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ALGEBRAIC RANKS OF THE FUNDAMENTAL GROUPS OF HIGH DIMENSIONAL GRAPH MANIFOLDS

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Abstract. The fundamental group of a high dimensional graph manifold canonically has a graph of groups structure. We analyze the group action on the associated Bass-Serre tree and study the algebraic ranks of the fundamental groups of high dimensional graph manifolds.

1. Introduction

1.1. High dimensional graph manifolds

Generalizing 3-manifolds supporting non-positively curved metrics, Frigerio, Lafont and Sisto introduced the notion of high dimensional graph manifolds. In their monograph, they describe various, including topological and quasi-isometric, rigidities and prove the existence of infinitely many graph manifolds not supporting a locally CAT(0) metric, even if they are built up with pieces supporting locally CAT(0) metric. For details, see [6].

They also studied various algorithmic and algebraic properties of the fundamental groups of high dimensional graph manifolds. In this paper, we analyze the group action on the Bass-Serre tree associated to a graph of groups structure on the fundamental groups of high dimensional graph manifolds and study the algebraic ranks. Let's begin with the definition of high dimensional graph manifolds.

Fix $n \geq 3, k \in \mathbb{N}$ and $n_i \in \mathbb{N}$ with $3 \leq n_i \leq n$ for $i = 1, \ldots, k$. For every $i = 1, \ldots, k$, let N_i be a complete finite-volume non-compact hyperbolic n_i-manifold with toric cusps. It is well-known that each cusp of N_i supports a canonical smooth foliation by closed tori and defines a diffeomorphism between the cusp and $T^{n_i-1} \times [0, \infty)$, where $T^{n_i-1} = \mathbb{R}^{n_i-1}/\mathbb{Z}^{n_i-1}$ is the standard torus. Truncate the cusps of N_i by setting $\overline{N}_i = N_i \setminus \cup_{j=1}^{a_i} T_j^{n_i-1} \times (4, \infty)$, where $T_j^{n_i-1} \times [0, \infty), j = 1, \ldots, a_i$ are the cusps of N_i. Let $V_i = \overline{N}_i \times T^{n-n_i}$

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and fix a subset $\mathcal{B}$ of the set of boundary components of the V_i's. Finally, glue the V_i's along affine diffeomorphisms between the paired tori in $\mathcal{B}$. The (connected) manifold M obtained in this way is called *a graph n-manifold*.

The manifolds $V_1, \dots, V_k$ are called the *pieces* of M. For every i, $\overline{N}_i$ (or N_i) is the *base* of V_i and if $p \in \overline{N}_i$, the set $\{p\} \times T^{n-n_i}$ is a *fiber* of V_i. The toric hypersurfaces of M corresponding to the tori in $\mathcal{B}$ are called *internal walls* and the components of the boundary of M are called *boundary walls*.

Definition. Let M be a graph manifold, and V^+, V^- a pair of adjacent pieces of M. We say that two pieces have *transverse fibers* along the common internal wall T if, under the gluing diffeomorphism $\psi : T^+ \to T^-$ of the paired boundary tori corresponding to T, the image of the fiber subgroup of $\pi_1(T^+)$ under ψ_* intersects the fiber subgroup of $\pi_1(T^-)$ trivially.

Definition. A graph manifold is *irreducible* if every pair of adjacent pieces has tranverse fibers along every common internal wall.

As mentioned earlier, there are infinitely many examples of graph manifolds not supporting any locally CAT(0) metrics. It should be noted that it is much harder (but still possible) to find examples of irreducible graph manifolds not supporting any locally CAT(0) metrics. See [6, Chapter 11] for details.

1.2. A graph of groups

We briefly describe the graph of groups structure on the fundamental group of a graph manifold. See [12] for the definition and basic results on the fundamental group of a graph of groups. The structure of a graph of groups $\mathcal{G}$ induced by the decomposition of a graph manifold M into pieces $V_1, \dots, V_k$ is described as follows: every vertex group is the fundamental group of the corresponding piece $\pi_1(V_i) = \pi_1(\overline{N}_i) \times \mathbb{Z}^{n-n_i}$, every edge group is isomorphic to $\mathbb{Z}^{n-1}$ and the homomorphism of an edge group into the adjacent vertex group is induced by the inclusion of the corresponding boundary component of a piece V into V. A basic result on Bass-Serre theory tells us that $\pi_1(M)$ is isomorphic to the fundamental group of the graph of groups $\pi_1(\mathcal{G})$.

