

ADMISSIBLE INERTIAL MANIFOLDS FOR INFINITE DELAY EVOLUTION EQUATIONS

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ABSTRACT. The aim of this paper is to prove the existence of an admissible inertial manifold for mild solutions to infinite delay evolution equation of the form

$$\begin{cases} \frac{du}{dt} + Au = F(t, u_t), & t \geq s, \\ u_s(\theta) = \phi(\theta), \quad \forall \theta \in (-\infty, 0], & s \in \mathbb{R}, \end{cases}$$

where A is positive definite and self-adjoint with a discrete spectrum, the Lipschitz coefficient of the nonlinear part F may depend on time and belongs to some admissible function space defined on the whole line. The proof is based on the Lyapunov-Perron equation in combination with admissibility and duality estimates.

1. Introduction

The new concept of inertial manifold called *admissible inertial manifolds* for evolution equations was first introduced by Huy in [5]. These manifolds are constituted by trajectories of the solutions which belong to rescaledly admissible function spaces which contain wide classes of function spaces like weighted L_p spaces, the Lorentz spaces $L_{p,q}$ and many other rescaling function spaces occurring in interpolation theory. The important property of these manifolds is their exponential attracting all solutions of considered evolution equations (see [1, 3, 4, 6]). This fact allows us to apply the reduction principle to study the asymptotic behavior of the partial differential equation by determining the structures of its induced solutions belonging to such an inertial manifold, which turn out to be solutions of ordinary differential equations.

In [5], Huy proved the existence of admissible inertial manifold for a class of semi-linear evolution equations without delay of the form

$$\frac{dx}{dt} + Ax = f(t, x), \quad t > s, \quad x(s) = x_s, \quad s \in \mathbb{R},$$

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where A is positive definite and self-adjoint with a discrete spectrum on a separable Hilbert space X and $f : \mathbb{R} \times \mathcal{D}(A^\beta) \rightarrow X$ is φ -Lipschitz for $0 \leq \beta < 1$. Later, for the differential operator A as in [5], Huy and the author [7] proved the existence of admissible inertial manifolds for a class of finite delay evolution equations which has the form

$$\frac{dx}{dt} + Ax = f(t, x_t), \quad t > s, \quad x_s(\cdot) = \phi(\cdot) \in \mathcal{C}_\beta, \quad s \in \mathbb{R}.$$

Here, $f : \mathbb{R} \times \mathcal{C}_\beta \rightarrow X$ is a nonlinear operator satisfying φ -Lipschitz with $C_\beta := C([-r, 0], \mathcal{D}(A^\beta))$ being the infinite-dimensional Banach space of all continuous functions from $[-r, 0]$ into $\mathcal{D}(A^\beta)$ equipped with the norm

$$\|x\|_{\mathcal{C}_\beta} := \sup_{-r \leq \theta \leq 0} \|A^\beta x\|, \quad \forall x \in \mathcal{C}_\beta,$$

x_t is the history function which defined in finite interval $[-r, 0]$ by the formula $x_t(\theta) = x(t + \theta)$ for all $-r \leq \theta \leq 0$.

In this paper, motivated by the results in [5, 7], we prove the existence of an admissible inertial manifolds for mild solutions of the following infinite delay evolution equation

$$(1) \quad \begin{cases} \frac{du}{dt} + Au = F(t, u_t), & t \geq s, \\ u_s(\theta) = \phi(\theta), \quad \forall \theta \in (-\infty, 0], \end{cases}$$

where $A : X \supset \mathcal{D}(A) \rightarrow X$ is a positive definite and self-adjoint with a discrete spectrum on a separable Hilbert space X ; $F : \mathbb{R} \times \mathcal{C}_g^\beta \rightarrow X$ is a nonlinear operator with

$$\mathcal{C}_g^\beta := \left\{ \phi \in C((-\infty, 0], X_\beta) : \sup_{\theta \leq 0} \frac{\|A^\beta \phi(\theta)\|}{g(\theta)} < +\infty \right\}$$

being the Banach space with respect to the norm

$$\|\phi\|_{\mathcal{C}_g^\beta} = \sup_{\theta \leq 0} \frac{\|A^\beta \phi(\theta)\|}{g(\theta)}, \quad \forall \phi \in \mathcal{C}_g^\beta,$$

and $X_\beta := \mathcal{D}(A^\beta)$ is the domain of the fractional power A^β for $0 \leq \beta < 1$, $g : (-\infty, 0] \rightarrow [1, +\infty)$ is the given continuous function, and u_t is the *history function* defined by

$$u_t(\theta) := u(t + \theta) \quad \text{for all } -\infty < \theta \leq 0.$$

This paper is organized as follows. In Section 2, for convenience of the reader, we recall some background materials on the semigroup e^{-tA} generated by the operator A and admissible function spaces. In Section 3, we give the notion of admissible inertial manifold and prove the existence of such manifold for mild solutions to Equation (1). In the last section, we give an example to illustrate the obtained result. Our main result is contained in Lemma 3 and Theorem 3.4 which extends the results in [5, 7] to the case of infinite delay evolution equations. This result can be applied to a wide class of infinite delay

evolution equations such as: Lotka-Volterra models with diffusion, population dynamics, biological models, ...

2. Preliminaries

2.1. Semigroups

Throughout this paper, let X be a separable Hilbert space and suppose that A is a closed linear operator on X satisfying the following hypothesis.

Hypothesis 2.1. *A is a positive definite, self-adjoint operator with a discrete spectrum, say*

$$0 < \lambda_1 \leq \lambda_2 \leq \dots, \quad \text{each with finite multiplicity and } \lim_{k \rightarrow \infty} \lambda_k = \infty,$$

and assume that $\{e_k\}_{k=1}^{\infty}$ is the orthonormal basis in X consisting of the corresponding eigenfunctions of the operator A , i.e., $Ae_k = \lambda_k e_k$.

Now, for a non-zero natural number N , let λ_N and λ_{N+1} be two successive and distinct eigenvalues such that

$$(2) \quad \lambda_N < \lambda_{N+1} \quad \text{and} \quad \sup_{\theta \leq 0} \frac{e^{-\lambda_{N+1}\theta}}{g(\theta)} < +\infty.$$

Furthermore, let P be the orthogonal projection onto the first N eigenvectors of the operator A , and $(e^{-tA})_{t \geq 0}$ be the semigroup generated by $-A$. Since PX is of finite dimension, it follows that the restriction $(e^{-tA}P)_{t \geq 0}$ of the semigroup $(e^{-tA})_{t \geq 0}$ to PX can be extended to the whole line \mathbb{R} .

For $0 \leq \beta < 1$ we then recall the following dichotomy estimates (see [2]):

$$(3) \quad \begin{aligned} \|e^{-tA}P\| &\leq e^{\lambda_N|t|}, \\ \|A^\beta e^{-tA}P\| &\leq \lambda_N^\beta e^{\lambda_N|t|}, \quad t \in \mathbb{R}, \\ \|e^{-tA}(I-P)\| &\leq e^{-\lambda_{N+1}t}, \quad t \geq 0, \\ \|A^\beta e^{-tA}(I-P)\| &\leq \left[\left(\frac{\beta}{t}\right)^\beta + \lambda_{N+1}^\beta \right] e^{-\lambda_{N+1}t}, \quad t > 0, \beta > 0. \end{aligned}$$

Now, we can define the Green function $\mathcal{G} : X \rightarrow X_\beta$ as follows.

$$(4) \quad \mathcal{G}(t, \tau) := \begin{cases} e^{-(t-\tau)A}[I-P] & \text{for } t > \tau, \\ -e^{-(t-\tau)A}P & \text{for } t \leq \tau. \end{cases}$$

Then, by the dichotomy estimates given in (3) we have

$$(5) \quad \|e^{\gamma(t-\tau)}A^\beta \mathcal{G}(t, \tau)\| \leq K(t, \tau)e^{-\alpha|t-\tau|} \quad \text{for all } t \neq \tau,$$

where $\gamma = \frac{\lambda_{N+1} + \lambda_N}{2}$, $\alpha = \frac{\lambda_{N+1} - \lambda_N}{2}$ and

$$K(t, \tau) = \begin{cases} \left(\frac{\beta}{t-\tau}\right)^\beta + \lambda_{N+1}^\beta & \text{if } t > \tau, \\ \lambda_N^\beta & \text{if } t \leq \tau. \end{cases}$$

2.2. Admissible Banach spaces

Now, let $\mathbb{I} = \mathbb{R}$ or $\mathbb{I} = (-\infty, t_0]$ for $t_0 \in \mathbb{R}$, we recall some concepts and notions on admissibility for later use (see [5, 7] and the reference therein).

Denote by \mathcal{B} the Borel algebra and by λ the Lebesgue measure on \mathbb{R} . The space $L_{1,\text{loc}}(\mathbb{R})$ of real-valued locally integrable functions on \mathbb{R} (modulo λ -nullfunctions) becomes a Frechet space for the seminorms $p_n(f) = \int_{J_n} |f(t)| dt$, where $J_n = [n, n+1]$ for each $n \in \mathbb{Z}$.

