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ADDITIVE MAPS OF SEMIPRIME RINGS SATISFYING AN ENGEL CONDITION

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Abstract. Let R be a semiprime ring with maximal right ring of quotients $Q_{mr}(R)$, and let $n_1, n_2, \ldots, n_k$ be k fixed positive integers. Suppose that R is $(n_1+n_2+\cdots+n_k)$!-torsion free, and that $f\colon \rho \to Q_{mr}(R)$ is an additive map, where ρ is a nonzero right ideal of R. It is proved that if $\Big[\big[\ldots[f(x), x^{n_1}], \ldots\big], x^{n_k}\Big] = 0$ for all $x \in \rho$, then $\big[f(x), x\big] = 0$ for all $x \in \rho$. This gives the result of Beidar et al. [2] for semiprime rings. Moreover, it is also proved that if R is p-torsion, where p is a prime integer with $p = \sum_{i=1}^{k} n_i$, and if $f\colon R \to Q_{mr}(R)$ is an additive map satisfying $\Big[\big[\ldots[f(x), x^{n_1}], \ldots\big], x^{n_k}\Big] = 0$ for all $x \in R$, then $\big[f(x), x\big] = 0$ for all $x \in R$.

1. Results

Throughout the paper, unless specially stated, R denotes a semiprime ring (i.e., for $a \in R$, $aRa = 0$ implies $a = 0$) with maximal right ring of quotients $Q_{mr}(R)$. The center of $Q_{mr}(R)$, denoted by C, is called the extended centroid of R. It is known that C is a commutative regular self-injective ring. Moreover, C is a field if and only if R is a prime ring (i.e., for $a, b \in R$, $aRb = 0$ implies either $a = 0$ or $b = 0$). We refer the reader to the book [3] for details.

For $a, b \in R$, we let $[a, b] := ab - ba$, the additive commutator of a and b. Given an additive subgroup S of R, a map $f\colon S \to R$ is called m-power commuting if $[f(x), x^m] = 0$ for all $x \in S$, where m is a fixed positive integer. A 1-power commuting map is called a commuting map for brevity. Additive m-power commuting maps of prime rings or semiprime rings have been studied by a lot of scholars in the literature (see [2, 4–12, 15, 19, 21–24, 26] etc.).

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To state the main result of the paper we follow some notations from [2]. Given positive integers $n_1, \dots, n_k$, we let $\overline{n} = (n_1, n_2, \dots, n_k)$ and define $[x, y^{\overline{n}}]_k$ inductively as follows:

$$[x, y^{\overline{n}}]_0 = x, \ [x, y^{\overline{n}}]_1 = [x, y^{n_1}] \text{ and } \left[x, y^{\overline{n}}\right]_{t+1} = \left[[x, y^{\overline{n}}]_t, y^{n_{t+1}}\right]$$

for $t = 1, 2, \dots, k-1$. If $n_1 = n_2 = \cdots = n_k = m$, then we shall write $[x, y^m]_k$ in place of $[x, y^{\overline{n}}]_k$. In 1997 Beidar et al. [2] proved the following.

Theorem 1.1. *Let R be a prime ring with ρ a nonzero right ideal and $\overline{n} = (n_1, n_2, \dots, n_k)$. Suppose that $f\colon \rho \to RC$ is an additive map satisfying the Engel condition $\left[f(x), x^{\overline{n}}\right]_k = 0$ for all $x \in \rho$. Then $[f(x), x] = 0$ for all $x \in \rho$ provided that either* $\operatorname{char}(R) = 0$ *or a prime $p > n_1 + n_2 + \cdots + n_k$.*

Some generalizations of the theorem above have been obtained for $n!$-torsion free semiprime rings R with an additive map $f\colon R \to R$. M. Fošner et al. proved that if $[f(x), x]_n = 0$ for all $x \in R$, then f is commuting (see [12, Theorem 1]). Also, A. Fošner and Nadeem-ur-Rehman proved that if $[f(x), x^n] = 0$ for all $x \in R$, then f is commuting (see [11, Main Theorem]). The first aim of the paper is then to extend Theorem 1.1 to semiprime rings in its full generality. Applying the same argument given in the proof of Theorem 1.1, we remark that Theorem 1.1 keeps true when the additive map $f\colon \rho \to RC$ is replaced by $f\colon \rho \to Q_{mr}(R)$. The fact will be used in our proofs. We are now ready to state the first main theorem of the paper.