A graph of groups $\mathcal{G}$ determines a tree T on which $G = \pi_1(\mathcal{G})$ acts by isometries. Such a tree T is called the *Bass-Serre tree* of $\mathcal{G}$. A vertex stabilizer in T is the conjugate of a vertex group in $\mathcal{G}$ and an edge stabilizer in T is the conjugate of an edge group in $\mathcal{G}$. An element in G is *elliptic* if it fixes a vertex in T, *hyperbolic* otherwise. If $g \in G$ is hyperbolic, there exists a unique g-invariant subtree T' of T isomorphic to the real line, called the *axis of g*, on which g acts by a non-trivial translation. Following Delzant [5]:

Definition. Let G be the fundamental group of a graph of groups $\mathcal{G}$ and T be the Bass-Serre tree of $\mathcal{G}$. The G-action on T is *K-acylindrical* if there exists a constant K such that any element which pointwise fixes any path in T of length $\geq K$ is automatically trivial. The action is *acylindrical* if it is *K-acylindrical* for some $K \geq 0$.

One of the main consequences of irreducibility for graph manifolds is the following theorem appeared in [6].

Proposition 1.1 ([6] Proposition 6.4)**.** *Let M be a graph manifold containing at least one internal wall and $G = \pi_1(M)$. Denote by T the Bass-Serre tree associated to the decomposition of M into pieces. Then M is irreducible if and only if the G-action on T is acylindrical.*

Remark 1.2. They actually proved that if M is irreducible, then the action is 3-acylindrical.

1.3. Algebraic ranks of groups

In [11], Prasad and Raghunathan introduced the notion of the algebraic rank, $\mathrm{Rank}(G)$, of a group G, and Ballmann and Eberlein, in [2], proved that for a complete Riemannian manifold M of bounded nonpositive sectional curvature, the geometric rank of M is equal to the algebraic rank of $\pi_1(M)$. Invoking the celebrated Rank-Rigidity theorem, due to Ballmann [1], Burns-Spatzier [4], they classified the fundamental groups of complete Riemannian manifolds of bounded nonpositive sectional curvature. See [2] for details. There is an analogous notion of *geometric rank* for CAT(0) spaces using closed convex subsets isometric to Euclidean spaces. By the close analogy of non-positively curved manifolds to CAT(0) spaces, it is natural to ask the following.

Conjecture 1.3. *A* CAT(0) *space X has geometric rank bigger than one if and only if any group G acting geometrically on X has algebraic rank bigger than one.*

As mentioned above, we are interested in algebraic ranks of the fundamental group of irreducible graph manifolds, which are "almost" CAT(0) in the sense that there are infinitely many examples of irreducible graph manifolds not admitting CAT(0) metrics. Within the author's knowledge, there is no proper definition of higher geometric rank beyond CAT(0) spaces. But it is worthy of trying to extend the notion of the geometric ranks of spaces beyond the category of CAT(0) spaces and study the relation to the algebraic ranks of the corresponding groups. Below we briefly introduce the algebraic rank of a group and state our main theorem.

For an abstract group G, we denote by $\mathcal{A}_i(G)$ the set of elements g such that the centralizer $C_G(g)$ of g contains a free abelian subgroup of finite index of rank $\leq i$. Define

$$r(G) = \min\left\{ i \,|\, \text{there exist finitely many elements}\, g_j \,\text{such that}\, G = \bigcup_j g_j \mathcal{A}_i(G) \right\}.$$

The algebraic rank of G is defined by

$$\mathrm{Rank}(G) = \sup\{ r(G^*) \,|\, G^* \leq G \,\text{is a finite index subgroup}\}.$$

We allow the possibility that $\mathrm{Rank}(G) = 0$. For example, if G is a finite group, then $\mathrm{Rank}(G) = 0$. On the other hand, if G is torsion-free and non-trivial, then $\mathrm{Rank}(G) > 0$. Suppose that $\mathrm{Rank}(G) = 0$. Then, in particular, $r(G) = 0$. But every element in $\mathcal{A}_0(G)$ is of finite order, therefore $\mathcal{A}_0(G)$ is empty if G is torsion-free. In fact, the case $\mathcal{A}_0(G) = \{1\}$ is easily seen to lead to a contradiction.

