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GORENSTEIN PROJECTIVE DIMENSIONS OF COMPLEXES UNDER BASE CHANGE WITH RESPECT TO A SEMIDUALIZING MODULE

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Abstract. Let $R \to S$ be a ring homomorphism. The relations of Gorenstein projective dimension with respect to a semidualizing module of homologically bounded complexes between $U \otimes^{\mathbf{L}}_R X$ and X are considered, where X is an R-complex and U is an S-complex. Some sufficient conditions are given under which the equality $\mathcal{GP}_{\widetilde{C}}\text{-pd}_S(S \otimes^{\mathbf{L}}_R X) = \mathcal{GP}_C\text{-pd}_R(X)$ holds. As an application it is shown that the Auslander-Buchsbaum formula holds for G_C-projective dimension.

1. Introduction

The classical theory of homological dimensions is very important to commutative algebra. In particular, it is useful that there are a number of finiteness conditions on these dimensions which characterize regular rings. For example, if the projective dimension of each finitely generated R-module is finite, then R is a regular ring.

Semidualizing modules (cf. Definition 2) have been considered by many authors (see, for example, [4, 9, 10, 12–15]). For any commutative noetherian ring R, any semidualizing R-module C and any complex Z with bounded and finitely generated homology, Christensen introduced the dimension $G\text{-dim}_C Z$ in [4], and developed a satisfactory theory for this new invariant, which characterized Cohen-Macaulay rings in a way one could hope for. However, Christensen's $G\text{-dim}_C(-)$ only works when the argument has bounded and finitely generated homology. To circumvent this shortcoming, Holm and Jϕrgensen proposed to study a homological dimension based on a larger class of complexes: $\mathcal{GP}_C$-projective dimension of X, $\mathcal{GP}_C\text{-pd}_R X$, for every homologically right-bounded complex X (see [9]). It was already known from [9] that for

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complexes with bounded and finitely generated homology, the $\mathcal{GP}_C$-$\mathrm{pd}_R(-)$ agrees with Christensen's G-$\dim_C(-)$.

Transfer of homological properties along ring homomorphisms is a classical field of study (see, for instance, [1, 2, 5–7, 18]). The main goal of this paper is to study the properties of $\mathcal{GP}_C$-projective dimensions for complexes over ring homomorphisms.

In this paper, all rings are commutative, unital, and noetherian.

2. Ring homomorphisms and G_C-projective dimensions

In this section, the Gorenstein projective dimension of complexes with respect to a semidualizing module is considered. First, we recall the following definitions for later use.

Definition. Let $\varphi : R \to S$ be a ring homomorphism. φ is said to be of *finite flat dimension* if flat dimension of S is finite as an R-module. We say φ is *faithfully flat* if S is a faithfully flat R-module (that is, S_R satisfies the condition that $0 \to A \to B$ is an exact sequence of R-modules if and only if $0 \to S \otimes_R A \to S \otimes_R B$ is exact). We call φ *finite* if it makes S a finite R-module, and we say that φ is *local* if R and S are local rings and $\varphi(\mathfrak{m}) \subseteq \mathfrak{n}$, where $\mathfrak{m}$ and $\mathfrak{n}$ are the maximal ideals of R and S.

Definition. An *R-complex* X is a sequence of R-modules X_i and R-linear maps $\partial_i^X : X_i \to X_{i-1}$, $i \in \mathbb{Z}$. If $X_i = 0$ for $i \neq 0$ we identify X with the module in degree 0, and an R-module M is thought of as a complex $0 \to M \to 0$, with M in degree 0. The homological position of a complex is captured by the numbers *supremum* and *infimum* defined by $\sup X = \sup\{i \in \mathbb{Z} | \mathrm{H}_i(X) \neq 0\}$ and $\inf X = \inf\{i \in \mathbb{Z} | \mathrm{H}_i(X) \neq 0\}$. By convention $\sup X = -\infty$ and $\inf X = \infty$ if $X \simeq 0$.

The category of R-complexes is denoted by $\mathcal{C}(R)$, and we use subscripts $\sqsubset$, $\sqsupset$ and $\square$ to denote boundedness conditions. For example, $\mathcal{C}_\square(R)$ is the full subcategory of $\mathcal{C}(R)$ of bounded complexes.