Theorem 1.2. *Let R be an $\big(n_1 + n_2 + \cdots + n_k\big)!$-torsion free semiprime ring, and let $f\colon \rho \to Q_{mr}(R)$ be an additive map, where ρ is a nonzero right ideal of R. Suppose that $\left[f(x), x^{\overline{n}}\right]_k = 0$ for all $x \in \rho$, where $\overline{n} = (n_1, n_2, \dots, n_k)$. Then $\big[f(x), x\big] = 0$ for all $x \in \rho$.*

Given a right ideal ρ of a semiprime ring R, T.-K. Lee and T.-C. Lee gave a complete characterization of additive commuting maps from ρ to $Q_{mr}(R)$ (see [20, Theorem 1]). Moreover, Theorem 1.2 is false if we drop the assumption that R is $\big(n_1 + n_2 + \cdots + n_k\big)!$-torsion free (see [21, Example 1.2]). The second goal of the paper is to study an exceptional case of Theorem 1.2 as follows.

Theorem 1.3. *Let R be a semiprime ring with $pR = 0$, where p is a prime integer. Suppose that $f\colon R \to Q_{mr}(R)$ is an additive map satisfying $\left[f(x), x^{\overline{n}}\right]_k = 0$ for all $x \in R$, where $\overline{n} = (n_1, n_2, \dots, n_k)$ and $p = \sum_{i=1}^k n_i$. Then $\big[f(x), x\big] = 0$ for all $x \in R$.*

2. Proofs

Throughout, R denotes a semiprime ring with extended centroid C. The set $\mathcal{B}$ of all idempotents of C forms a Boolean algebra with respect to the operations $e \dot{+} h := e + h - 2eh$ and $e \cdot h := eh$ for all $e, h \in \mathcal{B}$. It is complete with respect to the partial order $e \leq h$ (defined by $eh = e$) in the sense that any subset S of $\mathcal{B}$ has a supremum and an infimum.

A subset $\{e_\nu \in \mathcal{B} \mid \nu \in \Lambda\}$ of $\mathcal{B}$ is called orthogonal if $e_\nu e_\mu = 0$ for $\nu \neq \mu$ and is a dense subset if $\sum_{\nu\in\Lambda} e_\nu C$ is an essential ideal of C. A subset T of $Q_{ml}(R)$, where $0 \in T$, is called orthogonally complete in the following sense. Given any dense orthogonal subset $\{e_\nu \in \mathcal{B} \mid \nu \in \Lambda\}$ of $\mathcal{B}$, there exists a one-to-one correspondence between T and the direct product $\prod_{\nu\in\Lambda} Te_\nu$ via the map $x \mapsto \langle xe_\nu\rangle$ for $x \in T$. Therefore, given any subset $\{a_\nu \in T \mid \nu \in \Lambda\}$, there exists a unique $a \in T$ such that $a \mapsto \langle a_\nu e_\nu\rangle$. The element a is written as $\sum^{\perp}_{\nu\in\Lambda} a_\nu e_\nu$ and is characterized by the property that $ae_\nu = a_\nu e_\nu$ for all $\nu \in \Lambda$.

Given a subset T of $Q_{mr}(R)$, we denote by $\widehat{T}$ the intersection of all orthogonally complete subsets of $Q_{mr}(R)$ containing T. It is known that $\widehat{T}$ is itself orthogonally complete and is called the orthogonal completion of T (see [3, Chapter 3]).

In view of [3, Proposition 3.1.10], $Q_{mr}(R)$ is orthogonally complete. Moreover, P is a minimal prime ideal of $Q_{mr}(R)$ if and only if $P = \mathbf{m}Q_{mr}(R)$ for some $\mathbf{m} \in \mathrm{Spec}(\mathcal{B})$, the spectrum of $\mathcal{B}$ (i.e., the set of all maximal ideals of $\mathcal{B}$) (see [3, Theorem 3.2.15]). In particular, it follows from the semiprimeness of $Q_{mr}(R)$ that $\bigcap_{\mathbf{m}\in\mathrm{Spec}(\mathcal{B})} \mathbf{m}Q_{mr}(R) = 0$. We refer the reader to the book [3] for details. Throughout, we set $Q := Q_{mr}(R)$ for simplicity of notation.

To begin with the proof of Theorem 1.2, we need the following, which has the same proof as that of [18, Lemma 2.1].

Lemma 2.1. *Let R be an $m!$-torsion free semiprime ring, where m is a positive integer. Then* $\mathrm{char}\,(Q/\mathbf{m}Q) = 0$ *or a prime $p > m$ for any* $\mathbf{m} \in \mathrm{Spec}(\mathcal{B})$.

Given an ideal I of R, for $q \in R$ it is clear that $qI = 0$ if and only if $Iq = 0$. Thus, $\mathrm{Ann}_R(I) := \{q \in R \mid qI = 0\}$ is an ideal of R. Moreover, an ideal I of R is essential if $\mathrm{Ann}_R(I) = 0$. The following is well-known (see, for instance, [19, Lemma 2.10] with replacing the Martindale symmetric ring of quotients of R by $Q_{mr}(R)$).