We set $\mathrm{Rank}(G) = \infty$ if the set $\mathcal{A}_i(G)$ are empty, or if G cannot be covered by finitely many translates of any of the sets $\mathcal{A}_i(G)$. For example, if a group G has infinitely generated free abelian center, then $\mathrm{Rank}(G) = \infty$. In fact, we can find finitely presented examples of such groups. More precisely, Hall obtained a finitely generated group having infinitely generated free abelian center in [7]. Using [10], we can obtain a finitely presented group having infinitely generated free abelian center.

Finally, $r(G)$ is not necessarily equal to $\mathrm{Rank}(G)$. For example, the fundamental group G of the flat Klein bottle, acting on $\mathbb{E}^2$ by isometries, can be covered as a union of the set $\mathcal{A}_1(G)$ of orientation reversing elements and $\gamma\mathcal{A}_1(G)$ for any $\gamma \in \mathcal{A}_1(G)$, so that $r(G) = 1$. But, being the fundamental group of a manifold of non-positive sectional curvature, $\mathrm{Rank}(G) = 2$. See [2, Section 4] for more such groups.

The following says that algebraic ranks of groups behave well under products and taking finite index subgroups.

Proposition 1.4 ([2] Proposition 2.1)**.** *Let G be an abstract group.*

(1) *If G' is a finite index subgroup of G, then $r(G) \le r(G')$ and* $\mathrm{rank}(G) = \mathrm{rank}(G')$.
(2) *If $G = G_1 \times \cdots \times G_n$, then*

$$r(G) = \sum_{i=1}^{n} r(G_i), \qquad \mathrm{rank}(G) = \sum_{i=1}^{n} \mathrm{rank}(G_i).$$

The main theorem of the paper is as follows.

Theorem 1.5. *Let M be an irreducible graph manifold and set $G = \pi_1(M)$. Then: if M consists of a single piece with k-dimensional fiber without internal walls, the* $\mathrm{Rank}(G) = 1 + k$; *otherwise,* $\mathrm{Rank}(G) = 1$.

2. Algebraic ranks of $\pi_1(M)$

Let M be an irreducible graph manifold of dimension n, $G = \pi_1(M)$ be the fundamental group of M, $\mathcal{G}$ be the graph of groups associated to the decomposition of M into pieces $V_1, \ldots, V_k$, and T be the Bass-Serre tree associated to $\mathcal{G}$.

Lemma 2.1. $\mathrm{Rank}(G) > 0$.

Proof. Every element of finite order in G has a bounded orbit in T, therefore it fixes a vertex in T. It follows that such an element can be viewed as an element in a vertex group or an edge group. Note that an edge group is free abelian, so that it is torsion-free. Consider a vertex group $\pi_1(V_i) = \pi_1(\overline{N}_i) \times \mathbb{Z}^{n-n_i}$ in $\mathcal{G}$. Since $\pi_1(\overline{N}_i)$ acts freely on $\widehat{\mathbb{H}}^{n_i}$, the universal covering of N_i, $\pi_1(\overline{N}_i)$ is torsion-free, so that a vertex group is also torsion-free. It follows that G is torsion-free, and by the remark in the previous section, $\mathrm{Rank}(G) > 0$. □

It is well-known that the fundamental group of $\pi_1(\overline{N}_i)$ is hyperbolic relative to cusp subgroups.

Remark 2.2. In [8], the author proved that a relatively hyperbolic group with 2 or more peripheral subgroups has the algebraic rank of ≤ 1. The idea is, roughly speaking, one can find two distinct elements of infinite order from two distinct maximal parabolic subgroups, stabilizing two distinct horoballs (See in [3, Proposition 6.13]), to prove that the set of parabolic elements of infinite order is contained in the union of translates of $\mathcal{A}_1(G)$ by some powers of these two elements. The assumption on the number of peripheral subgroups enables us to choose such two elements. It is easily seen that the argument still works, without the assumption on the number of peripheral subgroups, if one can choose two distinct elements of infinite order from two distinct maximal parabolic subgroups. In our case below, due to the malnormality of subgroups in a relatively hyperbolic group, we can always choose such two elements to prove that relative hyperbolic groups have the algebraic rank ≤ 1. See [9, Proposition 2.3.6].

Before proceeding to the general case, we study the case when M consists of a single piece without internal walls.