Definition. The *derived category* of the category of R-modules is the category of R-complexes localized at the class of all quasi-isomorphisms, it is denoted by $\mathcal{D}(R)$. The symbol "$\simeq$" is used to designate isomorphisms in $\mathcal{D}(R)$ and quasi-isomorphisms in $\mathcal{C}(R)$, and we use subscripts $\sqsubset$, $\sqsupset$ and $\square$ to denote homological boundedness conditions. Superscript "f" signifies that the homology is degreewise finitely generated. Thus, $\mathcal{D}^f_\sqsupset(R)$ denotes the full subcategory of $\mathcal{D}(R)$ of homologically right-bounded complexes with finitely generated homology modules.

Definition. The *left derived functor* of the tensor product functor of R-complexes is denoted by $-\otimes^{\mathbf{L}}_R-$, and $\mathbf{R}\mathrm{Hom}_R(-,-)$ denotes the *right derived functor* of the homomorphism functor of complexes. For $X, Y \in \mathcal{D}(R)$ and $i \in \mathbb{Z}$, we set $\mathrm{Tor}_i^R(X,Y) = \mathrm{H}_i(X \otimes^{\mathbf{L}}_R Y)$ and $\mathrm{Ext}^i_R(X,Y) = \mathrm{H}_{-i}(\mathbf{R}\mathrm{Hom}_R(X,Y))$. For modules X and Y this agrees with the notation of classical homological algebra.

Definition. A complex $X \in \mathcal{D}_\Box(R)$ is said to be of *finite projective* (*or flat*) *dimension* if $X \simeq U$, where U is a complex of projective (or flat) modules and $U_i = 0$ for $|i| \gg 0$. By $\mathbf{P}(R)$ and $\mathbf{F}(R)$ we denote the full subcategories of $\mathcal{D}_\Box(R)$ whose objects are complexes of finite projective and flat dimension, respectively. Note that $\mathbf{P}_0(R)$ and $\mathbf{F}_0(R)$ are equivalent, respectively, to the full subcategories of modules of finite projective or flat dimension. We use two-letter abbreviations pd, fd for the homological dimensions.

Definition. A finitely generated R-module C is *semidualizing* if

(a) The natural homothety morphism $R \to \operatorname{Hom}_R(C, C)$ is an isomorphism,

(b) $\operatorname{Ext}_R^{\geq 1}(C, C) = 0$.

Let C be a semidualizing R-module. Set

$$\mathcal{P}_C(R) = \text{the subcategory of modules } C \otimes_R P \text{ where } P \text{ is } R\text{-projective},$$

$$\mathcal{F}_C(R) = \text{the subcategory of modules } C \otimes_R F \text{ where } F \text{ is } R\text{-flat}.$$

Modules in $\mathcal{P}_C(R)$ and $\mathcal{F}_C(R)$ are called *C-projective* and *C-flat*, respectively.

A free R-module of rank one is semidualizing. If R admits a dualizing module D, then D is semidualizing.

Setting $C = R$ in the definition above we see that $\mathcal{P}_R(R)$ and $\mathcal{F}_R(R)$ are the classes of ordinary projective and flat R-modules, which we usually denote $\mathcal{P}(R)$ and $\mathcal{F}(R)$, respectively.

Definition. Let $\mathcal{X}$ be a class of R-modules and M an R-module. An *$\mathcal{X}$-resolution* of M is a complex of R-modules in $\mathcal{X}$ of the form

$$X = \cdots \to X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 \to 0$$

such that $\mathrm{H}_0(X) \cong M$ and $\mathrm{H}_n(X) = 0$ for $n \geq 1$. The *$\mathcal{X}$-projective dimension* of M is the quantity

$$\mathcal{X}\text{-pd}_R(M) = \inf\{\sup\{n \geq 0 \mid X_n \neq 0\} \mid X \text{ is an } \mathcal{X}\text{-resolution of } M\}.$$

In particular, one has $\mathcal{X}\text{-pd}_R(0) = -\infty$. The modules of $\mathcal{X}$-projective dimension 0 are the nonzero modules of $\mathcal{X}$.

The $\mathcal{P}_C$-projective dimension and $\mathcal{F}_C$-projective dimension of M are defined as above in [13], which are called *C-projective* and *C-flat dimension* of M, respectively.