Lemma 2.2. *Every annihilator ideal of Q is generated by one central idempotent.*

Let A, B be subsets of R. We let $[A, B]$ (resp. AB) denote the additive subgroup of R generated by all $[a, b]$ (resp. ab) for $a \in A$ and $b \in B$. If $A = \{a\}$, we write $[a, B]$ (resp. aB) in place of $[\{a\}, B]$ (resp. $\{a\}B$). The following is a special case of [17, Main Theorem].

Lemma 2.3. *Let ρ be a right ideal of R, $a \in R$. Suppose that $\left[a, x^{\overline{n}}\right]_k = 0$ for all $x \in \rho$, where $\overline{n} = (n_1, n_2, \ldots, n_k)$. Then $[\rho, R][a, R] = 0$.*

Proof. Let $a, x \in R$ and let s, t be positive integers. Suppose that $\left[[a, x^s], x^t\right] = 0$. Then $\left[[a, x^s], x^{st}\right] = 0$ and so $\left[[a, x^{st}], x^s\right] = 0$. Thus, $\left[[a, x^{st}], x^{st}\right] = 0$. Since $\left[a, x^{\overline{n}}\right]_k = 0$ for all $x \in \rho$ where $\overline{n} = (n_1, \ldots, n_k)$, it follows that $\left[a, x^m\right]_k = 0$ for all $x \in \rho$, where $m := n_1 \cdots n_k$. In view of [17, Main Theorem], we have $[\rho, R][a, R] = 0$. □

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. In view of Lemma 2.2, there exists $e = e^2 \in C$ such that $\mathrm{Ann}_Q\big(Q[\rho, Q]Q\big) = e\,Q$. Since C is a commutative self-injective ring, C is a direct summand of the C-module Q. There exists a C-submodule W of Q such that $Q = C \oplus W$. Let $\pi\colon Q \to Q$ be the projection along W. In particular, π is a C-module map. Set $e' := 1 - e$. Let $g\colon \rho \to Q$ be the additive map defined by

$$g(x) = \pi\big(e' f(x)\big)$$

for all $x \in \rho$. Then $g(x) - e'f(x) \in C$ for all $x \in \rho$. Therefore, $\big[g(x), x^{\overline{n}}\big]_k = 0$ for all $x \in \rho$.

$$\widehat{\rho} = \{\sum_{\alpha\in I}^{\perp} x_\alpha e_\alpha \mid x_\alpha \in \rho, \sum_{\alpha\in I} e_\alpha C \text{ is a dense ideal of } C\}.$$

Then $\widehat{\rho}$ is a right ideal of $\widehat{R}$ and $Q_{mr}(\widehat{R}) = Q$. We claim that g can be extended to an additive map, denoted by $\widehat{g}$, from $\widehat{\rho}$ to Q by

$$\widehat{g}\Big(\sum_{\alpha\in I}^{\perp} x_\alpha e_\alpha\Big) = \sum_{\alpha\in I}^{\perp} g\big(x_\alpha\big) e_\alpha \tag{1}$$

for all $x_\alpha \in \rho$. To prove $\widehat{g}$ to be well-defined, it suffices to claim that if $x_\alpha e_\alpha = 0$, where $x_\alpha \in \rho$ and $e_\alpha \in \mathcal{B}$, then $g\big(x_\alpha\big)e_\alpha = 0$. Indeed, let $y \in \rho$. Multiplying $\big[f(x_\alpha + y), (x_\alpha + y)^{\overline{n}}\big]_k = 0$ by e_α, we get $\big[f(x_\alpha + y), y^{\overline{n}} e_\alpha\big]_k = 0$. Since $\big[f(y), y^{\overline{n}}\big]_k = 0$, we see that

$$\big[f(x_\alpha)e_\alpha, y^{\overline{n}}\,\big]_k = 0. \tag{2}$$

Since R and Q satisfy the same GPIs (see [3, Theorem 6.4.1]) and $\rho R \subseteq \rho \subseteq \rho Q$, (2) holds for all $y \in \rho Q$. Applying Lemma 2.3 to (2), we get

$$\big[\rho Q, Q\big]\big[f(x_\alpha)e_\alpha, Q\big] = 0 \text{ and so } \big[\rho, Q^2\big]\big[f(x_\alpha)e_\alpha, Q\big] = 0.$$

This implies that $Q\big[\rho, Q\big]Q\big[f(x_\alpha)e_\alpha, Q\big] = 0$. Thus,

$$\big[f(x_\alpha)e_\alpha, Q\big] \subseteq \mathrm{Ann}_Q\big(Q[\rho, Q]Q\big) = eQ$$

and so $\big[e'f(x_\alpha)e_\alpha, Q\big] = 0$. That is, $e'f(x_\alpha)e_\alpha \in C$. Hence, $\pi\big(e'f(x_\alpha)\big)e_\alpha = 0$, i.e., $g(x_\alpha)e_\alpha = 0$, as asserted. This proves that $\widehat{g}$ is well-defined. Clearly, the map $\widehat{g}\colon \widehat{\rho} \to Q$ is additive.