Proposition 2.3. *If $M = \overline{N} \times \mathbb{Z}^k$ consists of a single piece without internal walls,* $\mathrm{Rank}(\pi_1(M)) = 1 + k$.

Proof. By Remark 2.2, $\mathrm{Rank}(\pi_1(\overline{N})) \leq 1$. The same argument as in Lemma 2.1 implies that $\mathrm{Rank}(\pi_1(\overline{N})) = 1$. Finally, the additivity of algebraic ranks of groups in Proposition 1.4 implies that $\mathrm{Rank}(\pi_1(M)) = 1 + k$. □

We briefly recall some definitions about graphs of groups. An edge in the corresponding graph to a graph of groups is *trivial* if it has distinct endpoints and at least one of the two monomorphisms associated to the edge is an isomorphism. A graph of groups is *reduced* if no edge is trivial. Finally, a graph of groups is *non-trivial* if it is reduced and the corresponding graph has at least one edge. It is a basic result on Bass-Serre theory that if a graph of groups is reduced and non-trivial, the fundamental group has at least one hyperbolic element.

From now on, we assume that M is irreducible and has at least one internal wall. We claim $\mathrm{Rank}(G) = 1$. In order to prove that, we need to describe the set $\mathcal{A}_1(G)$, at first. Let $\mathcal{H}$ be the set of hyperbolic elements in G. Note that edge groups have infinite index in the adjacent vertex groups and we assumed

M has at least one internal wall. Since the graph of groups is reduced and non-trivial, by the argument in the previous paragraph, $\mathcal{H}$ is non-empty.

Lemma 2.4. *The centralizer of a hyperbolic element is infinite cyclic. In particular,* $\mathcal{H} \subset \mathcal{A}_1(G)$.

Proof. Let g be a hyperbolic element with the (unique) axis γ in T and $h \in C_G(g)$. Since $gh(\gamma) = hg(\gamma) = h(\gamma)$, h also stabilizes the axis γ. Since h acts on T without inversions, it follows that this action defines a map from $C_G(g)$ to $\mathbb{Z}$ with image $p\mathbb{Z}$, where p is the minimal non-zero translation length in $C_G(h)$. If H denotes the kernel of this map, $C_G(g) = H \rtimes \mathbb{Z}$ and H fixes γ pointwise. Since the action is acylindrical, H must be trivial. Therefore, $C_G(g)$ is infinite cyclic. □

Proposition 2.5. *Let v, w be vertices in T with distance ≥ 3, $[v, w]$ be the geodesic segment connecting from v to w. Let g and h be non-trivial elements in vertex groups G_v and G_w, respectively. If g and h fix no vertices inside of $[v, w]$, then gh is hyperbolic.*

Proof. Note that since g is an element of G_v, g fixes the vertex v. The hypothesis says exactly that v is the only vertex fixed by g in the geodesic segment $[v, w]$. By the same reason, w is the only vertex fixed by h in the geodesic segment $[v, w]$.

Suppose that $gh.v = v = g.v$, i.e., gh fixes the vertex v. Then $h.v = v$. This implies that h fixes both v and w, and by 3-acylindrical action of the group, h must be the identity. Therefore, gh does not fix the vertex v. Suppose that $gh.w = w$, i.e., gh fixes the vertex w. In this case, $gh.w = g.w = w$ and g fixes the vertex w. Again, by 3-acylindrical action of the group, g must be the identity. Therefore, gh does not fix the vertex w. Suppose that gh fixes a vertex, say $\alpha (\neq v, w)$, inside of $[v, w]$. Consider the geodesic segment $[\alpha, w]$. Since h acts on T by an isometry and w is the only vertex fixed by h in $[v, w]$, $d(v, h.\alpha) > d(v, w)$, where $d(-,-)$ is the distance between two vertices in T. Since g acts on T by an isometry and fixes the vertex v,

$$d(v, w) < d(v, h.\alpha) = d(g.v, gh.\alpha) = d(v, \alpha).$$

This contradicts to the assumption that α is a vertex inside of $[v, w]$. It follows that gh fixes no vertex in $[v, w]$.