Lemma 2.1 ([8, Lem. 3.2]). *Let $\varphi : R \to S$ be a ring homomorphism of finite flat dimension and C a semidualizing R-module. Then $\widetilde{C} = C \otimes_R S$ is a semidualizing S-module.*

Definition ([15]). Let C be a semidualizing R-module.

A *complete $\mathcal{PP}_C$-resolution* is a complex X of R-modules satisfying the following:

(1) X is exact and $\operatorname{Hom}_R(-, \mathcal{P}_C(R))$-exact, and

(2) X_i is projective if $i \geq 0$ and X_i is C-projective if $i < 0$.

An R-module M is G_C*-projective* if there exists a complete $\mathcal{PP}_C$-resolution X such that $M \cong \mathrm{Coker}\partial_1^X$, in which case X is a complete $\mathcal{PP}_C$-resolution of M.

We set

$$\mathcal{GP}_C(R) = \text{the subcategory of } G_C\text{-projective } R\text{-modules.}$$

In the special case $C = R$, we set $\mathcal{GP}_R(R) = \mathcal{GP}(R)$, and $\mathcal{GP}_R(R)\text{-pd}_R(-) = \mathrm{Gpd}_R(-)$.

Example 2.2. ([9, Exam. 2.8]) Projective and C-projective R-modules are G_C-projective.

Remark 2.3 ([9]). An R-module M is G_C-projective if and only if

(P1) $\mathrm{Ext}_R^{\geq 1}(M, C \otimes_R P) = 0$ for any projective R-module P, and

(P2) there exist projective R-modules P_{-1}, P_{-2}, $\cdots$ together with an exact sequence:

$$X = 0 \to M \to C \otimes_R P_{-1} \to C \otimes_R P_{-2} \to \cdots$$

such that this sequence stays exact when we apply the functor $\mathrm{Hom}_R(-, C \otimes_R P)$ to it for any projective R-module P (i.e., M admits a proper $\mathcal{P}_C(R)$-coresolution).

By Example 2.2, there exists for every homologically bounded below complex X a bounded below complex A of G_C-projective R-modules with $A \simeq X$ in $\mathcal{D}(R)$ (as one could take A to be a projective resolution of X). Every such A is called a G_C-projective resolution of X.

We proceed by recalling the definition of G_C-projective dimensions from [16].

Definition. The G_C*-projective dimension*, $\mathcal{GP}_C\text{-pd}_R(X)$, of $X \in \mathcal{D}_{\sqsupset}(R)$ is defined as

$$\mathcal{GP}_C\text{-pd}_R(X) = \inf\{\sup\{l \in \mathbb{Z} \mid A_l \neq 0\} \mid X \simeq A \in \mathcal{C}_{\sqsupset}^{\mathcal{GP}_C}(R)\}.$$

For modules, this dimension above agree with Definition 2, see [16].

The following result is one of the main results in this paper.

Theorem 2.4. *Let* $\varphi : R \to S$ *be a ring homomorphism of finite flat dimension. Assume that* $X \in \mathcal{D}_{\square}(R)$*. If* U *is a complex of finite projective dimension, i.e.,* $U \in \mathbf{P}(S)$*, then*

$$\mathcal{GP}_{\widetilde{C}}\text{-pd}_S(U \otimes_R^{\mathbf{L}} X) \leq \mathcal{GP}_C\text{-pd}_R(X) + \mathrm{pd}_S U$$

provided $\mathbf{F}_0(S) \subseteq \mathbf{P}_0(R)$*.*

Proof. If $U \simeq 0$ or $X \simeq 0$, the $\mathcal{GP}_{\widetilde{C}}\text{-pd}_S(U \otimes_R^{\mathbf{L}} X) = -\infty$ and so the result is clear. If $\mathcal{GP}_C\text{-pd}_R(X) = \infty$, then there is nothing to do. So we assume that $U \not\simeq 0$ and $X \not\simeq 0$ and $\mathcal{GP}_C\text{-pd}_R(X) < \infty$. Denote $\mathcal{GP}_C\text{-pd}_R(X) = g \in \mathbb{Z}$. Then there exists a complex $A \in \mathcal{C}_{\square}^{\mathcal{GP}_C}(R)$ which is equivalent to X in $\mathcal{D}(R)$ and has $A_l = 0$ for $l > g$ by [16, Thm. 3.5]. Since $U \in \mathbf{P}(S)$, there exists a bounded complex P of projective S-modules such that $U \simeq P$ and $P_l = 0$

when $l < v = \inf U$ or $l > u = \mathrm{pd}_S U$. It is easy to see that U and P are quasi-isomorphism as complexes of R-modules.