Let $x \in \widehat{\rho}$. There exists a dense orthogonal subset $\{e_\nu \in \mathcal{B} \mid \nu \in \Lambda\}$ of $\mathcal{B}$ such that $x = \sum_{\nu\in\Lambda}^{\perp} x_\nu e_\nu$, where $x_\alpha \in \rho$ for $\alpha \in \Lambda$. It follows from (1) that

$$\Big[\widehat{g}(x), x^{\overline{n}}\Big]_k = \sum_{\alpha\in\Lambda}^{\perp} \big[g(x_\alpha), x_\alpha^{\overline{n}}\big]_k e_\alpha = 0. \tag{3}$$

Let $\mathbf{m} \in \mathrm{Spec}(\mathcal{B})$. We claim that $\widehat{g}\big(\widehat{\rho} \cap \mathbf{m}\widehat{R}\big) \subseteq \mathbf{m}Q$. Indeed, let $x \in \widehat{\rho} \cap \mathbf{m}\widehat{R}$ and $y \in \widehat{\rho}$. Then $hx = 0$ for some $h \in \mathcal{B} \setminus \mathbf{m}$. By (3) we have

$$\big[\widehat{g}(x + y), (x + y)^{\overline{n}}\big]_k = 0.$$

Multiplying it by h, we get $\big[\widehat{g}(x+y), y^{\overline{n}}h\big]_k = 0$ and so $\big[\widehat{g}(x)h, y^{\overline{n}}\big]_k = 0$. As before, it follows from Lemma 2.3 that $\big[\widehat{\rho}, \widehat{R}\big]\big[\widehat{g}(x)h, Q\big] = 0$. This implies that $Q\big[\widehat{\rho}, \widehat{R}\big]Q\big[\widehat{g}(x)h, Q\big] = 0$ and so

$$Q\big[\rho, Q\big]Q\big[\widehat{g}(x)h, Q\big] = 0.$$

Hence, $\big[\widehat{g}(x)h, Q\big] \subseteq eQ$ and so $\big[\widehat{g}(x)h, Q\big] = 0$, since $\widehat{g}(x) \in e'Q$. Thus, $\widehat{g}(x)h \in C \cap W = \{0\}$. That is, $\widehat{g}(x) \in \mathbf{m}Q$, as asserted.

This means that $\widehat{g}$ canonically induces the following additive map

$$\widehat{g}_{\mathbf{m}} \colon \big(\widehat{\rho} + \mathbf{m}\widehat{R}\big)/\mathbf{m}\widehat{R} \to Q/\mathbf{m}Q,$$

which is defined by $\widehat{g}_{\mathbf{m}}\big(\overline{x}\big) = \overline{\widehat{g}(x)}$ for $x \in \widehat{\rho}$, where

$$\overline{x} := x + \mathbf{m}\widehat{R} \in \big(\widehat{\rho} + \mathbf{m}\widehat{R}\big)/\mathbf{m}\widehat{R}.$$

Clearly, by (3) we have $\big[\widehat{g}_{\mathbf{m}}(\overline{x}), \overline{x}^{\overline{n}}\big]_k = 0$ for all $\overline{x} \in \big(\widehat{\rho} + \mathbf{m}\widehat{R}\big)/\mathbf{m}\widehat{R}$. Note that $\big(\widehat{\rho} + \mathbf{m}\widehat{R}\big)/\mathbf{m}\widehat{R}$ is a right ideal of the prime ring $\widehat{R}/\mathbf{m}\widehat{R}$ and $Q/\mathbf{m}Q$ is contained in the maximal right ring of quotients of $\widehat{R}/\mathbf{m}\widehat{R}$. In view of Lemma 2.1, $\operatorname{char}\big(\widehat{R}/\mathbf{m}\widehat{R}\big) = 0$ or a prime $p > n_1 + \cdots + n_k$. It follows from Theorem 1.1 that $\big[\widehat{g}_{\mathbf{m}}(\overline{x}), \overline{x}\big] = 0$ for all $\overline{x} \in \big(\widehat{\rho} + \mathbf{m}\widehat{R}\big)/\mathbf{m}\widehat{R}$. In particular, $[g(x), x] \in \mathbf{m}Q$ for all $x \in \rho$.