Since v is the only vertex fixed by g in $[v, w]$, $[w, v] \cup g.[v, w]$ is the geodesic segment connecting w and $g.w$. Note that $g.w = gh.w$. Therefore, $[w, v] \cup g.[v, w] = [w, v] \cup [v, gh.w]$ is the geodesic segment connecting w and $gh.w$. Since w is the only vertex fixed by h^{-1} in $[v, w]$, the geodesic segments $[v, w]$ and $h^{-1}.[v, w] = [h^{-1}v, w]$ overlap only at w. Therefore the geodesic segments $gh.[v, w]$ and $ghh^{-1}.[v, w] = g.[v, w] = [v, gw] = [v, gh.w]$ overlap only at the vertex $gh.w$. Finally, the same argument above implies that the geodesic segment $gh.[v, gh.w]$ overlaps with $gh.[v, w]$ only at the vertex $gh.v$ and, since T is a tree, it does not meet the geodesic segment $[w, v] \cup [v, gh.w]$. Therefore,

$[gh.w, gh.v] \cup [gh.v, (gh)^2 w]$ is the geodesic segment extending $[w, v] \cup [v, gh.w]$. It follows that gh acts on the infinite geodesic

$$\cdots \cup ([w, v] \cup [v, gh.w]) \cup (gh).([w, v] \cup [v, gh.w]) \cup (gh)^2([w, v] \cup [v, gh.w]) \cup \cdots$$

in T by translation (with translation length of $2d(v, w)$). This concludes that gh is hyperbolic. □

Proposition 2.6. $r(G) \leq 1$.

Proof. Let v, w be vertices in T with distance $\geq 14, g_1$ and g_2 be non-trivial elements in vertex groups G_v and G_w, respectively. Let $h \in G$ be an elliptic element and u be a vertex fixed by h. Then, by construction, either the distance between v and u or between w and u is ≥ 7. Without loss of generality, suppose that the distance between v and u is ≥ 7. Since the action G on T is 3-acylindrical, one can assume that, in the geodesic segment $[v, u]$ in T connecting v and u, there exists v_1, fixed by g_1 and u_1, fixed by h such that the distance between v_1 and u_1 is ≥ 3. It follows from that vertices fixed by g_1 (and h, respectively) has the distance from v (and u, respectively) less than 3. Viewing g_1^{-1} and h as elements in appropriate vertex groups and by Proposition 2.5, $g_1^{-1}h$ is hyperbolic. In particular, $h \in g_1\mathcal{A}_1(G)$. It follows that

$$G \subset \mathcal{A}_1(G) \cup g_1\mathcal{A}_1(G) \cup g_2\mathcal{A}_1(G).$$ □

In order to prove Rank$(G) = 1$, it remains to show that $r(G^{'}) \leq 1$ for a finite index subgroup $G^{'}$ in G.

Theorem 2.7. *Let M be an irreducible graph manifold with at least one internal wall and G be the fundamental group of M. Then G has an algebraic rank of* 1.

Proof. Let $G^{'}$ be a finite index subgroup of G. Note that since $G^{'}$ is of finite index in G, the normal core of $G^{'}$ is also a finite index normal subgroup of G. By Proposition 1.4, we can assume that $G^{'}$ is normal. Let $\mathcal{H}' = G^{'} \cap \mathcal{H}$. Since $G^{'}$ also acts on T by isometries and $\mathcal{H}$ is non-empty, $\mathcal{H}'$ is non-empty and every element in $\mathcal{H}'$ has infinite cyclic centralizer. It follows that $\mathcal{H}' \subset \mathcal{A}_1(G^{'})$. Since the intersection of $G^{'}$ and a vertex group (respectively, an edge group) is also of finite index in a vertex group (respectively, an edge group), one can choose two distinct elements $g_1^{'}, g_2^{'}$ such that for every elliptic element $h^{'}$ in $G^{'}$, either $(g_1^{'})^{-1}h^{'}$ or $(g_2^{'})^{-1}h^{'}$ is an element in $\mathcal{H}'$. It follows that

$$G' \subset \mathcal{A}_1(G') \cup g_1^{'}\mathcal{A}_1(G') \cup g_2^{'}\mathcal{A}_1(G')$$

and $r(G^{'}) \leq 1$. □

Remark 2.8. The anonymous referee indicated the following fact: If G' is a finite index subgroup of G and M' is the finite covering of M associated to G', then M' still is an irreducible graph manifold. Combined with the same argument as used to prove $r(G) \leq 1$, this fact easily provides the alternative proof of the above theorem.