Note that $U \otimes_R^{\mathbf{L}} X$ is represented by the complex $P \otimes_R A$ by [16, Cor. 2.14] and for any $l \in \mathbb{Z}$,

$$(2.1) \qquad (P \otimes_R A)_l = \oplus_{t\in\mathbb{Z}} P_t \otimes_R A_{l-t} = \oplus_{v\leq t\leq u, l-t\leq g} P_t \otimes_R A_{l-t}$$

is a $G_{\widetilde{C}}$-projective S-module by [8, Prop. 4.12], and direct sums of $G_{\widetilde{C}}$-projective S-modules are $G_{\widetilde{C}}$-projective by [15, Prop. 2.4]. So $P \otimes_R A \in \mathcal{C}^{\mathcal{GP}_{\tilde{C}}}(S)$. Furthermore, it is easy to see that $P \otimes_R A$ is bounded: by (2.1), we have $(P \otimes_R A)_l = 0$ for $g + u < l < g + v$. That is, $P \otimes_R A \in \mathcal{C}_{\square}^{\mathcal{GP}_{\tilde{C}}}(S)$, and therefore, $\mathcal{GP}_{\widetilde{C}}\text{-pd}_S(U \otimes_R^{\mathbf{L}} X) \leq g + u = \mathcal{GP}_C\text{-pd}_R(X) + \mathrm{pd}_S U$ as desired. □

Corollary 2.5. *Let $\varphi : R \to S$ be a ring homomorphism of finite flat dimension, and assume that* $\dim R$ *is finite. For every $X \in \mathcal{D}_{\square}(R)$, there is an inequality*

$$\mathcal{GP}_{\widetilde{C}}\text{-pd}_S(S \otimes_R^{\mathbf{L}} X) \leq \mathcal{GP}_C\text{-pd}_R(X).$$

Proof. Note that under the condition that $\varphi : R \to S$ is a ring homomorphism of finite flat dimension and $\dim R$ is finite, one has every S-module of finite flat dimension is of finite projective dimension over R via φ. Now the result follows from Theorem 2.4. □

Next, we consider when the equality in Corollary 2.5 holds. To this end we need the next two lemmas.

Lemma 2.6 ([7, Lem. 3.2])**.** *Let $\varphi : R \to S$ be a faithfully flat finite ring homomorphism. If P is a projective R-module, then it is a direct summand* (*as an R-module*) *of the projective S-module $S \otimes_R P$.*

Lemma 2.7. *Let $\varphi : R \to S$ be a faithfully flat ring homomorphism. Assume that* $\dim R$ *is finite. Then an R-module M is G_C-projective if and only if $S \otimes_R M$ is a $G_{\widetilde{C}}$-projective S-module and* $\mathrm{Ext}_R^i(M, C \otimes_R P) = 0$ *for all $i > 0$ and all projective R-modules P.*

Proof. The necessity follows from Remark 2.3 and [8, Prop. 4.12(3)]. The sufficiency follows from [17, Thm. 3.10, Cor. 3.11]. □

Note that if $\varphi : R \to S$ is a faithfully flat ring homomorphism and $\dim S$ is finite, one has $\dim R$ is finite. Then we have:

Theorem 2.8. *Let $\varphi : R \to S$ be a faithfully flat finite ring homomorphism. If* $\dim S$ *is finite, then for every $X \in \mathcal{D}_{\square}(R)$, there is an equality*

$$\mathcal{GP}_{\widetilde{C}}\text{-pd}_S(S \otimes_R^{\mathbf{L}} X) = \mathcal{GP}_C\text{-pd}_R(X).$$

Proof. By Corollary 2.5, it is enough to show that

$$\mathcal{GP}_C\text{-pd}_R(X) \leq \mathcal{GP}_{\widetilde{C}}\text{-pd}_S(S \otimes_R^{\mathbf{L}} X).$$

Assume that $\mathcal{GP}_{\widetilde{C}}\text{-pd}_S(S\otimes^{\mathbf{L}}_R X) = g < \infty$. Then by [16, Thm. 3.5], $\sup(S\otimes^{\mathbf{L}}_R X) \leq g$ and for every bounded complex $A \simeq S \otimes^{\mathbf{L}}_R X$ of $G_{\widetilde{C}}$-projective S-modules, the module C^A_g is $G_{\widetilde{C}}$-projective.