Note that $\bigcap_{\mathbf{m} \in \operatorname{Spec}(\mathcal{B})} \mathbf{m}Q = 0$. We get $[g(x), x] = 0$ for all $x \in \rho$. That is, $[\pi(e'f(x)), x] = 0$ for all $x \in \rho$. Since $e'f(x) - \pi(e'f(x)) \in C$ for $x \in \rho$, we get $[e'f(x), x] = 0$ and so

$$0 = [e'f(x), x] = [f(x), x] - [ef(x), x] = [f(x), x]$$

for all $x \in \rho$, because $[ef(x), x] \in [eQ, \rho] = 0$ (by the fact that $\operatorname{Ann}_Q\big(Q[\rho, Q]Q\big) = e\,Q$). That is, $[f(x), x] = 0$ for all $x \in \rho$, as desired. $\square$

We next turn to the proof of Theorem 1.3. The first step is to handle the prime case by beginning with a preliminary lemma. For $x \in R$, a prime ring with extended centroid C, we define $\deg(x)$ to be the minimal algebraic degree over C if x is algebraic over the field C and $\deg(x) = \infty$, otherwise. For a subset T of R, we define $\deg(T) = \sup\{\deg(t) \mid t \in T\}$. It is known that, for a positive integer m, $\deg(R) \le m$ if and only if $\dim_C RC \le m^2$. We denote by S_n the permutation group on the set $\{1, 2, \ldots, n\}$. The following is well-known. We give its statement without proof. We denote by $Z(R)$ the center of R.

Lemma 2.4. *Let R be a semiprime ring, $a \in R$. If $\big[a, [R, R]\big] = 0$, then $a \in Z(R)$.*

Lemma 2.5. *Let R be a prime PI-ring with center $Z(R)$, $a \in R$, and* $\operatorname{char}(R) = p > 0$. *Suppose that*

$$\text{(4)} \quad \sum_{\sigma \in S_p} \Big[\big[\ldots[a, x_{\sigma(1)}x_{\sigma(2)}\cdots x_{\sigma(m_1)}], \ldots\big], x_{\sigma(m_{k-1}+1)}x_{\sigma(m_{k-1}+2)}\cdots x_{\sigma(m_k)}\Big] = 0$$

for all $x_i \in R$, where $n_1, \ldots, n_k$ are positive integers with $p = \sum_{i=1}^{k} n_i$ and where $m_1 := n_1$ and $m_j := m_{j-1} + n_j$ for $j = 2, \ldots, k$. Then $a \in Z(R)$.

Proof. To prove $a \in Z(R)$ we may assume that R is noncommutative. By Posner-Rowen's theorem, $RC \cong \mathrm{M}_n(D)$, where D is a finite dimensional central division C-algebra (see [27, Corollary 1]). Choose F to be a maximal subfield of D. Note that $F = C$ if D is a field, and C is an infinite field, otherwise. It is known that $RC \otimes_C F \cong \mathrm{M}_m(F)$ for some $m > 1$, and, applying a standard argument, we get that (4) holds for all $x_i \in RC \otimes_C F$. Thus, we may assume from the start that $R = \mathrm{M}_m(F)$ for some field F and some $m > 1$.

For the case that $p = 2$, we get $[a, xy + yx] = 0$ for all $x, y \in R$ or $\big[[a, x], y\big] + \big[[a, y], x\big] = 0$ for all $x, y \in R$. Since $\mathrm{char}(R) = 2$, the latter case implies that $\big[a, [x, y]\big] = 0$ for all $x, y \in R$. In either case, we get $\big[a, [R, R]\big] = 0$. In view of Lemma 2.6, we have $a \in F$, as desired.

From now on, we assume that $p > 2$. We first consider the case that $k = 1$. That is,

$$\big[a, \sum_{\sigma \in S_p} x_{\sigma(1)} \cdots x_{\sigma(p)}\big] = 0 \tag{5}$$

for all $x_i \in R$. Replacing x_1 by x and x_j by a fixed idempotent e for $j > 1$ in (5), we get

$$\big[a, (p-1)!(xe + ex) + (p-2)(p-1)!exe\big] = 0 \tag{6}$$

for all $x \in R$. Multiplying both sides of (6) by $1 - e$ and applying the fact that $\mathrm{char}(R) = p$, we get $(1-e)aex(1-e) = (1-e)x(1-e)ae$ for all $x \in R$. In view of [25, Theorem 1], $(1-e)ae \in F(1-e)$, implying that $(1-e)ae = 0$. Since e is an arbitrary idempotent of R, replacing e by $1-e$, we also get $ea(1-e) = 0$. Therefore, $[a, e] = 0$. Note that R is generated by idempotents as a vector space over F. This implies that $a \in F$, as desired.

