \|\mathbf{1}_{Q_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{-1}$ , we find that, for any  $x \in 4Q_j$ ,

$$(2.12) \quad M_z^{(q)}(a_j)(x) \leq \|a_j\|_{L^\infty(\mathbb{R}^n)} \leq \|\mathbf{1}_{Q_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{-1}.$$

For  $x \notin 4Q_j$  and  $\phi \in F_x(\Omega)$  with  $\text{supp}(\phi) \subset Q$ , from the fact that  $\int_{\mathbb{R}^n} a_j(y) dy = 0$ ,  $\|a\|_{L^\infty(\mathbb{R}^n)} \leq \|\mathbf{1}_{Q_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{-1}$ , and similar to that of [8, (2.20)], it follows that, for any  $x \notin 4Q_j$ ,

$$(2.13) \quad \begin{aligned} M_z^{(q)}(a_j)(x) &\lesssim \left[ \frac{\ell(Q_j)}{|x - c_{Q_j}|} \right]^{1+n/q'} \|\mathbf{1}_{Q_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{-1} \\ &\lesssim [M(\mathbf{1}_{Q_j})(x)]^{(1/n+1/q')} \|\mathbf{1}_{Q_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{-1}. \end{aligned}$$

Moreover, by the fact that  $p^* = \frac{np_-}{n-p_-}$  and  $\frac{p^*}{p^*-1} < q < \infty$ , we know that

$$\theta := 1/n + 1/q' = 1/n + 1 - 1/q > 1/n + 1/p^* = 1/p_- \geq 1.$$

Thus, by (2.12), (2.13) with  $\theta > 1$ , and using the same estimates of (2.6) and (2.8), we find that  $\|M_z^{(q)}(f)\|_{L^{p(\cdot)}(\Omega)} \lesssim \|f\|_{H_z^{p(\cdot)}(\Omega)} < \infty$ . This finishes the proof of Theorem 1.3(iii).  $\square$

*Proof of Theorem 1.4(i).* Let  $f \in H_r^{p(\cdot)}(\Omega)$ . From the definition of  $H_r^{p(\cdot)}(\Omega)$ , we choose an extension  $F$  of  $f$  on  $\mathbb{R}^n$  such that  $\|F\|_{H^{p(\cdot)}(\mathbb{R}^n)} \leq 2\|f\|_{H_r^{p(\cdot)}(\Omega)}$ . Moreover, by Lemma 2.6 and the proof of [8, Theorem 1.2], we conclude that, for all  $x \in \Omega$ ,  $M_r(f)(x) \lesssim M_z(F)(x)$ , and hence  $\|M_r(f)\|_{L^{p(\cdot)}(\Omega)} \lesssim \|M_z(F)\|_{L^{p(\cdot)}(\mathbb{R}^n)}$ , which, combined with Lemma 2.7, implies that

$$\|M_r(f)\|_{L^{p(\cdot)}(\Omega)} \lesssim \|M_z(F)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \|F\|_{H^{p(\cdot)}(\mathbb{R}^n)} \lesssim \|f\|_{H_r^{p(\cdot)}(\Omega)}.$$

On the other hand, proceeding as in the proof of [12, Theorem 1.12], we know that  $\|f\|_{H_r^{p(\cdot)}(\Omega)} \lesssim \|f\|_{H^{p(\cdot)}(\Omega)}$  when  $\Omega$  is a bounded Lipschitz domain. Furthermore, by the proof of [12, Theorem 1.5], we find that  $\|f\|_{H^{p(\cdot)}(\Omega)} \lesssim \|M_r(f)\|_{L^{p(\cdot)}(\Omega)}$  when  $\Omega$  is a proper open subset. Thus, we have  $\|f\|_{H_r^{p(\cdot)}(\Omega)} \lesssim \|M_r(f)\|_{L^{p(\cdot)}(\Omega)}$ . This finishes the proof of Theorem 1.4(i).  $\square$

*Proof of Theorem 1.4(ii).* From the fact that  $G_x(\Omega) \subset G_x^q(\Omega)$ , it follows that  $M_r(f)(x) \leq M_r^{(q)}(f)(x)$  for all  $x \in \Omega$ , which together with Theorem 1.4(i), implies that  $\|f\|_{H_r^{p(\cdot)}(\Omega)} \lesssim \|M_r^{(q)}(f)\|_{L^{p(\cdot)}(\Omega)}$ . Conversely, by the proofs of [8, Theorem 2.12] and Theorem 1.3(iii), we can get the desired results and leave the details to the interested readers. This completes the proof of Theorem 1.4(ii).  $\square$

### 3. Proof of Theorem 1.9

In this section, we give the proof of Theorem 1.9. Let us recall the following notion of local  $(p(\cdot), q)$ -atoms (see [17, Definitions 1.3 and 1.4]).