Consider a G_C-projective resolution $G \xrightarrow{\simeq} X$ over R. Then by [16, Cor. 2.14], $S \otimes^{\mathbf{L}}_R X \simeq S \otimes_R G$. Clearly, $S \otimes_R G$ is a complex of $G_{\widetilde{C}}$-projective S-modules by [8, Cor. 4.17]. Then $S \otimes_R G$ is a $G_{\widetilde{C}}$-projective resolution of $S \otimes_R X$, and so $\sup(S \otimes_R G) \leq g$. Hence the sequence

$$\cdots \to S \otimes_R G_{g+2} \to S \otimes_R G_{g+1} \to S \otimes_R G_g$$

is exact. Clearly, it is exact as a sequence of R-modules. Since S is a faithfully flat R-module, the sequence

$$\cdots \to G_{g+2} \to G_{g+1} \to G_g$$

is exact. Consequently, one has $\sup G \leq g$ and so $\sup X \leq g$.

Next, we prove that C^G_g is G_C-projective. For $i > g$, one has $\mathrm{H}_i(S\otimes_R G) = 0$. Right-exactness of the functor $S \otimes_R -$ yields an isomorphism $\mathrm{Coker}\partial^{S\otimes_R G}_n \cong S\otimes_R \mathrm{Coker}\partial^G_n$ for each n. Set $K = \mathrm{C}^G_g$. By [16, Thm. 3.5], one has the S-module $\mathrm{C}^{S\otimes_R G}_g \cong S \otimes_R K$ is $G_{\widetilde{C}}$-projective. For every projective R-module P, one has P is a direct summand of a projective S-module Q by Lemma 2.6. Let $\mathbb{P}$ be a projective resolution of K. For all $i \geq 1$, one has $\widetilde{C} \otimes_S Q \cong (C \otimes_R S) \otimes_S Q \cong C \otimes_R Q$, then we have

$$\begin{aligned}
\mathrm{Ext}^i_R(K, C \otimes_R Q) &= \mathrm{H}_{-i}(\mathrm{Hom}_R(\mathbb{P}, C \otimes_R Q)) \\
&= \mathrm{H}_{-i}(\mathrm{Hom}_R(\mathbb{P}, \mathrm{Hom}_S(S, \widetilde{C} \otimes_S Q))) \\
&= \mathrm{H}_{-i}(\mathrm{Hom}_S(S \otimes_R \mathbb{P}, \widetilde{C} \otimes_S Q)) \\
&= \mathrm{Ext}^i_S(S \otimes_R K, \widetilde{C} \otimes_S Q) \\
&= 0.
\end{aligned}$$

Therefor, one has $\mathrm{Ext}^i_R(K, C\otimes_R P) = 0$ and so K is a G_C-projective R-module by Lemma 2.7. It follows from [16, Thm. 3.5] that $\mathcal{GP}_C\text{-pd}_R(X) < \infty$.

To prove the equality, using [16, Thm. 3.5], choose a projective R-module Q such that $\mathcal{GP}_C\text{-pd}_R(X) = -\inf \mathbf{R}\mathrm{Hom}_R(X, C \otimes_R Q)$. Since Q is a direct summand of a projective S-module $\overline{Q}$ by Lemma 2.6, hence one has

$$\begin{aligned}
\mathcal{GP}_{\widetilde{C}}\text{-pd}_S(S \otimes^{\mathbf{L}}_R X) &\geq -\inf \mathbf{R}\mathrm{Hom}_S(S \otimes^{\mathbf{L}}_R X, \widetilde{C} \otimes_S \overline{Q}) \\
&= -\inf \mathbf{R}\mathrm{Hom}_R(X, \mathbf{R}\mathrm{Hom}_S(S, \widetilde{C} \otimes_S \overline{Q})) \\
&= -\inf \mathbf{R}\mathrm{Hom}_R(X, \widetilde{C} \otimes_S \overline{Q}) \\
&\geq -\inf \mathbf{R}\mathrm{Hom}_R(X, \widetilde{C} \otimes_S Q) \\
&= -\inf \mathbf{R}\mathrm{Hom}_R(X, C \otimes_R Q) \\
&= \mathcal{GP}_C\text{-pd}_R(X).
\end{aligned}$$