We then define Banach function spaces as follows.

Definition. A vector space $E_{\mathbb{I}}$ of real-valued Borel-measurable functions on \mathbb{I} (modulo λ -null-functions) is *Banach function space* (over $(\mathbb{I}, \mathcal{B}, \lambda)$) if

- (1) $E_{\mathbb{I}}$ is Banach lattice with respect to a norm $\|\cdot\|_{E_{\mathbb{I}}}$, i.e., $(E_{\mathbb{I}}, \|\cdot\|_{E_{\mathbb{I}}})$ is a Banach space, and if $\varphi \in E_{\mathbb{I}}$, ψ is a real-valued Borel-measurable function such that $|\psi(\cdot)| \leq |\varphi(\cdot)|$, λ -a.e., then $\psi \in E_{\mathbb{I}}$ and $\|\psi\|_{E_{\mathbb{I}}} \leq \|\varphi\|_{E_{\mathbb{I}}}$,
- (2) the characteristic functions χ_A belong to $E_{\mathbb{I}}$ for all $A \in \mathcal{B}$ of finite measure, and

$$\sup_{t \in \mathbb{I}} \|\chi_{[t-1, t]}\|_{E_{\mathbb{I}}} < \infty \quad ; \quad \inf_{t \in \mathbb{I}} \|\chi_{[t-1, t]}\|_{E_{\mathbb{I}}} > 0,$$

- (3) $E_{\mathbb{I}} \hookrightarrow L_{1,\text{loc}}(\mathbb{I})$, i.e., for each compact interval $\mathbb{J} \subset \mathbb{I}$ there exists a number $\beta_{\mathbb{J}} \geq 0$ such that $\int_{\mathbb{J}} |f(t)| dt \leq \beta_{\mathbb{J}} \|f\|_{E_{\mathbb{I}}}$ for all $f \in E_{\mathbb{I}}$.

Definition (Admissibility). The Banach function space $E_{\mathbb{I}}$ is called *admissible* if the following hold:

- (i) there is a constant $M \geq 1$ such that for every compact interval $[a, b] \subset \mathbb{I}$, and for all $\varphi \in E_{\mathbb{I}}$ we have

$$\int_a^b |\varphi(t)| dt \leq \frac{M(b-a)}{\|\chi_{[a,b]}\|_{E_{\mathbb{I}}}} \|\varphi\|_{E_{\mathbb{I}}}.$$

- (ii) for all $\varphi \in E_{\mathbb{I}}$, the function $\Lambda_1 \in E_{\mathbb{I}}$ where $(\Lambda_1 \varphi)(t) = \int_{t-1}^t \varphi(\tau) d\tau$.

- (iii) $E_{\mathbb{I}}$ is T_{τ}^+ -invariant for all $\tau \in \mathbb{I}$, where

- if $\mathbb{I} = (-\infty, t_0]$ and for some $t_0 \in \mathbb{R}$, then

$$(T_{\tau}^+ \varphi)(t) = \begin{cases} \varphi(t - \tau) & \text{for } t \leq \tau + t_0, \\ 0 & \text{for } t > \tau + t_0; \end{cases}$$

- if $\mathbb{I} = \mathbb{R}$, then

$$(T_{\tau}^+ \varphi)(t) = \varphi(t - \tau) \text{ for } t \in \mathbb{R}.$$

- (iv) $E_{\mathbb{I}}$ is T_{τ}^- -invariant for all $\tau \in \mathbb{I}$, where

- if $\mathbb{I} = (-\infty, t_0]$ and for some $t_0 \in \mathbb{R}$, then

$$(T_{\tau}^- \varphi)(t) = \begin{cases} \varphi(t + \tau) & \text{for } t \leq t_0 - \tau, \\ 0 & \text{for } t > t_0 - \tau; \end{cases}$$

- if $\mathbb{I} = \mathbb{R}$, then

$$(T_\tau^- \varphi)(t) = \varphi(t + \tau) \text{ for } t \in \mathbb{R}.$$

Furthermore, there are constants N_1, N_2 such that $\|T_\tau^+\| \leq N_1$, $\|T_\tau^-\| \leq N_2$ for all $\tau \in \mathbb{I}$.

Example 2.2 ([5]). The spaces $L_p(\mathbb{R})$, $1 \leq p \leq \infty$, the space

$$\mathbf{M}(\mathbb{R}) := \left\{ f \in L_{1,\text{loc}}(\mathbb{R}) : \sup_{t \in \mathbb{R}} \int_{t-1}^t |f(\tau)| d\tau < \infty \right\}$$

endowed with the norm

$$\|f\|_{\mathbf{M}} := \sup_{t \in \mathbb{R}} \int_{t-1}^t |f(\tau)| d\tau,$$

and many other function spaces occurring in interpolation theory, e.g. the Lorentz spaces $L_{p,q}$, $1 < p < \infty$, $1 < q < \infty, \dots$ are admissible Banach function spaces.

Remark 2.3. If $E_{\mathbb{I}}$ is the admissible Banach function space, then $E_{\mathbb{I}} \hookrightarrow \mathbf{M}(\mathbb{I})$.

Proposition 2.4. *Let $E_{\mathbb{I}}$ be an admissible Banach function space. Then the following assertions hold.*

- (i) *Let $\varphi \in L_{1,\text{loc}}(\mathbb{I})$ such that $\varphi \geq 0$ and $\Lambda_1 \varphi \in E_{\mathbb{I}}$. For $\sigma > 0$, we define functions $\Lambda'_\sigma \varphi$, $\Lambda''_\sigma \varphi$ by*

$$(\Lambda'_\sigma \varphi)(t) = \int_{-\infty}^t e^{-\sigma(t-s)} \varphi(s) ds,$$

and

$$(\Lambda''_\sigma \varphi)(t) = \begin{cases} \int_t^\infty e^{-\sigma(s-t)} \varphi(s) ds, & \text{if } \mathbb{I} = \mathbb{R}, \\ \int_t^{t_0} e^{-\sigma(s-t)} \varphi(s) ds & \text{if } \mathbb{I} = (-\infty, t_0]. \end{cases}$$

Then, $\Lambda'_\sigma \varphi$ and $\Lambda''_\sigma \varphi$ belong to $E_{\mathbb{I}}$. Moreover, we have

$$\|\Lambda'_\sigma \varphi\|_{E_{\mathbb{I}}} \leq \frac{N_1}{1 - e^{-\sigma}} \|\Lambda_1 \varphi\|_{E_{\mathbb{I}}} \text{ and } \|\Lambda''_\sigma \varphi\|_{E_{\mathbb{I}}} \leq \frac{N_2}{1 - e^{-\sigma}} \|\Lambda_1 \varphi\|_{E_{\mathbb{I}}}$$

for constants N_1, N_2 are defined as in Definition 2.2.

- (ii) $E_{\mathbb{I}}$ contains exponentially decaying functions $e^{-a|t|}$ for all $t \in \mathbb{I}$ and any fixed constant $a > 0$.
- (iii) $E_{\mathbb{I}}$ does not contain exponentially growing functions $e^{b|t|}$ for all $t \in \mathbb{I}$ and any fixed constant $b > 0$.

We next recall the definition of associate spaces of admissible Banach spaces on \mathbb{I} as follows:

Definition. Let $E_{\mathbb{I}}$ be an admissible Banach space and denote by $S(E_{\mathbb{I}})$ the unit sphere in $E_{\mathbb{I}}$. Consider, the set $E'_{\mathbb{I}}$ of all measurable real-valued functions ψ on \mathbb{I} such that

$$\varphi\psi \in L_1(\mathbb{I}), \quad \int_{\mathbb{I}} |\varphi(t)\psi(t)| dt \leq k, \quad \forall \varphi \in S(E_{\mathbb{I}}),$$

where k depends only on ψ and

$$L_1(\mathbb{I}) = \left\{ g : \mathbb{I} \rightarrow \mathbb{I} : g \text{ is measurable and } \int_{\mathbb{I}} |g(t)| dt < \infty \right\}.$$

Then, $E'_{\mathbb{I}}$ is normed space with the norm given by

$$\|\psi\|_{E'_i} := \sup \left\{ \int_{\mathbb{I}} |\varphi(t)\psi(t)| dt : \varphi \in S(E_{\mathbb{I}}) \right\} \quad \text{for } \psi \in E'_i,$$

and we call $E'_{\mathbb{I}}$ *associate space* of $E_{\mathbb{I}}$.

Remark 2.5. Let $E_{\mathbb{I}}$ be an admissible Banach function space and $E'_{\mathbb{I}}$ be its associate space. Then, we have following Hölder inequality

$$\int_{\mathbb{I}} |\varphi(t)\psi(t)| dt \leq \|\varphi\|_{E_{\mathbb{I}}} \|\psi\|_{E'_i}, \quad \forall \varphi \in E_{\mathbb{I}}, \psi \in E'_i.$$

Remark 2.6. In the case $\mathbb{I} = \mathbb{R}$ we write E, \mathcal{E} instead of $E_{\mathbb{R}}$ and $\mathcal{E}_{\mathbb{R}}$.