Let M be a graph manifold, H be an internal wall and $M' = M \setminus N(H)$, where $N(H)$ is an open regular neighborhood of H in M. Then M' is either a graph manifold, if H doesn't separate M, or the disjoint union of two graph manifolds, if H separates M. By collapsing graph manifold(s) outside the wall H to point(s), $\pi_1(M)$ has the following a graph of groups structure, denoted by $\mathcal{G}'$: a vertex group is the fundamental group of the graph manifold appeared in M', an edge group is isomorphic to a free abelian group and the homomorphism from an edge group into a vertex group is induced by the gluing along H. Then, by construction, $\pi_1(\mathcal{G}') = \pi_1(M)$. Note that $\mathcal{G}'$ is a realization of $\pi_1(M)$ as an HNN-extension or an amalgamated product.

Lemma 2.9 ([6] Lemma 6.5)**.** *If H is an internal wall of M with transverse fibers, then the $\pi_1(\mathcal{G}')$-action on the Bass-Serre tree associated to $\mathcal{G}'$ is 3-acylindrical.*

Remark 2.10. The same argument as above can be, without any change, applied to show that if M has at least one transverse fiber, then $\mathrm{Rank}(\pi_1(M)) = 1$. More precisely,

(1) Since $\mathcal{G}'$ is non-trivial and reduced, $\pi_1(\mathcal{G}')$ has at least one hyperbolic element. Therefore, if we set $\mathcal{H}$ the set of hyperbolic elements, then $\mathcal{H} \subset \mathcal{A}_1(\pi_1(M))$.

(2) One can choose two non-trivial elements g_1, g_2 from two distinct vertex groups such that for every elliptic element h, either $g_1^{-1}h$ or $g_2^{-1}h$ is hyperbolic. It follows that

$$\pi_1(M) \subset \mathcal{A}_1(\pi_1(M)) \cup g_1\mathcal{A}_1(\pi_1(M)) \cup g_2\mathcal{A}_1(\pi_1(M))$$

and therefore $r(\pi_1(M)) \leq 1$.

(3) For any finite index subgroup G' of $\pi_1(M)$, $r(G') \leq 1$.

References

[1] W. Ballmann, *Nonpositively curved manifolds of higher rank*, Ann. of Math. (2) **122** (1985), no. 3, 597–609. https://doi.org/10.2307/1971331

[2] W. Ballmann and P. Eberlein, *Fundamental groups of manifolds of nonpositive curvature*, J. Differential Geom. **25** (1987), no. 1, 1–22. http://projecteuclid.org/euclid.jdg/1214440722

[3] B. H. Bowditch, *Relatively hyperbolic groups*, Internat. J. Algebra Comput. **22** (2012), no. 3, 1250016, 66 pp. https://doi.org/10.1142/S0218196712500166

[4] K. Burns and R. Spatzier, *Manifolds of nonpositive curvature and their buildings*, Inst. Hautes Études Sci. Publ. Math. No. 65 (1987), 35–59.

[5] T. Delzant, *Sur l'accessibilité acylindrique des groupes de présentation finie*, Ann. Inst. Fourier (Grenoble) **49** (1999), no. 4, 1215–1224.

[6] R. Frigerio, J.-F. Lafont, and A. Sisto, *Rigidity of high dimensional graph manifolds*, Astérisque No. 372 (2015), xxi+177 pp.

[7] P. Hall, *Finiteness conditions for soluble groups*, Proc. London Math. Soc. (3) **4** (1954), 419–436. https://doi.org/10.1112/plms/s3-4.1.419

[8] R. Kim, *Algebraic ranks of CAT(0) groups*, Algebraic & Geometric Topology **14** (2014), no. 3, 1627–1648.
[9] D. V. Osin, *Relatively hyperbolic groups: intrinsic geometry, algebraic properties, and algorithmic problems*, Mem. Amer. Math. Soc. **179** (2006), no. 843, vi+100 pp. https://doi.org/10.1090/memo/0843
[10] A. Ould Houcine, *Embeddings in finitely presented groups which preserve the center*, J. Algebra **307** (2007), no. 1, 1–23. https://doi.org/10.1016/j.jalgebra.2006.07.015
[11] G. Prasad and M. S. Raghunathan, *Cartan subgroups and lattices in semi-simple groups*, Ann. of Math. (2) **96** (1972), 296–317. https://doi.org/10.2307/1970790
[12] J.-P. Serre, *Trees*, translated from the French by John Stillwell, Springer-Verlag, Berlin, 1980.

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