**Definition 3.1.** Let  $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$  and  $q \in (1, \infty]$ . Assume that  $p_-$  and  $p$  are as in (1.1). Fix an integer  $d \geq d_{p(\cdot)} := \min\{d \in \mathbb{Z}_+ : p_-(n + d + 1) > n\}$ . A measurable function  $a$  on  $\mathbb{R}^n$  is called a *type (a) local  $(p(\cdot), q)$ -atom* if there exists a cube  $Q$  such that

- (i)  $\text{supp}(a) \subset Q$  with  $0 < \ell(Q) < 1$ ;
- (ii)  $\|a\|_{L^q(\mathbb{R}^n)} \leq |Q|^{1/q} \|\mathbf{1}_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{-1}$ ;
- (iii)  $\int_{\mathbb{R}^n} a(x)x^\alpha dx = 0$  for  $|\alpha| \leq d$ .

Moreover, a measurable function  $b$  on  $\mathbb{R}^n$  is called a *type (b) local  $(p(\cdot), q)$ -atom* if  $\text{supp}(b) \subset \tilde{Q}$  with  $\ell(\tilde{Q}) \geq 1$  and  $\|b\|_{L^q(\mathbb{R}^n)} \leq |\tilde{Q}|^{1/q} \|\mathbf{1}_{\tilde{Q}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{-1}$ .

For sequences of  $\{\lambda_j\}_{j=1}^\infty \subset \mathbb{C}$  and cubes  $\{Q_j\}_{j=1}^\infty$ , define that

$$\mathcal{A}'(\{\lambda_j\}_{j=1}^\infty, \{Q_j\}_{j=1}^\infty) := \left\| \left[ \sum_{j=1}^\infty \left( \frac{|\lambda_j| \mathbf{1}_{Q_j}}{\|\mathbf{1}_{Q_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \right)^p \right]^{1/p} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

*Proof of Theorem 1.9.* Let  $f \in h_r^{p(\cdot)}(\Omega)$ . Then by the proof of [12, Theorem 1.12(i)], we have the atomic decomposition of  $h_r^{p(\cdot)}(\Omega)$ . Conversely, this part of the proof largely follows [7, Theorem 2.7] and we include it here primarily for the reader's convenience. Then, we know that there exist two sequences  $\{\lambda_j\}_{j=1}^\infty \subset \mathbb{C}$  and  $\{\kappa_j\}_{j=1}^\infty \subset \mathbb{C}$ , a sequence  $\{a_j\}_{j=1}^\infty$  of type (a)  $(p(\cdot), q)_\Omega$ -atoms and a sequence  $\{b_j\}_{j=1}^\infty$  of type (b)  $(p(\cdot), q)_\Omega$ -atoms such that

$$(3.1) \quad f = \sum_{j=1}^\infty \lambda_j a_j + \sum_{j=1}^\infty \kappa_j b_j \quad \text{in } \mathcal{D}'(\mathbb{R}^n),$$

and

$$(3.2) \quad \mathcal{A}(\{\lambda_j\}_{j=1}^\infty, \{Q_j\}_{j=1}^\infty) + \mathcal{A}(\{\kappa_j\}_{j=1}^\infty, \{\tilde{Q}_j\}_{j=1}^\infty) < \infty,$$

where  $\{Q_j\}_{j=1}^\infty$  and  $\{\tilde{Q}_j\}_{j=1}^\infty$ , respectively, denote the supports of  $\{a_j\}_{j=1}^\infty$  and  $\{b_j\}_{j=1}^\infty$ . The type (a)  $(p(\cdot), q)_\Omega$ -atoms  $a_j$  in (3.1) are already type (a) local  $(p(\cdot), q)$ -atoms and

$$(3.3) \quad \mathcal{A}'(\{\lambda_j\}_{j=1}^\infty, \{Q_j\}_{j=1}^\infty) = \mathcal{A}(\{\lambda_j\}_{j=1}^\infty, \{Q_j\}_{j=1}^\infty) < \infty,$$

hence we let those atoms stand unchanged. For type (b)  $(p(\cdot), q)_\Omega$ -atoms  $b_j$  in (3.1), there are two different cases.