In order to get the existence of an admissible inertial manifold of \mathcal{E} -class, it is necessary to put some restrictions on Banach function space $E_{\mathbb{I}}$ as follows.

Hypothesis 2.7. (1) *The Banach function space $E_{\mathbb{I}}$ and its associate space $E'_{\mathbb{I}}$ are admissible spaces.*
 (2) *The function space*

$$E_{\mathbb{I}}^{\beta} := \{u \in E_{\mathbb{I}} \mid |u|^{\frac{1+\beta}{1-\beta}} \in E_{\mathbb{I}}\} \quad \text{for } 0 \leq \beta < 1$$

is also an admissible Banach function space with the norm

$$\|u\|_{\beta} := \max \left\{ \|u\|_{E_{\mathbb{I}}}, \left\| |u|^{\frac{1+\beta}{1-\beta}} \right\|_{E_{\mathbb{I}}}^{\frac{1-\beta}{1+\beta}} \right\}.$$

(3) *For the function $\varphi \geq 0$ and for a fixed $\nu > 0$, the functions h_{ν} and Θ_{ν} defined by*

$$h_{\nu}(t) := \|e^{-\nu|t-\cdot|} \varphi(\cdot)\|_{E'_i}, \quad t \in \mathbb{R},$$

$$\Theta_{\nu}(t) := \|e^{-\nu \frac{1+\beta}{1-\beta} |t-\cdot|} \varphi^{\frac{1+\beta}{1-\beta}}(\cdot)\|_{E'_i}^{\frac{1-\beta}{1+\beta}}, \quad t \in \mathbb{R}$$

belong to $E_{\mathbb{I}}$.

Definition. A function $u \in C((-\infty, T], X_{\beta})$ is said to be a mild solution of Equation (1) on the interval $(-\infty, T]$ if $u_s(\theta) = \phi(\theta)$ for $\theta \in (-\infty, 0]$ and

$$(6) \quad u(t) = e^{-(t-s)A} u(s) + \int_s^t e^{-(t-\tau)A} F(\tau, u_{\tau}) d\tau$$

for all $t \in [s, T]$.

From now on, instead of Equation (1) we will consider the integral equation (6). We also need the φ -Lipschitz property defined as follows.

Definition (φ -Lipschitz function). Let E be an admissible Banach function space and φ be a positive function belonging to E . A function $F : \mathbb{R} \times \mathcal{C}_g^\beta \rightarrow X$ is said to be φ -Lipschitz if F satisfies

- (i) $\|F(t, \phi)\| \leq \varphi(t)(1 + \|\phi\|_{\mathcal{C}_g^\beta})$, $\forall t \in \mathbb{R}$,
- (ii) $\|F(t, \phi_1) - F(t, \phi_2)\| \leq \varphi(t)\|\phi_1 - \phi_2\|_{\mathcal{C}_g^\beta}$, $\forall t \in \mathbb{R}$, $\forall \phi_1, \phi_2 \in \mathcal{C}_g^\beta$.

3. Admissible inertial manifolds

Now, on \mathcal{C}_g^β , we define the projection \hat{P} by

$$(\hat{P}\phi)(\theta) = \sum_{k=1}^N e^{-\lambda_k \theta} \langle \phi(0), e_k \rangle e_k = e^{-\theta A} P\phi(0), \quad \forall -\infty < \theta \leq 0,$$

where $\phi = \phi(\theta)$ is an element of \mathcal{C}_g^β . Then, we give the notion of admissible inertial manifolds in the following definition.

Definition. An admissible inertial manifold of \mathcal{E} -class for Equation (6) is a collection of surfaces $\mathcal{M} = (\mathcal{M}_t)_{t \in \mathbb{R}}$ in \mathcal{C}_g^β of the form

$$\mathcal{M}_t = \{\hat{p} + \Phi_t(\hat{p}(0)) : \hat{p} \in \hat{P}\mathcal{C}_g^\beta\} \subset \mathcal{C}_g^\beta,$$

where $\Phi_t : PX \rightarrow (I - \hat{P})\mathcal{C}_g^\beta$ is a Lipschitz mapping, and the following conditions are satisfied:

- (i) The Lipschitz constants of Φ_t are independent of t , i.e., there exists a constant C independent of t such that

$$\|\Phi_t(x_1) - \Phi_t(x_2)\|_{\mathcal{C}_g^\beta} \leq C \|A^\beta(x_1 - x_2)\|, \quad \forall x_1, x_2 \in X_\beta, \quad \forall t \in \mathbb{R}.$$

- (ii) There exists $\gamma > 0$ such that to each $\phi \in \mathcal{M}_{t_0}$ there corresponds one and only one solution $u(\cdot)$ to Equation (6) on $(-\infty, t_0]$ satisfying that $u_{t_0} = \phi$ and the function

$$t \mapsto e^{-\gamma(t_0-t)} \|u_t\|_{\mathcal{C}_g^\beta}, \quad t \leq t_0$$

belongs to $E_{(-\infty, t_0]}$.

- (iii) \mathcal{M} is positively invariant under Equation (6), i.e., if $u(\cdot)$ is a solution of Equation (6) satisfies $u_s = \phi \in \mathcal{M}_s$, then we have that $u_t \in \mathcal{M}_t$ for all $t \geq s$.

- (iv) \mathcal{M} exponentially attracts all the solutions to Equation (6), i.e., for any solution $u(\cdot)$ of (6) and any fixed $s \in \mathbb{R}$, there exists a positive constant H and a solution u_t^* lying in \mathcal{M} such that

$$(7) \quad \|u_t - u_t^*\|_{\mathcal{C}_g^\beta} \leq H e^{-\gamma(t-s)} \quad \text{for } t \geq s.$$

We can now construct the form of solutions to Equation (6) which belongs to rescaled admissible spaces on the half-line $(-\infty, t_0]$ in the following lemma.

Lemma 3.1. *Let A satisfy Hypothesis 2.1. Let E, E' and $\varphi \in E'$ be as in Hypothesis 2.7. Suppose that $F : \mathbb{R} \times \mathcal{C}_g^\beta \rightarrow X$ is φ -Lipschitz. For fixed $t_0 \in \mathbb{R}$ let $u(\cdot)$ be a solution to Equation (6) such that $u(t) \in X_\beta$ for all $t \leq t_0$ and the function*

$$z(t) = e^{-\gamma(t_0-t)} \|u_t\|_{\mathcal{C}_g^\beta}, \quad t \leq t_0,$$

belongs to $E_{(-\infty, t_0]}$. Then,

$$(8) \quad u(t) = e^{-(t-t_0)A} p + \int_{-\infty}^{t_0} \mathcal{G}(t, \tau) F(\tau, u_\tau) d\tau, \quad \forall t \leq t_0,$$

where $p \in PX$ and $\mathcal{G}(t, \tau)$ is the Green function defined as in (4).

Proof. By the definition of $\mathcal{G}(\cdot, \cdot)$ one can see that

$$v(t) := \int_{-\infty}^{t_0} \mathcal{G}(t, \tau) F(\tau, u_\tau) d\tau \in X_\beta, \quad \forall t \leq t_0.$$

Furthermore, for $-\infty < \theta \leq 0$ we have

$$(9) \quad \begin{aligned} & e^{-\gamma(t_0-t)} \|A^\beta v(t+\theta)\| \\ & \leq \int_{-\infty}^{t_0} \|e^{\gamma(t-\tau)} A^\beta \mathcal{G}(t+\theta, \tau)\| \|\varphi(\tau) e^{-\gamma(t_0-\tau)} (1 + \|u_\tau\|_{\mathcal{C}_g^\beta}) d\tau \\ & \leq e^{-\gamma\theta} \int_{-\infty}^{t_0} \|e^{\gamma(t+\theta-\tau)} A^\beta \mathcal{G}(t+\theta, \tau)\| \|\varphi(\tau) w(\tau) d\tau. \end{aligned}$$

Here, $w(\cdot) := e^{-\gamma(t_0-\cdot)} + \|z(\cdot)\| \in E_{(-\infty, t_0]}$.