The first step is by [16, Thm. 3.5], the second one is from Hom-tensor adjointness, the fourth one follows from Q is a direct summand of a projective S-module $\overline{Q}$ and the last one comes from the choice of Q. This completes the proof. □

3. An application

Let $(R, \mathfrak{m}, k)$ be a local ring. Recall that the depth of an R-complex X is defined as

$$\mathrm{depth}_R X = -\sup \mathbf{R}\mathrm{Hom}_R(k, X).$$

The following equality is well-known as the Auslander-Buchsbaum formula: for any $X \in \mathbf{P}^f(R)$, there is an equality

$$\mathrm{pd}_R X = \mathrm{depth} R - \mathrm{depth}_R X. \tag{3.1}$$

For homologically bounded complex with finite homology, for finite modules in particular, the G_C-projective dimension coincides with Christensen's notion of G_C-dimension; see [9, Prop. 3.1]. Then we have the next equality, which is the Auslander-Buchsbaum formula of G_C-projective dimension.

Theorem 3.1. *Let R be local and $X \in \mathcal{D}^f_\square(R)$. If $G_C\text{-}\dim_R X$ is finite, then there is an equality*

$$G_C\text{-}\dim_R X = \mathrm{depth} R - \mathrm{depth}_R X. \tag{3.2}$$

Proof. By [9, Thm. 2.6], $G_C\text{-}\dim_R X = \mathrm{Gpd}_{R\propto C} X$, where $R \propto C$ is the trivial extension of R by C. On the other hand $\mathrm{Gpd}_{R\propto C} X = \mathrm{depth} R \propto C - \mathrm{depth}_{R\propto C} X$ by [4, Thm. 3.14] since $\mathrm{Gpd}_{R\propto C} X < \infty$. Note that

$$\mathrm{depth} R \propto C = \min\{\mathrm{depth} R, \mathrm{depth}_R C\} = \mathrm{depth} R$$

since $\mathrm{depth}_R C = \mathrm{depth} R$ by [12, Thm. 2.2.6] and $\mathrm{depth}_{R\propto C} X = \mathrm{depth}_R X$ by [3, Exercise 1.2.26]. □

Then we have the following result for modules, and which recovers [11, Thm. 3.12] and [14, Thm. 2.5].

Corollary 3.2. *Let R be a local ring. Then for every finitely generated R-module $M \neq 0$ of finite G_C-dimension, there is an equality*

$$G_C\text{-}\dim_R M = \mathrm{depth} R - \mathrm{depth}_R M.$$

Corollary 3.3. *Let $\varphi : R \to S$ be a local ring homomorphism of finite flat dimension. Assume that $X \in \mathcal{D}^f_\square(R)$ with $G_C\text{-}\dim_R X$ finite and $U \in \mathbf{P}^f(S)$, then the following equality holds*

$$G_C\text{-}\dim_S(U \otimes^{\mathbf{L}}_R X) = G_C\text{-}\dim_R X + \mathrm{pd}_S U.$$

Proof. By Theorem 2.4, one has $G_C\text{-}\dim_S(U \otimes^{\mathbf{L}}_R X)$ is finite. By hypothesis, it is not hard to see that $U \otimes^{\mathbf{L}}_R X \in \mathcal{D}^f_\square(R)$ and $U \in \mathbf{P}(R)$. Since $G_C\text{-}\dim_R X$ is finite, the complex X is homologically bounded above. Now the first equality in the computation below follows from (3.2), the second one follows by [5, Thm. 6.2(i)] and the last one follows from (3.2) and the Auslander-Buchsbaum formula (3.1).

$$\begin{aligned} G_C\text{-dim}_S(U \otimes^{\mathbf{L}}_R X) &= \operatorname{depth} S - \operatorname{depth}_S(U \otimes^{\mathbf{L}}_R X) \\ &= \operatorname{depth} S - \operatorname{depth}_S U - \operatorname{depth}_R X + \operatorname{depth} R \\ &= G_C\text{-dim}_R X + \operatorname{pd}_S U. \end{aligned}$$

□

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