We next assume that $k > 1$. Note that

$$m_1 := n_1 \text{ and } m_j := m_{j-1} + n_j$$

for $j = 2, \ldots, k$. Thus, $m_k = p$. For $\sigma \in S_p$, by hypothesis we have

$$\sum_{\sigma \in S_p} A_\sigma = 0,$$

where

$$A_\sigma := \big[B_\sigma, C_\sigma\big], \tag{7}$$

where

$$B_\sigma := \Big[\big[\ldots[a, x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(m_1)}], \ldots\big], x_{\sigma(m_{k-2}+1)} x_{\sigma(m_{k-2}+2)} \cdots x_{\sigma(m_{k-1})}\Big]$$

and

$$C_\sigma := x_{\sigma(m_{k-1}+1)} x_{\sigma(m_{k-1}+2)} \cdots x_{\sigma(m_k)}.$$

Given an idempotent $e \in R$, let $x_i = e$ for $i = 2, \ldots, p$. Suppose that $(1-e)A_\sigma(1-e) \neq 0$. Then we have either $\sigma(m_{k-1}+1) = 1$ or $\sigma(m_k) = 1$, and in this case,

$$B_\sigma = [a,e]_{k-1}, \text{ and } C_\sigma = ex_1, \; x_1 e \; \text{ or } \; x_1,$$

where $C_\sigma = x_1$ occurs only when $n_k = 1$.

Case 1: $n_k > 1$. Therefore, we get

$$0 = \sum_{\sigma \in S_p} (1-e)A_\sigma(1-e) = (p-1)!(1-e)\big[[a,e]_{k-1}, ex_1 + x_1 e\big](1-e).$$

Since p is not a divisor of $(p-1)!$, this implies

$$(1-e)[a,e]_{k-1}ex_1(1-e) = (1-e)x_1 e[a,e]_{k-1}(1-e) \tag{8}$$

for all $x_1 \in R$. In view of [25, Theorem 2(a)], either $(1-e)[a,e]_{k-1}e \in F(1-e)$ or $e[a,e]_{k-1}(1-e) \in F(1-e)$; that is, either $(1-e)[a,e]_{k-1}e = 0$ or $e[a,e]_{k-1}(1-e) = 0$. In view of (8), if $(1-e)[a,e]_{k-1}e = 0$, we get $e[a,e]_{k-1}(1-e) = 0$, and conversely. Thus we get

$$(1-e)[a,e]_{k-1}e = 0 = e[a,e]_{k-1}(1-e). \tag{9}$$

However, $[a,e]_{k-1} = ae - ea$ if $k-1$ is odd, and $[a,e]_{k-1} = [a,e]_2 = ae - 2eae + ea$ if $k-1$ is even. It follows from (9) that $ea = eae = ae$ and so $[a,e] = 0$. Since R is generated by idempotents as a vector space over F, we have $a \in F$, as desired.

Case 2: $n_k = 1$. Therefore, we get

$$0 = \sum_{\sigma \in S_p} (1-e)A_\sigma(1-e) = (p-1)!(1-e)\big[[a,e]_{k-1}, x_1\big](1-e).$$

Applying an analogous argument as given in Case 1, we can conclude that $a \in F$. □

Let $R^n := \{(x_1, \ldots, x_n) \,|\, x_i \in R \text{ for } i = 1, \ldots, n\}$. A map $\pi \colon R^n \to R$ is called an n-additive map if, for $i = 1, \ldots, n$, we have

$$\begin{aligned} &\pi(x_1, \ldots, x_{i-1}, x_i + x_i', x_{i+1}, \ldots, x_n) \\ &= \pi(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) + \pi(x_1, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_n) \end{aligned}$$

for all $x_i, x_i' \in R$. The following lemma is essentially well-known and is referred to [16, pages 15–17], [14, Lemma 1] and [13, Lemma 2].

Lemma 2.6. *Let $\pi \colon R^n \to R$ be an n-additive map. If $\pi(x, x, \ldots, x) = 0$ for all $x \in R$, then $\sum_{\sigma \in S_n} \pi\big(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}\big) = 0$ for all $x_i \in R$.*

Proof of Theorem 1.3. Applying an analogous argument as given in the proof of Theorem 1.2 (in fact, it is easier), we must only prove the prime case. That is, we may assume that R is a noncommutative prime ring with $\operatorname{char}(R) = p > 0$.

Suppose first that $\operatorname{char}(R) = 2$. Then either $[f(x), x^2] = 0$ for all $x \in R$ or $\big[[f(x), x], x\big] = 0$ for all $x \in R$. In either case, $[f(x), x^2] = 0$ for all $x \in R$.

In view of [15, Theorem 1.5], $[f(x), x] = 0$ for all $x \in R$, as asserted. Suppose next that $\mathrm{char}\,(R) = p > 2$. We separate the proof into two cases.

Case 1: R is not a PI-ring. In this case, $\deg(R) = \infty$. By hypothesis,

$$\sum_{i=1}^{p} x^i f_i(x) x^{p-i} = 0$$

for all $x \in R$, where f_i's are additive maps of R, $0 \le i \le p$, with $f_0 = f$ and $f_p := (-1)^k f$. Invoking [1, Theorem 4.4], there exist $a, b \in Q_{mr}(R)$, and additive maps $\mu, \nu\colon R \to C$ such that $f_0(x) = xa + \mu(x)$ and $f_p(x) = bx + \nu(x)$ for all $x \in R$. Since $f_0 = f$ and $f_p = (-1)^k f$, we get $xa + (-1)^{k+1} bx \in C$ for all $x \in R$. Since R is not a PI-ring, R is not commutative. It is easy to prove that $a = (-1)^k b \in C$. Thus, f is commuting.