Now, by using properties of Green function one has

$$\begin{aligned} & \int_{-\infty}^{t_0} \|e^{\gamma(t+\theta-\tau)} A^\beta \mathcal{G}(t+\theta, \tau)\| \|\varphi(\tau) w(\tau) d\tau \\ & \leq \int_{-\infty}^{t+\theta} \left(\left(\frac{\beta}{t+\theta-\tau} \right)^\beta + \lambda_{N+1}^\beta \right) e^{-\alpha(t+\theta-\tau)} \varphi(\tau) w(\tau) d\tau \\ & \quad + \int_{t+\theta}^{t_0} \lambda_N^\beta e^{-\alpha(\tau-t-\theta)} \varphi(\tau) w(\tau) d\tau \\ & \leq \int_{-\infty}^{t+\theta} \left(\frac{\beta}{t+\theta-\tau} \right)^\beta e^{-\alpha(t+\theta-\tau)} \varphi(\tau) w(\tau) d\tau + \\ & \quad + \left(e^{-\alpha\theta} \lambda_{N+1}^\beta + e^{\alpha\theta} \lambda_N^\beta \right) \|e^{-\alpha(t-\cdot)} \varphi(\cdot)\|_{E'_{(-\infty, t_0]}} \|w\|_{E_{(-\infty, t_0]}} \end{aligned}$$

and

$$\begin{aligned} & \int_{-\infty}^{t+\theta} \left(\frac{\beta}{t+\theta-\tau} \right)^\beta e^{-\alpha(t+\theta-\tau)} \varphi(\tau) w(\tau) d\tau \\ & = \int_{-\infty}^{t+\theta-1} \left(\frac{\beta}{t+\theta-\tau} \right)^\beta e^{-\alpha(t+\theta-\tau)} \varphi(\tau) w(\tau) d\tau \end{aligned}$$

$$\begin{aligned}
 & + \int_{t+\theta-1}^{t+\theta} \left(\frac{\beta}{t+\theta-\tau} \right)^\beta e^{-\alpha(t+\theta-\tau)} \varphi(\tau) w(\tau) d\tau \\
 \leq & \beta^\beta e^{-\alpha\theta} \int_{-\infty}^{t+\theta-1} e^{-\alpha(t-\tau)} \varphi(\tau) w(\tau) d\tau \\
 & + \beta^\beta e^{-\alpha\theta} \left(\int_{t+\theta-1}^{t+\theta} \frac{1}{(t+\theta-\tau)^{\frac{1+\beta}{2}}} d\tau \right)^{\frac{2\beta}{1+\beta}} \\
 & \times \left(\int_{t+\theta-1}^{t+\theta} e^{-\alpha \frac{1+\beta}{1-\beta} (t-\tau)} (\varphi(\tau) w(\tau))^{\frac{1+\beta}{1-\beta}} d\tau \right)^{\frac{1-\beta}{1+\beta}} \\
 \leq & \beta^\beta e^{-\alpha\theta} \|e^{-\alpha(\cdot-\cdot)} \varphi(\cdot)\|_{E'_{(-\infty, t_0)}} \|w\|_{E_{(-\infty, t_0)}} \\
 & + \beta^\beta e^{-\alpha\theta} \left(\frac{2}{1-\beta} \right)^{\frac{2\beta}{1+\beta}} \|e^{-\alpha \frac{1+\beta}{1-\beta} (t-\cdot)} \varphi^{\frac{1+\beta}{1-\beta}}(\cdot)\|_{E'_{(-\infty, t_0)}}^{\frac{1-\beta}{1+\beta}} \|w^{\frac{1+\beta}{1-\beta}}\|_{E_{(-\infty, t_0)}}^{\frac{1-\beta}{1+\beta}}.
 \end{aligned}$$

The inequalities just mentioned show that

$$\begin{aligned}
 & \int_{-\infty}^{t_0} \|e^{\gamma(t+\theta-\tau)} A^\beta \mathcal{G}(t+\theta, \tau) \varphi(\tau) w(\tau) d\tau \\
 (10) \quad & \leq k(t, \theta) \max \left\{ \|w\|_{E_{(-\infty, t_0)}}, \|w^{\frac{1+\beta}{1-\beta}}\|_{E_{(-\infty, t_0)}}^{\frac{1-\beta}{1+\beta}} \right\} \\
 & \leq k(t, \theta) \|w\|_\beta, \quad \forall t \leq t_0
 \end{aligned}$$

with

$$\begin{aligned}
 k(t, \theta) = & \beta^\beta e^{-\alpha\theta} \left[\|e^{-\alpha|t-\cdot|} \varphi(\cdot)\|_{E'} + \left(\frac{2}{1-\beta} \right)^{\frac{2\beta}{1+\beta}} \|e^{-\alpha \frac{1+\beta}{1-\beta} |t-\cdot|} \varphi^{\frac{1+\beta}{1-\beta}}(\cdot)\|_{E'}^{\frac{1-\beta}{1+\beta}} \right] \\
 & + \left(e^{-\alpha\theta} \lambda_{N+1}^\beta + e^{\alpha\theta} \lambda_N^\beta \right) \|e^{-\alpha(t-\cdot)} \varphi(\cdot)\|_{E'}.
 \end{aligned}$$

Plugging (10) into (9), we have

$$\begin{aligned}
 & e^{\gamma(t_0-t)} \|v_t\|_{\mathcal{E}_g^\beta} = e^{\gamma(t_0-t)} \sup_{\theta \leq 0} \frac{\|A^\beta v_t(\theta)\|}{g(\theta)} \\
 (11) \quad & = e^{\gamma(t_0-t)} \sup_{\theta \leq 0} \frac{\|A^\beta v(t+\theta)\|}{g(\theta)} \\
 & \leq \ell(t) \|w\|_\beta,
 \end{aligned}$$

where

$$\begin{aligned}
 \ell(t) = & \sup_{\theta \leq 0} \frac{e^{-\lambda_{N+1}\theta}}{g(\theta)} \left[\left(\beta^\beta + \lambda_{N+1}^\beta + \lambda_N^\beta \right) \cdot \|e^{-\alpha|t-\cdot|} \varphi(\cdot)\|_{E'} \right. \\
 & \left. + \beta^\beta \left(\frac{2}{1-\beta} \right)^{\frac{2\beta}{1+\beta}} \|e^{-\alpha \frac{1+\beta}{1-\beta} |t-\cdot|} \varphi^{\frac{1+\beta}{1-\beta}}(\cdot)\|_{E'}^{\frac{1-\beta}{1+\beta}} \right].
 \end{aligned}$$

Since $\ell(\cdot) \in E_{(-\infty, t_0]}$, and by the admissibility of $E_{(-\infty, t_0]}$ we arrive at

$$e^{\gamma(t_0-t)} \|v_t\|_{\mathcal{G}_g^\beta} \in E_{(-\infty, t_0]}.$$

It is clear that $v(\cdot)$ satisfies the following integral equation

$$v(t_0) = e^{-(t_0-t)A} v(t) + \int_t^{t_0} e^{-(t_0-\tau)A} F(\tau, u_\tau) d\tau \quad \text{for } t \leq t_0.$$

On the other hand,

$$u(t_0) = e^{-(t_0-t)A} u(t) + \int_t^{t_0} e^{-(t_0-\tau)A} F(\tau, u_\tau) d\tau.$$

Hence,

$$(12) \quad u(t_0) - v(t_0) = e^{-(t_0-t)A} [u(t) - v(t)].$$

Applying the operator $A^\beta(I - P)$ to (12), we have

$$\begin{aligned} & \|A^\beta(I - P)[u(t_0) - v(t_0)]\| \\ &= \|e^{-(t_0-t)A} A^\beta(I - P)[u(t) - v(t)]\| \\ &\leq N e^{-(\lambda_{N+1}-\gamma)(t_0-t)} \|I - P\| \|e^{-\gamma(t_0-t)} A^\beta(u(t) - v(t))\|. \end{aligned}$$

Since $\text{esssup}_{t \leq t_0} \|e^{-\gamma(t_0-t)} A^\beta(u(t) - v(t))\| < \infty$, letting $t \rightarrow -\infty$ we obtain that

$$\|A^\beta(I - P)[u(t_0) - y(t_0)]\| = 0, \quad \text{hence } A^\beta(I - P)[u(t_0) - y(t_0)] = 0.$$

Since A^β is injective, it follows that $(I - P)[u(t_0) - y(t_0)] = 0$. Thus,

$$p := u(t_0) - y(t_0) \in PX.$$

Using the fact that the restriction of $e^{-(t_0-t)A}$ to PX is invertible with the inverse $e^{-(t-t_0)A}$ we obtain that

$$u(t) = e^{-(t-t_0)A} p + v(t) = e^{-(t-t_0)A} p + \int_{-\infty}^{t_0} \mathcal{G}(t, \tau) F(\tau, u_\tau) d\tau \quad \text{for } t \leq t_0.$$

This finishes the proof. \square

Remark 3.2. Equation (8) is called *Lyapunov-Perron equation* which will be used to determine the admissible inertial manifold for Equation (6). By direct computation, one can see that the converse of Lemma 3.1 is also true. It means, all solutions of Equation (8) satisfies Equation (6) for $t \leq t_0$.

We now show the existence of rescaling solutions to Equation (6) on negative half-line which belong to an admissible Banach function space in the following lemma.