Case 1: If  $b_j$  is a type (b)  $(p(\cdot), q)_\Omega$ -atom as in (3.1) and is supported on a type (b) cube  $Q$  with  $\ell(Q) < 1$ , then we find a cube  $\tilde{Q} \subset (\bar{\Omega})^c$  which has the

same size as  $Q$ . We further consider the extension  $(b_j)_1$  of the function  $b_j$  as follows:

$$(b_j)_1 := \begin{cases} b_j(x), & \text{for } x \in Q, \\ -\frac{1}{|Q|} \int_Q b_j(y) dy, & \text{for } x \in \tilde{Q}. \end{cases}$$

From this, we deduce that the function  $(b_j)_1$  is supported on  $Q \cup \tilde{Q}$ . Since the distance of  $Q$  and  $\tilde{Q}$  to  $\partial\Omega$  are comparable to  $\ell(Q)$ , we may find another cube  $\hat{Q}$  such that  $(Q \cup \tilde{Q}) \subset \hat{Q}$  and  $|Q| \leq |\hat{Q}| \lesssim |Q|$ . By this and the Hölder inequality, we know that

$$\begin{aligned} \|(b_j)_1\|_{L^q(\mathbb{R}^n)} &\leq \|b_j\|_{L^q(\mathbb{R}^n)} + \left\| \left( -\frac{1}{|Q|} \int_Q b_j(y) dy \right) \mathbf{1}_{\tilde{Q}} \right\|_{L^q(\mathbb{R}^n)} \\ &\leq \|b_j\|_{L^q(\mathbb{R}^n)} + |\tilde{Q}|^{1/q} |Q|^{-1} \left| \int_Q b_j(y) dy \right| \\ &\leq \|b_j\|_{L^q(\mathbb{R}^n)} + |\tilde{Q}|^{1/q} |Q|^{-1+1/q'} \|b_j\|_{L^q(\mathbb{R}^n)} \lesssim \frac{|\hat{Q}|^{1/q}}{\|\mathbf{1}_{\hat{Q}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \end{aligned}$$

and

$$\begin{aligned} \int_{\mathbb{R}^n} (b_j)_1(x) dx &= \int_Q b_j(x) dx - \int_{\tilde{Q}} \left( \frac{1}{|Q|} \int_Q b_j(x) dx \right) \mathbf{1}_{\tilde{Q}}(y) dy \\ &= \int_Q b_j(x) dx - |\tilde{Q}| |Q|^{-1} \int_Q b_j(x) dx = 0, \end{aligned}$$

which further implies that  $(b_j)_1$  is an acceptable type  $(a)$  local  $(p(\cdot), q)$ -atom in  $\mathbb{R}^n$  and

$$(3.4) \quad \mathcal{A}'(\{\kappa_Q\}_{Q \in Q_1}, \{Q\}_{Q \in Q_1}) = \mathcal{A}(\{\kappa_Q\}_{Q \in Q_1}, \{Q\}_{Q \in Q_1}) < \infty,$$

where  $Q_1 := \{Q \subset \Omega : Q \text{ be a type } (b) \text{ cube with } \ell(Q) < 1\}$ .

Case 2: If  $b_j$  is a type  $(b)$   $(p(\cdot), q)_\Omega$ -atom as in (3.1) and is supported on a type  $(b)$  cube  $Q$  with  $\ell(Q) \geq 1$ . Now we just define

$$(b_j)_2 := \begin{cases} b_j(x), & \text{for } x \in Q, \\ 0, & \text{elsewhere,} \end{cases}$$

which implies that  $(b_j)_2$  is an acceptable type  $(b)$  local  $(p(\cdot), q)$ -atom in  $\mathbb{R}^n$  and

$$(3.5) \quad \mathcal{A}'(\{\kappa_Q\}_{Q \in Q_2}, \{Q\}_{Q \in Q_2}) = \mathcal{A}(\{\kappa_Q\}_{Q \in Q_2}, \{Q\}_{Q \in Q_2}) < \infty,$$

where  $Q_2 := \{Q \subset \Omega : Q \text{ be a type } (b) \text{ cube with } \ell(Q) \geq 1\}$ . Moreover, from Definition 1.7, [17, Theorem 1.5], (3.3), (3.4), (3.5) and (3.2), we deduce that

$$\begin{aligned} \|f\|_{h_r^{p(\cdot)}(\Omega)} &\leq \|F\|_{h_r^{p(\cdot)}(\mathbb{R}^n)} \lesssim \mathcal{A}'(\{\lambda_j\}_{j=1}^\infty, \{Q_j\}_{j=1}^\infty) + \mathcal{A}'(\{\kappa_Q\}_{Q \in Q_1}, \{Q\}_{Q \in Q_1}) \\ &\quad + \mathcal{A}'(\{\kappa_Q\}_{Q \in Q_2}, \{Q\}_{Q \in Q_2}) < \infty, \end{aligned}$$

which implies that  $f \in h_r^{p(\cdot)}(\Omega)$ . This finishes the proof of Theorem 1.9.  $\square$

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