Case 2: R is a PI-ring. In this case, $Q_{mr}(R) = RC$. Set $m_1 := n_1$ and $m_j := m_{j-1} + n_j$ for $j = 2, \ldots, k$. Set $\ell := 1 + \sum_{i=1}^k n_i = 1 + p$ and let $\pi\colon R^\ell \to R$ be defined as

$$\begin{aligned} &\pi\big(x_1, x_2, \ldots, x_\ell\big) \\ &= \Big[\big[\ldots[f(x_1), x_2 x_3 \cdots x_{m_1+1}], \ldots\big], x_{m_{k-1}+2} x_{m_{k-1}+3} \cdots x_{m_k+1}\Big] \end{aligned}$$

for all $x_i \in R$. Since $\big[f(x), x^{\overline{n}}\big]_k = 0$ for all $x \in R$, we get $\pi(x, x, \ldots, x) = 0$ for all $x \in R$. In view of Lemma 2.6, we get

$$\begin{aligned} &\sum_{\sigma \in S_{p+1}, \sigma(1)=1} \Big[\big[\ldots[f(x_1), x_{\sigma(2)} x_{\sigma(3)} \cdots x_{\sigma(m_1+1)}], \ldots\big], \\ &\qquad\qquad x_{\sigma(m_{k-1}+2)} x_{\sigma(m_{k-1}+3)} \cdots x_{\sigma(m_k+1)}\Big] \\ &+ \sum_{\sigma \in S_{p+1}, \sigma(1)\neq 1} \Big[\big[\ldots[f(x_{\sigma(1)}), x_{\sigma(2)} x_{\sigma(3)} \cdots x_{\sigma(m_1+1)}], \ldots\big], \\ &\qquad\qquad x_{\sigma(m_{k-1}+2)} x_{\sigma(m_{k-1}+3)} \cdots x_{\sigma(m_k+1)}\Big] = 0 \end{aligned} \tag{10}$$

for all $x_1, \ldots, x_{p+1} \in R$. Let $\beta \in Z(R)$, the center of R. It follows from (10) that

$$\begin{aligned} &\sum_{\sigma \in S_{p+1}, \sigma(1)=1} \Big[\big[\ldots[f(\beta x_1) - \beta f(x_1), x_{\sigma(2)} x_{\sigma(3)} \cdots x_{\sigma(m_1+1)}], \ldots\big], \\ &\qquad\qquad x_{\sigma(m_{k-1}+2)} x_{\sigma(m_{k-1}+3)} \cdots x_{\sigma(m_k+1)}\Big] = 0 \end{aligned} \tag{11}$$

for all $x_1, \ldots, x_{p+1} \in R$. In view of Lemma 2.5, $f(\beta x_1) - \beta f(x_1) \in C$ for all $x_1 \in R$. Write $RC = W \oplus C$ as C-spaces, where W is a C-subspace of RC. Let $\pi\colon R \to RC$ be the projection along W, and let $\widetilde{f} := \pi \circ f$. Then $\widetilde{f}$ is a $Z(R)$-linear map satisfying $\big[\widetilde{f}(x), x^{\overline{n}}\big]_k = 0$ for all $x \in R$. Set $w := n_1 \cdots n_k$ and choose a positive integer v such that $p^v \ge k$. Then $\big[\widetilde{f}(x), x^w\big]_k = 0$ and so $\big[\widetilde{f}(x), x^w\big]_{p^v} = 0$; that is, $\big[\widetilde{f}(x), x^{wp^v}\big] = 0$ for all $x \in R$. In view of

[21, Theorem 1.1], $\big[\widetilde{f}(x), x\big] = 0$ for all $x \in R$, and so $\big[f(x), x\big] = 0$ for all $x \in R$ since char $(R) = p > 2$. □

References

[1] K. I. Beidar, *On functional identities and commuting additive mappings*, Comm. Algebra **26** (1998), no. 6, 1819–1850. https://doi.org/10.1080/00927879808826241

[2] K. I. Beidar, Y. Fong, P. Lee, and T. Wong, *On additive maps of prime rings satisfying the Engel condition*, Comm. Algebra **25** (1997), no. 12, 3889–3902. https://doi.org/10.1080/00927879708826093

[3] K. I. Beidar, W. S. Martindale, III, and A. V. Mikhalev, *Rings with generalized identities*, Monographs and Textbooks in Pure and Applied Mathematics, 196, Marcel Dekker, Inc., New York, 1996.