Lemma 3.3. *Let A satisfy Hypothesis 2.1. Let E , E' and $\varphi \in E'$ be as in Hypothesis 2.7. For $0 \leq \beta < 1$, we define the function $\ell : \mathbb{R} \rightarrow E$ by*

$$(13) \quad \begin{aligned} \ell(t) = \sup_{\theta \leq 0} \frac{e^{-\lambda_{N+1}\theta}}{g(\theta)} & \left[\left(\beta^\beta + \lambda_{N+1}^\beta + \lambda_N^\beta \right) \cdot \|e^{-\alpha|t-\cdot|} \varphi(\cdot)\|_{E'} \right. \\ & \left. + \beta^\beta \left(\frac{2}{1-\beta} \right)^{\frac{2\beta}{1+\beta}} \|e^{-\alpha \frac{1+\beta}{1-\beta} |t-\cdot|} \varphi^{\frac{1+\beta}{1-\beta}}(\cdot)\|_{E'}^{\frac{1-\beta}{1+\beta}} \right]. \end{aligned}$$

Let $F : \mathbb{R} \times \mathcal{C}_g^\beta \rightarrow X$ be φ -Lipschitz such that $\|\ell\|_\beta < 1$ where the norm $\|\cdot\|_\beta$ is defined as in Hypothesis 2.7. Then, there corresponds to each $p \in PX$ one and only one solution $u(\cdot)$ of Equation (6) on $(-\infty, t_0]$ satisfying the conditions that $Pu(t_0) = p$ and the function

$$z(t) := e^{-\gamma(t_0-t)} \|u_t\|_{\mathcal{C}_g^\beta}, \quad t \leq t_0,$$

belongs to $E_{(-\infty, t_0]}$.

Proof. We set

$$\begin{aligned} \mathcal{E}_g^{\gamma, t_0, \beta} := \{h : (-\infty, t_0] \rightarrow X_\beta \mid h \text{ is strongly measurable,} \\ \text{and } e^{-\gamma(t_0-\cdot)} \|h\|_{\mathcal{C}_g^\beta} \in E_{(-\infty, t_0]}^\beta \} \end{aligned}$$

with the norm

$$\|h\|_{\gamma, \beta} := \|e^{-\gamma(t_0-\cdot)} h\|_{\mathcal{C}_g^\beta} \|_\beta.$$

For each $t_0 \in \mathbb{R}$, $u \in \mathcal{E}_g^{\gamma, t_0, \beta}$ and $p \in PX$ we define

$$T(p, u)(t) = e^{-(t-t_0)A} p + \int_{-\infty}^{t_0} \mathcal{G}(t, \tau) F(\tau, u_\tau) d\tau \quad \text{for } t \leq t_0.$$

Then, for $p \in PX$, $u \in \mathcal{E}_g^{\gamma, t_0, \beta}$ and $t \leq t_0$ we have

$$\begin{aligned} & \|e^{-\gamma(t_0-t)} A^\beta T(p, u)(t + \theta)\| \\ & \leq \lambda_N^\beta e^{-\lambda_N \theta} e^{-\alpha(t_0-t)} \|A^\beta p\| + e^{-\gamma \theta} \int_{-\infty}^{t_0} \|e^{\gamma(t+\theta-\tau)} A^\beta \mathcal{G}(t + \theta, \tau)\| \varphi(\tau) w(\tau) d\tau. \end{aligned}$$

Noting that, $w(\cdot) := e^{-\gamma(t_0-\cdot)} + \|z(\cdot)\| \in E_{(-\infty, t_0]}$.

By using (10) and (11) we obtain that

$$e^{-\gamma(t_0-t)} \| [T(p, u)]_t \|_{\mathcal{C}_g^\beta} \leq \lambda_N^\beta \cdot \sup_{\theta \leq 0} \frac{e^{-\lambda_N \theta}}{g(\theta)} \cdot \|e^{-\alpha(t_0-\cdot)}\|_\beta \cdot \|A^\beta p\| + \ell(t) \|w\|_\beta$$

and therefore,

$$\|T(p, u)\|_{\gamma, \beta} \leq \lambda_N^\beta \sup_{\theta \leq 0} \frac{e^{-\lambda_N \theta}}{g(\theta)} \cdot \|e^{-\alpha(t_0-\cdot)}\|_\beta \cdot \|A^\beta p\| + \|\ell\|_\beta \|w\|_\beta,$$

i.e., the transformation T acts from $PX \times \mathcal{E}_g^{\gamma, t_0, \beta}$ into $\mathcal{E}_g^{\gamma, t_0, \beta}$.

Next, for any $u, v \in \mathcal{E}_g^{\gamma, t_0, \beta}$ and $p := Pu(t_0), q := Pv(t_0)$, we have

$$\begin{aligned} & \|e^{-\gamma(t_0-t)} A^\beta [T(p, u) - T(q, v)](t + \theta)\| \\ & \leq \lambda_N^\beta e^{-\lambda_N \theta} \|e^{-\alpha(t_0-\cdot)}\|_\beta \|A^\beta(p - q)\| \\ & \quad + \int_{-\infty}^{t_0} \|e^{-\gamma(t_0-t)} A^\beta \mathcal{G}(t + \theta, \tau) [F(\tau, u_\tau) - F(\tau, v_\tau)]\| d\tau. \end{aligned}$$

It follows that

$$\|T(p, u) - T(q, v)\|_{\gamma, \beta} \leq \lambda_N^\beta \cdot \sup_{\theta \leq 0} \frac{e^{-\lambda_N \theta}}{g(\theta)} \|e^{-\alpha(t_0-\cdot)}\|_\beta \|A^\beta(p - q)\| + \|\ell\|_\beta \|u - v\|_{\gamma, \beta}.$$

Therefore, the condition $\|m(\cdot)\|_\beta < 1$ implies that T is a strict contraction in $\mathcal{E}_g^{\gamma, t_0, \beta}$, uniformly in PX (if $p = q$). Thus, there exists a unique $u \in \mathcal{E}_g^{\gamma, t_0, \beta}$ such that $T(u, p) = u$, and by definition of T we have that $u(\cdot)$ is the unique solution in $\mathcal{E}_g^{\gamma, t_0, \beta}$ of Equation (8) for $t \leq t_0$. Lemma 3.1 and Remark 3.2 show that $u(\cdot)$ is the unique solution in $\mathcal{E}_g^{\gamma, t_0, \beta}$ of Equation (6) for $t \leq t_0$. \square

Theorem 3.4. *Let A satisfy Hypothesis 2.1. Let E, E' and $\varphi \in E'$ be as in Hypothesis 2.7. Set*

$$\begin{aligned} \kappa = \frac{\|\Lambda_1 \varphi\|_\infty}{1 - e^{-\alpha}} & \left[\frac{N_1 N_2 \lambda_N^{2\beta} \|e_\alpha\| \|n\|_\beta}{1 - \|n\|_\beta} \sup_{\theta \leq 0} \frac{e^{-\lambda_N \theta}}{g(\theta)} \left(\sup_{\theta \leq 0} \frac{e^{-\gamma \theta}}{g(\theta)} \right)^2 \right. \\ & \left. + N_1 \beta^\beta + N_1 \lambda_{N+1}^\beta + N_2 \lambda_N^\beta \right] + \beta \left(\frac{2}{1 - \beta} \right)^{\frac{2\beta}{1+\beta}} \|\Lambda_1 \varphi\|_{\frac{1+\beta}{1-\beta}}^{\frac{1+\beta}{1-\beta}} \|\cdot\|_\infty^{\frac{1-\beta}{1+\beta}}, \end{aligned}$$

and

$$(14) \quad \Delta = \sup_{\theta \leq 0} \frac{e^{-\gamma \theta}}{g(\theta)} \cdot \kappa.$$

Let F be φ -Lipschitz and suppose that

$$(15) \quad \max \{ \|\ell\|_\beta, \Delta \} < 1$$

where the function ℓ is defined by (13). Then, Equation (6) has an admissible inertial manifold of \mathcal{E} -class.

Proof. We start by defining a collection of surfaces $\{\mathcal{M}_{t_0}\}_{t_0 \in \mathbb{R}}$ by

$$\mathcal{M}_{t_0} = \left\{ \hat{p} + \Phi_{t_0}(\hat{p}(0)) \mid \hat{p} \in \hat{P}\mathcal{E}_g^\beta \right\} \subset \mathcal{E}_g^\beta.$$

Here, for each $t_0 \in \mathbb{R}$ the mapping $\Phi_{t_0} : PX \rightarrow (I - \hat{P})\mathcal{E}_g^\beta$ is defined by

$$\Phi_{t_0}(p)(\theta) = \int_{-\infty}^{t_0} \mathcal{G}(t_0 + \theta, \tau) F(\tau, u_\tau) d\tau, \quad \forall p \in PX, \quad \forall \theta \leq 0,$$

where $u(\cdot)$ is the unique solution in $\mathcal{E}_g^{\gamma, t_0, \beta}$ of Equation (6) satisfying that $Pu(t_0) = p$.

Noting that, the existence and uniqueness of $u(\cdot)$ is proved in Lemma 3.3.

Next, we prove that Φ_{t_0} is Lipschitz continuous with Lipschitz constant independent of t_0 . For this purpose, taking any p and q in PX , letting $u(\cdot)$ and $v(\cdot)$ be the solutions to Equation (8) with $Pu(t_0) = p$ and $Pv(t_0) = q$ respectively, and using the formula (8) for $u(\cdot)$ and $v(\cdot)$ we then have

$$\begin{aligned} & e^{-\gamma(t_0-t)} \|A^\beta [u(t+\theta) - v(t+\theta)]\| \\ & \leq e^{-\lambda_N \theta} \lambda_N^\beta e^{-\alpha(t-t_0)} \|A^\beta(p-q)\| + \ell(t) \|u-v\|_{\gamma,\beta}. \end{aligned}$$

Therefore,

$$\|u-v\|_{\gamma,\beta} \leq N_1 \lambda_N^\beta \sup_{\theta \leq 0} \frac{e^{-\lambda_N \theta}}{g(\theta)} \|e_\alpha\| \|A^\beta(p-q)\| + \|\ell\|_\beta \|u-v\|_{\gamma,\beta}.$$

Since $\|\ell\|_\beta < 1$, we arrive at

$$(16) \quad \|u-v\|_{\gamma,\beta} \leq \frac{N_1 \lambda_N^\beta \|e_\alpha\|}{1 - \|\ell\|_\beta} \cdot \sup_{\theta \leq 0} \frac{e^{-\lambda_N \theta}}{g(\theta)} \cdot \|A^\beta(p-q)\|.$$

Next, from the definition of Φ_{t_0} it follows that

$$\begin{aligned} & \|A^\beta(\Phi_{t_0}(p)(\theta) - \Phi_{t_0}(q)(\theta))\| \\ & \leq \int_{-\infty}^{t_0} \|A^\beta \mathcal{G}(t_0 + \theta, \tau)\| \cdot \|F(\tau, u_\tau) - F(\tau, v_\tau)\| d\tau \\ & \leq e^{-\gamma\theta} \int_{-\infty}^{t_0} \|e^{\gamma(t_0+\theta-\tau)} A^\beta \mathcal{G}(t_0 + \theta, \tau)\| \cdot \varphi(\tau) e^{-\gamma(t_0-\tau)} \|u_\tau - v_\tau\|_{\mathcal{C}_g^\beta} d\tau \\ & \leq e^{-\gamma\theta} \|\ell\|_\beta \cdot \|u-v\|_{\gamma,\beta} \\ & \leq e^{-\gamma\theta} \cdot \frac{N_1 \lambda_N^\beta \|e_\alpha\| \|\ell\|_\beta}{1 - \|\ell\|_\beta} \cdot \sup_{\theta \leq 0} \frac{e^{-\lambda_N \theta}}{g(\theta)} \cdot \|A^\beta(p-q)\|. \end{aligned}$$

Here, we used the estimate (16).

Latter inequality shows that

$$\|\Phi_{t_0}(p) - \Phi_{t_0}(q)\|_{\mathcal{C}_g^\beta} \leq \frac{N_1 \lambda_N^\beta \|e_\alpha\| \|\ell\|_\beta}{1 - \|\ell\|_\beta} \cdot \sup_{\theta \leq 0} \frac{e^{-\lambda_N \theta}}{g(\theta)} \cdot \sup_{\theta \leq 0} \frac{e^{-\gamma\theta}}{g(\theta)} \cdot \|A^\beta(p-q)\|,$$

i.e., Φ_{t_0} is Lipschitzian and its Lipschitz constant

$$C = \frac{N_1 \lambda_N^\beta \|e_\alpha\| \|\ell\|_\beta}{1 - \|\ell\|_\beta} \cdot \sup_{\theta \leq 0} \frac{e^{-\lambda_N \theta}}{g(\theta)} \cdot \sup_{\theta \leq 0} \frac{e^{-\gamma\theta}}{g(\theta)}$$

is independent of t_0 .

The property (ii) follows from Lemma 3.1, Lemma 3.3 and Remark 3.2.

To prove the property (iii), we fix any $s \in \mathbb{R}$ and let $u(\cdot)$ be the solution to Equation (6) such that $u_s \in \mathcal{M}_s$, i.e.,

$$u_s(\theta) = e^{-\theta A} p_1 + \int_{-\infty}^s \mathcal{G}(s + \theta, \tau) F(\tau, u_\tau) d\tau, \quad \forall \theta \leq 0,$$

where $p_1 \in PX$. Then, we have to prove that $u_{t_0} \in \mathcal{M}_{t_0}$ for all $t_0 > s$. We fix an arbitrary number $t_0 \in (s, \infty)$ and define a function $w(t)$ on $(-\infty, t_0]$ by

$$w(t) = \begin{cases} u(t) & \text{if } t \in (s, t_0], \\ v(t) & \text{if } t \in (-\infty, s], \end{cases}$$

where $v(\cdot)$ is the unique solution in $\mathcal{E}_g^{\gamma, t_0, \beta}$ of Equation (6) with $v_s = u_s$.

It is clear that $w(t)$ is continuous, bounded on $(-\infty, t_0]$ and $u_{t_0} = w_{t_0}$, so we need to prove $w_{t_0} \in \mathcal{M}_{t_0}$.

For $t \in [s, t_0]$, we have

$$\begin{aligned} (17) \quad w(t) &= e^{-(t-s)A}u(s) + \int_s^t e^{-(t-\tau)A}F(\tau, w_\tau)d\tau \\ &= e^{-(t-s)A}p_1 + \int_{-\infty}^t e^{-(t-\tau)A}(I-P)F(\tau, w_\tau) \\ &\quad + \int_s^t e^{-(t-\tau)A}PF(\tau, w_\tau)d\tau \\ &= e^{-(t-t_0)A}p_2 + \int_{-\infty}^{t_0} \mathcal{G}(t, \tau)F(\tau, w_\tau)d\tau, \end{aligned}$$

where

$$p_2 = e^{-(t_0-s)A}p_1 + \int_s^{t_0} e^{-(t_0-\tau)A}(I-P)F(\tau, w_\tau)d\tau \in PX.$$

Obviously, Equation (17) also remains true for $t \in (-\infty, s]$. Therefore, for all $t_0 \geq s$, there exists $p_2 \in PX$ such that

$$w_{t_0}(\theta) = w(t_0 + \theta) = e^{-\theta A}p_2 + \int_{-\infty}^{t_0} \mathcal{G}(t_0 + \theta, \tau)F(\tau, w_\tau)d\tau.$$

This means $w_{t_0} \in \mathcal{M}_{t_0}$ and thus $u_{t_0} \in \mathcal{M}_{t_0}$ for all $t_0 \geq s$.

Lastly, we prove the property (iv) of the admissible inertial manifold. To do this, we will prove that for any solution $u(\cdot)$ to Equation (6) with $u_s \in \mathcal{M}_s$ there is a solution $u^*(\cdot)$ of Equation (6) such that $u_t^* \in \mathcal{M}_t$ for $t \geq s$ and

$$\|u_t - u_t^*\|_{\mathcal{E}_g^\beta} \leq H e^{-\gamma(t-s)}.$$

In this case, $u^*(\cdot)$ is called an induced trajectory for $u(\cdot)$ on the manifold $\{\mathcal{M}_t\}$. We will find the induced trajectory in the form $u^*(t) = u(t) + w(t)$ such that

$$\|w\|_{s,+} = \sup_{t \geq s} e^{\gamma(t-s)} \|A^\beta w(t)\| < +\infty.$$

Substituting $u^*(\cdot)$ to Equation (6) we obtain that $u^*(\cdot)$ is a solution to (6) for $t \geq s$ if and only if $w(\cdot)$ is a solution to the equation

$$(18) \quad w(t) = e^{-(t-s)A}w(s) + \int_s^t e^{-(t-\tau)A} [F(\tau, u_\tau + w_\tau) - F(\tau, u_\tau)] d\tau.$$

Put

$$\mathcal{F}(t, w_t) = F(t, u_t + w_t) - F(t, u_t),$$

and set

$$\mathcal{L}_{\gamma, s}^+ = \left\{ v \in C(\mathbb{R}; X_\beta) \mid \sup_{t \geq s} e^{\gamma(t-s)} \|A^\beta v(t)\| < +\infty \right\}.$$

Then, one can see that a function $w(\cdot) \in \mathcal{L}_{\gamma, s}^+$ is a solution to (18) if and only if it satisfies

$$(19) \quad w(t) = e^{-(t-s)A} \hat{q}(0) + \int_s^\infty \mathcal{G}(t, \tau) \mathcal{F}(\tau, w_\tau) d\tau$$

for $t \geq s$ and some $\hat{q} \in (I - \hat{P})\mathcal{C}_g^\beta$ is chosen such that $u_s^* = u_s + w_s \in \mathcal{M}_s$, i.e.,

$$(I - \hat{P})(u_s + w_s)(\theta) = \Phi_s(\hat{P}(u_s + w_s)(0))(\theta).$$

By (19) we claim that

$$w_s(\theta) = \hat{q}(\theta) - e^{-\theta A} \int_s^\infty e^{-(s-\tau)A} P \mathcal{F}(\tau, w_\tau) d\tau, \quad \forall \theta \leq 0.$$

Hence

$$\hat{P}(u_s + w_s)(\theta) = \hat{P}u_s(\theta) - e^{-\theta A} \int_s^\infty e^{-(s-\tau)A} P \mathcal{F}(\tau, w_\tau) d\tau$$

and therefore

$$\begin{aligned} \hat{q}(\theta) &= (I - \hat{P})w_s(\theta) \\ &= -(I - \hat{P})u_s(\theta) + \Phi_s \left(Pu(s) - \int_s^\infty e^{-(s-\tau)A} P \mathcal{F}(\tau, w_\tau) d\tau \right) (\theta). \end{aligned}$$