[4] M. Brešar, *Centralizing mappings and derivations in prime rings*, J. Algebra **156** (1993), no. 2, 385–394. https://doi.org/10.1006/jabr.1993.1080

[5] ———, *Applying the theorem on functional identities*, Nova J. Math. Game Theory Algebra **4** (1996), no. 1, 43–54.

[6] ———, *Commuting maps: a survey*, Taiwanese J. Math. **8** (2004), no. 3, 361–397. https://doi.org/10.11650/twjm/1500407660

[7] M. Brešar and B. Hvala, *On additive maps of prime rings. II*, Publ. Math. Debrecen **54** (1999), no. 1-2, 39–54.

[8] M. Chacron, *Commuting involution*, Comm. Algebra **44** (2016), no. 9, 3951–3965. https://doi.org/10.1080/00927872.2015.1087546

[9] ———, *Involution satisfying an Engel condition*, Comm. Algebra **44** (2016), no. 12, 5058–5073. https://doi.org/10.1080/00927872.2015.1130145

[10] M. Chacron and T.-K. Lee, *Open questions concerning antiautomorphisms of division rings with quasi-generalized Engel conditions*, J. Algebra Appl. **18** (2019), no. 9, 1950167, 11 pp. https://doi.org/10.1142/S0219498819501676

[11] A. Fošner and N. U. Rehman, *Identities with additive mappings in semiprime rings*, Bull. Korean Math. Soc. **51** (2014), no. 1, 207–211. https://doi.org/10.4134/BKMS.2014.51.1.207

[12] M. Fošner, N. U. Rehman, and J. Vukman, *An Engel condition with an additive mapping in semiprime rings*, Proc. Indian Acad. Sci. Math. Sci. **124** (2014), no. 4, 497–500. https://doi.org/10.1007/s12044-014-0205-4

[13] M. Gerstenhaber, *On nilalgebras and linear varieties of nilpotent matrices. II*, Duke Math. J. **27** (1960), 21–31. http://projecteuclid.org/euclid.dmj/1077468913

[14] I. N. Herstein, *Jordan homomorphisms*, Trans. Amer. Math. Soc. **81** (1956), 331–341. https://doi.org/10.2307/1992920

[15] H. G. Inceboz, M. T. Koşan, and T.-K. Lee, *m-power commuting maps on semiprime rings*, Comm. Algebra **42** (2014), no. 3, 1095–1110. https://doi.org/10.1080/00927872.2012.731623

[16] N. Jacobson, PI-*algebras*, Lecture Notes in Mathematics, Vol. 441, Springer-Verlag, Berlin, 1975.

[17] T.-K. Lee, *Semiprime rings with hypercentral derivations*, Canad. Math. Bull. **38** (1995), no. 4, 445–449. https://doi.org/10.4153/CMB-1995-065-2

[18] ———, *Ad-nilpotent elements of semiprime rings with involution*, Canad. Math. Bull. **61** (2018), no. 2, 318–327. https://doi.org/10.4153/CMB-2017-005-3

[19] ———, *Anti-automorphisms satisfying an Engel condition*, Comm. Algebra **45** (2017), no. 9, 4030–4036. https://doi.org/10.1080/00927872.2016.1255894

[20] T.-K. Lee and T.-C. Lee, *Commuting additive mappings in semiprime rings*, Bull. Inst. Math. Acad. Sinica **24** (1996), no. 4, 259–268.

[21] T.-K. Lee, K.-S. Liu, and W.-K. Shiue, *n-commuting maps on prime rings*, Publ. Math. Debrecen **64** (2004), no. 3-4, 463–472.
[22] P.-K. Liau and C.-K. Liu, *On automorphisms and commutativity in semiprime rings*, Canad. Math. Bull. **56** (2013), no. 3, 584–592. https://doi.org/10.4153/CMB-2011-185-5
[23] C.-K. Liu, *An Engel condition with automorphisms for left ideals*, J. Algebra Appl. **13** (2014), no. 2, 1350092, 14 pp. https://doi.org/10.1142/S0219498813500928
[24] ———, *Additive n-commuting maps on semiprime rings*, Proc. Edinb. Math. Soc. (2) **63** (2020), no. 1, 193–216. https://doi.org/10.1017/s001309151900018x
[25] W. S. Martindale, III, *Prime rings satisfying a generalized polynomial identity*, J. Algebra **12** (1969), 576–584. https://doi.org/10.1016/0021-8693(69)90029-5
[26] E. C. Posner, *Derivations in prime rings*, Proc. Amer. Math. Soc. **8** (1957), 1093–1100. https://doi.org/10.2307/2032686
[27] L. Rowen, *Some results on the center of a ring with polynomial identity*, Bull. Amer. Math. Soc. **79** (1973), 219–223. https://doi.org/10.1090/S0002-9904-1973-13162-3

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