Substituting this into (19) we have

$$(20) \quad \begin{aligned} w(t) &= e^{-(t-s)A} \left[-(I - \hat{P})u(s) + \Phi_s \left(Pu(s) - \int_s^\infty e^{-(s-\tau)A} P \mathcal{F}(\tau, w_\tau) d\tau \right) (0) \right] \\ &\quad + \int_s^\infty \mathcal{G}(t, \tau) \mathcal{F}(\tau, w_\tau) d\tau. \end{aligned}$$

In order to prove the existence of u^* satisfying (7) we have to show that (20) has a solution $w(\cdot)$ belongs to $\mathcal{L}_{\gamma, s}^+$. To this purpose, we will prove that the transformation \mathcal{T} defined by

$$(\mathcal{T}w)(t) = e^{-(t-s)A} \hat{q}(0) + \int_s^\infty \mathcal{G}(t, \tau) \mathcal{F}(\tau, w_\tau) d\tau \quad \text{for } t \geq s$$

acts from $QX \times \mathcal{L}_{\gamma, s}^+$ into $\mathcal{L}_{\gamma, s}^+$ and is a contraction in $\mathcal{L}_{\gamma, s}^+$.

Indeed, for $w(\cdot) \in \mathcal{L}_{\gamma, s}^+$, and for each $\theta \in (-\infty, 0]$ since

$$\begin{aligned} \|A^\beta w(t + \theta)\| &= e^{-\gamma(t+\theta-s)} \cdot e^{\gamma(t+\theta-s)} \|A^\beta w(t + \theta)\| \\ &\leq e^{-\gamma(t+\theta-s)} \sup_{t+\theta \geq s} e^{\gamma(t+\theta-s)} \|A^\beta w(t + \theta)\|, \end{aligned}$$

we have

$$\frac{\|A^\beta w(t+\theta)\|}{g(\theta)} \leq \frac{e^{-\gamma\theta}}{g(\theta)} \cdot e^{-\gamma(t-s)} \|w\|_{s,+}$$

and

$$(21) \quad \|\mathcal{F}(t, w_t)\| \leq \varphi(t) \|w_t\|_{\mathcal{C}_g^\beta} \leq \varphi(t) \cdot \sup_{\theta \leq 0} \frac{e^{-\gamma\theta}}{g(\theta)} \cdot e^{-\gamma(t-s)} \|w\|_{s,+}.$$

Therefore,

$$(22) \quad \begin{aligned} & e^{\gamma(t-s)} \|A^\beta(\mathcal{T}w)(t)\| \\ & \leq e^{-(\lambda_{N+1}-\gamma)(t-s)} \|A^\beta \hat{q}(0)\| \\ & \quad + \sup_{\theta \leq 0} \frac{e^{-\gamma\theta}}{g(\theta)} \cdot \|w\|_{s,+} \int_s^{+\infty} e^{\gamma(t-\tau)} \|A^\beta \mathcal{G}(t, \tau)\| \varphi(\tau) d\tau \\ & \leq \|A^\beta \hat{q}(0)\| + \sup_{\theta \leq 0} \frac{e^{-\gamma\theta}}{g(\theta)} \cdot \|w\|_{s,+} \int_s^{+\infty} e^{\gamma(t-\tau)} \|A^\beta \mathcal{G}(t, \tau)\| \varphi(\tau) d\tau. \end{aligned}$$

Set $\hat{Q} = I - \hat{P}$, we have

$$\begin{aligned} & \|A^\beta \hat{q}(0)\| \\ & = \|A^\beta \left[-\hat{Q}u_s(0) + \Phi_s \left(Pu(s) - \int_s^\infty e^{-(s-\tau)A} P \mathcal{F}(\tau, w_\tau) d\tau \right) (0) \right]\| \\ & \leq \|A^\beta (\Phi_s(Pu(s))(0) - \hat{Q}u_s(0))\| \\ & \quad + \|A^\beta \left[\Phi_s(Pu(s))(0) - \Phi_s \left(Pu(s)(0) - \int_s^\infty e^{-(s-\tau)A} P \mathcal{F}(\tau, w_\tau) d\tau \right) (0) \right]\| \\ & \leq \hbar + \frac{N_1 \lambda_N^\beta \|e_\alpha\| \|\ell\|_\beta}{1 - \|\ell\|_\beta} \cdot \sup_{\theta \leq 0} \frac{e^{-\lambda_N \theta}}{g(\theta)} \cdot \sup_{\theta \leq 0} \frac{e^{-\gamma\theta}}{g(\theta)} \int_s^\infty e^{-(s-\tau)A} P \mathcal{F}(\tau, w_\tau) d\tau \end{aligned}$$

where $\hbar = \|\Phi_s(Pu(s)) - \hat{Q}u_s\|_{\mathcal{C}_g^\beta}$.

Now, by (3), (5) and (21) we obtain that

$$(23) \quad \begin{aligned} & \|A^\beta \hat{q}(0)\| \\ & \leq \hbar + \frac{N_1 \lambda_N^\beta \|e_\alpha\| \|\ell\|_\beta}{1 - \|\ell\|_\beta} \sup_{\theta \leq 0} \frac{e^{-\lambda_N \theta}}{g(\theta)} \left[\sup_{\theta \leq 0} \frac{e^{-\gamma\theta}}{g(\theta)} \right]^2 \|w\|_{s,+} \int_s^\infty e^{\gamma(s-\tau)} \|A^\beta \mathcal{G}(s, \tau)\| \varphi(\tau) d\tau \\ & \leq \hbar + \frac{N_1 \lambda_N^{2\beta} \|e_\alpha\| \|\ell\|_\beta}{1 - \|\ell\|_\beta} \sup_{\theta \leq 0} \frac{e^{-\lambda_N \theta}}{g(\theta)} \left[\sup_{\theta \leq 0} \frac{e^{-\gamma\theta}}{g(\theta)} \right]^2 \|w\|_{s,+} \int_s^\infty e^{-\alpha(\tau-s)} \varphi(\tau) d\tau \\ & \leq \hbar + \frac{N_1 \lambda_N^{2\beta} \|e_\alpha\| \|\ell\|_\beta}{1 - \|\ell\|_\beta} \sup_{\theta \leq 0} \frac{e^{-\lambda_N \theta}}{g(\theta)} \left[\sup_{\theta \leq 0} \frac{e^{-\gamma\theta}}{g(\theta)} \right]^2 \frac{N_2}{1 - e^{-\alpha}} \|\Lambda_1 \varphi\|_\infty \|w\|_{s,+}. \end{aligned}$$

For the integral in the second term of (22), by using (5) we have

$$\begin{aligned}
 & \int_s^{+\infty} e^{\gamma(t-\tau)} \|A^\beta G(t, \tau)\| \varphi(\tau) d\tau \\
 & \leq \int_{-\infty}^t \|e^{\gamma(t-\tau)} A^\beta G(t, \tau)\| \varphi(\tau) d\tau + \int_t^{+\infty} \|e^{\gamma(t-\tau)} A^\beta G(t, \tau)\| \varphi(\tau) d\tau \\
 & \leq \int_{-\infty}^t \left(\frac{\beta}{t-\tau}\right)^\beta e^{-\alpha(t-\tau)} \varphi(\tau) d\tau + \frac{N_1 \lambda_{N+1}^\beta + N_2 \lambda_N^\beta}{1-e^{-\alpha}} \|\Lambda_1 \varphi\|_\infty \\
 (24) \quad & \leq \int_{-\infty}^{t-1} \left(\frac{\beta}{t-\tau}\right)^\beta e^{-\alpha(t-\tau)} \varphi(\tau) d\tau + \int_{t-1}^t \left(\frac{\beta}{t-\tau}\right)^\beta e^{-\alpha(t-\tau)} \varphi(\tau) d\tau \\
 & \quad + \frac{N_1 \lambda_{N+1}^\beta + N_2 \lambda_N^\beta}{1-e^{-\alpha}} \|\Lambda_1 \varphi\|_\infty \\
 & \leq \frac{N_1 \beta^\beta + N_1 \lambda_{N+1}^\beta + N_2 \lambda_N^\beta}{1-e^{-\alpha}} \|\Lambda_1 \varphi\|_\infty + \beta \left(\frac{2}{1-\beta}\right)^{\frac{2\beta}{1+\beta}} \|\Lambda_1 \varphi\|_\infty^{\frac{1+\beta}{1-\beta}} \|\varphi\|_\infty^{\frac{1-\beta}{1+\beta}}.
 \end{aligned}$$

Substituting the estimates (23) and (24) to (22) we obtain that

$$(25) \quad e^{\gamma(t-s)} \|A^\beta(\mathcal{T}w)(t)\| \leq \hbar + \Delta \|w\|_{s,+}, \quad \forall t \geq s$$

and that

$$\|\mathcal{T}w\|_{s,+} := \sup_{t \geq s} e^{\gamma(t-s)} \|A^\beta(\mathcal{T}w)(t)\| \leq \hbar + \Delta \|w\|_{s,+},$$

where Δ is defined as in (14). Therefore, $\mathcal{T} : QX \times \mathcal{L}_{\gamma,s}^+ \rightarrow \mathcal{L}_{\gamma,s}^+$.

Using now the inequality

$$\|\mathcal{F}(t, w_t^1) - \mathcal{F}(t, w_t^2)\| \leq \varphi(t) \cdot \sup_{\theta \leq 0} \frac{e^{-\gamma\theta}}{g(\theta)} \cdot e^{-\gamma(t-s)} \|w^1 - w^2\|_{s,+}, \quad \forall w^1, w^2 \in \mathcal{L}_{\gamma,s}^+,$$

we have

$$\begin{aligned}
 & e^{\gamma(t-s)} \|A^\beta((\mathcal{T}w^1)(t) - (\mathcal{T}w^2)(t))\| \\
 & \leq \|A^\beta(\hat{q}_1(0) - \hat{q}_2(0))\| \\
 & \quad + \sup_{\theta \leq 0} \frac{e^{-\gamma\theta}}{g(\theta)} \cdot \|w^1 - w^2\|_{s,+} \int_s^{+\infty} e^{\gamma(t-s)} \|A^\beta \mathcal{G}(t, \tau)\| \varphi(\tau) d\tau \\
 & \leq \|A^\beta(\hat{q}_1(0) - \hat{q}_2(0))\| + \Delta \|w^1 - w^2\|_{s,+}.
 \end{aligned}$$

Since $\Delta < 1$, \mathcal{T} is a contraction in $\mathcal{L}_{\gamma,s}^+$. Thus, there exists a unique $w(\cdot) \in \mathcal{L}_{\gamma,s}^+$ such that $\mathcal{T}w = w$. By the definition of \mathcal{T} we see that $w(\cdot)$ is the unique solution in $\mathcal{L}_{\gamma,s}^+$ to (20) for $t \geq s$. Also using (25) we have the following estimate for $\|w\|_{s,+}$

$$\|w\|_{s,+} \leq \frac{\hbar}{1-\Delta}.$$

By determination of w , we obtain the existence of the solution $u^* = u - w$ to Equation (6) such that $u_t^* \in \mathcal{M}_t$ for all $t \geq s$, and u^* satisfies

$$\begin{aligned} \|A^\beta [u_t^*(\theta) - u_t(\theta)]\| &= \|A^\beta w(t + \theta)\| \leq e^{-\gamma\theta} \cdot e^{-\gamma(t-s)} \|w\|_{s,+} \\ &\leq e^{-\gamma\theta} \cdot \frac{\hbar}{1 - \Delta} e^{-\gamma(t-s)}, \quad \forall t \geq s. \end{aligned}$$

This implies that

$$\|u_t - u_t^*\|_{\mathcal{C}_g^\beta} \leq H e^{-\gamma(t-s)}, \quad \forall t \geq s,$$

where

$$H := \sup_{\theta \leq 0} \frac{e^{-\gamma\theta}}{g(\theta)} \cdot \frac{\|\Phi_s(Pu(s)) - \hat{Q}u_s\|_{\mathcal{C}_g^\beta}}{1 - \Delta}.$$

Therefore, we conclude that $\{\mathcal{M}_t\}_{t \in \mathbb{R}}$ exponentially attracts every solution u of (6). \square

Remark 3.5. By the definition of the constant Δ , the condition (15) is fulfilled if the difference $\lambda_{N+1} - \lambda_N$ is sufficiently large, and/or the norm $\|\Lambda_1 \varphi\|_\infty = \sup_{t \in \mathbb{R}} \int_{t-1}^t \varphi(\tau) d\tau$ is sufficiently small.

4. An example

We now apply the obtained results to Mackey-Glass model with a distributed delay of the form

$$(26) \quad \begin{cases} \frac{\partial w(t, x)}{\partial t} = \frac{\partial^2 w(t, x)}{\partial x^2} - r w(t, x) + b(t) \int_{-\infty}^0 \frac{e^{-\theta^2 + \theta} |w(t + \theta, x)|}{1 + |w(t + \theta, x)|} d\theta, \\ \quad \quad \quad t > s, \quad 0 < x < \pi, \\ w(t, 0) = w(t, \pi) = 0, \quad t \in \mathbb{R}, \\ w(t, x) = \phi(t, x), \quad 0 \leq x \leq \pi, \quad t \leq 0, \end{cases}$$

where $r > 0$ is a constant, $b(t)$ is given by

$$b(t) = \begin{cases} n & \text{if } t \in [n - \frac{1}{2n+c}, n + \frac{1}{2n+c}] \text{ for } n = 1, 2, \dots \\ 0 & \text{otherwise.} \end{cases}$$

We choose the Hilbert space $X = L^2(0, \pi)$ and consider the operator $A : X \supset \mathcal{D}(A) \rightarrow X$ defined by

$$Au = -u'' + ru, \quad \forall u \in \mathcal{D}(A) = H_0^1(0, \pi) \cap H^2(0, \pi).$$

Then, A is a positive operator with discrete point spectrum

$$1^2 + r, 2^2 + r, \dots, n^2 + r, \dots$$

Now, we can choose $g(\theta) = e^{\theta^2}$ and $\beta = 0$. Then $X_0 = X$. In this case, we define the Banach space

$$\mathcal{C}_g^0 = \left\{ \phi \in C((-\infty, 0]; X) : \sup_{\theta \leq 0} \frac{\|\phi(\theta)\|}{e^{\theta^2}} < +\infty \right\},$$

endowed with the norm

$$\|\phi\|_{\mathcal{C}_g^0} := \sup_{\theta \leq 0} \frac{\|\phi(\theta)\|}{e^{\theta^2}},$$

and define $F : \mathbb{R} \times \mathcal{C}_g^0 \rightarrow X$ by

$$F(t, \phi) := b(t) \int_{-\infty}^0 \eta(s) \frac{\|\phi(s)\|}{1 + \|\phi(s)\|} ds, \quad \forall \phi \in \mathcal{C}_g^0.$$

For any $\phi_1, \phi_2 \in \mathcal{C}_g^0$ we have

$$\begin{aligned} \|F(t, \phi_1) - F(t, \phi_2)\| &\leq b(t) \left\| \int_{-\infty}^0 e^{-s^2+s} [\phi_1(s) - \phi_2(s)] ds \right\| \\ &\leq b(t) \int_{-\infty}^0 e^s \frac{\|\phi_1(s) - \phi_2(s)\|}{e^{s^2}} ds \\ &\leq b(t) \|\phi_1 - \phi_2\|_{\mathcal{C}_g^0} \int_{-\infty}^0 e^s ds \\ &\leq b(t) \|\phi_1 - \phi_2\|_{\mathcal{C}_g^0}. \end{aligned}$$

One can see that $\|F(t, \phi)\| \leq b(t)(1 + \|\phi\|_{\mathcal{C}_g^0})$, $\forall \phi \in \mathcal{C}_g^0$. This means F is φ -Lipschitz with $\varphi(t) = b(t)$.

By simple computation, one can see that

$$\sup_{\theta \leq 0} \frac{e^{-\lambda_{N+1}\theta}}{g(\theta)} = \sup_{\theta \leq 0} \frac{e^{-\lambda_{N+1}\theta}}{e^{\theta^2}} = e^{\frac{\lambda_{N+1}^2}{4}} < \infty$$

i.e., the condition (2) is fulfilled.

Furthermore, since φ can take any arbitrarily large value then $\varphi \notin L_\infty$.

Now, if we take $E = L^p(\mathbb{R})$ with $1 < p < \infty$, then $E' = L^q(\mathbb{R})$ for $\frac{1}{p} + \frac{1}{q} = 1$ and we have

$$\int_{\mathbb{R}} |\varphi(t)|^q dt = \sum_{n \in \mathbb{N}} \int_{n - \frac{1}{2^{n+c}}}^{n + \frac{1}{2^{n+c}}} n^q dt = \sum_{n \in \mathbb{N}} n^q \frac{1}{2^{n+c-1}} < +\infty$$

i.e., $\varphi \in E'$.

On the other hand

$$\begin{aligned} \|\Lambda_1 \varphi\|_\infty &= \sup_{t \in \mathbb{R}} \int_t^{t+1} \varphi(\tau) d\tau \\ &= \sup_{t \geq 0} \int_t^{t+1} a(\tau) d\tau \\ &\leq 2 \sup_{n \in \mathbb{N}} \int_{n - \frac{1}{2^{n+c}}}^{n + \frac{1}{2^{n+c}}} nd\tau \\ &\leq \frac{1}{2^{c-2}}. \end{aligned}$$

So, by Remark 3.5 the Equation (26) has an admissible inertial manifold of \mathcal{E} -class if N and/or c are large enough. Here, \mathcal{E} be the Banach space corresponding to $L^p(\mathbb{R})$.

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