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REMARKS ON THE INFINITY WAVE EQUATION

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Abstract. We propose the infinity wave equation which can be derived from the exponential wave equation through the limit $p \to \infty$. The solution of infinity Laplacian equation can be considered as a static solution of the infinity wave equation. We present basic observations and find some special solutions.

1. Introduction

We are interested in the following equation

$$\partial_\alpha u \, \partial_\beta u \, \partial^\alpha \partial^\beta u = 0, \tag{1.1}$$

where $u : \mathbb{R}^{n+1} \to \mathbb{R}$ is a scalar function and $\mathbb{R}^{n+1}$ is Minkowski spacetime with signature $(1, \ldots, 1, -1)$. We use the notation $\partial^j = \partial_j = \frac{\partial}{\partial x_j}$, $-\partial^0 = \partial_0 = \frac{\partial}{\partial t}$. The summation convention is used to sum over repeated indices. Greek indices are used to denote $0, 1, \ldots, n$, while Latin indices for $j = 1, 2, \ldots, n$. In particular, (1.1) reads as in $\mathbb{R}^{1+1}$

$$u_x^2 u_{xx} - 2u_x u_t u_{xt} + u_t^2 u_{tt} = 0. \tag{1.2}$$

The equation (1.1) is obtained by the formal limit $p \to \infty$ of the exponential wave equation [14]

$$\frac{1}{p}\partial_\alpha \partial^\alpha u + \partial_\alpha u \, \partial_\beta u \, \partial^\alpha \partial^\beta u = 0, \tag{1.3}$$

where p is a positive constant. We can check that the equation (1.3) is equivalent to

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\partial_i u \, \exp\left(\frac{p}{2}|\nabla u|^2 - \frac{p}{2}(\partial_t u)^2\right)\right) - \frac{\partial}{\partial t}\left(\partial_t u \, \exp\left(\frac{p}{2}|\nabla u|^2 - \frac{p}{2}(\partial_t u)^2\right)\right) = 0,$$

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which is the Euler-Lagrange equation of

$$\int_{\mathbb{R}^{n+1}} \frac{1}{p}\left(e^{\frac{p}{2}|\nabla u|^2-\frac{p}{2}(\partial_t u)^2}-1\right)dxdt.$$

For the static solution of (1.3), we have an equation

$$\frac{1}{p}\Delta u+\partial_i u\,\partial_j u\,\partial_i\partial_j u=0. \tag{1.4}$$

A function $u : \mathbb{R}^n \to \mathbb{R}$ is called an exponentially harmonic function if it satisfies (1.4). An exponentially harmonic map was introduced by J. Eells and L. Lemaire [9], which generalizes usual harmonic map. It was proved in [13] that any bounded exponentially harmonic function must be constant.

The authors in [15] investigated some exact solutions of exponential wave maps and described the applications of exponential wave maps in Kaluza-Klein gravity [19] by coupling gravitational fields with exponential scalar fields.

For a formal limit $p \to \infty$ in (1.4), we can obtain infinity Laplacian equation

$$\Delta_\infty u := \partial_i u\,\partial_j u\,\partial_i\partial_j u=0. \tag{1.5}$$

In fact, the solution of equation (1.5) was studied in [12] through the solution of (1.4). The equation (1.5) was proposed in [2–4]. Many authors have studied (1.5) from the viewpoint of elliptic PDE, in particular, boundary value problem through viscosity solution [5, 10, 12]. The regularity problem has been studied in [11,16,18]. A game theory point of view was provided in [20] for understanding the infinity Laplacian equation. Insights from game theory might help us understand this interesting equation. We refer to [1,5,10] for more information.

The equation (1.3) can be written as in $\mathbb{R}^{1+1}$

$$\frac{1}{p}(u_{xx}-u_{tt})+u_x^2u_{xx}-2u_xu_tu_{xt}+u_t^2u_{tt}=0. \tag{1.6}$$

The global existence of solution to (1.6) was proved in [14] for small initial data. We summarize the situation in the following table.

	Equation (1.4)	Equation (1.3)
$p<\infty$	Exponentially harmonic equation [6, 13, 15]	Exponential wave equation [7, 14, 22]
$p=\infty$	infinity Laplacian equation [1–5, 10, 12, 20]	

Here we want to fill out the above blank. That is to say, we take a formal limit $p \to \infty$ of the equation (1.3) to obtain (1.1) which will be called as *infinity wave* equation. The infinity Laplacian equation in $\mathbb{R}^2$ reads as

$$u_x^2u_{xx}+2u_xu_yu_{xy}+u_y^2u_{yy}=0. \tag{1.7}$$

Aronsson [3] considered (1.7) as a parabolic equation, in the sense of the definition in [8], because of $(u_xu_y)^2-u_x^2u_y^2=0$. Many authors deal with boundary value problem of (1.7). The equation (1.2) also satisfies $(u_xu_t)^2-u_x^2u_t^2=0$ but takes several characters of hyperbolic equation like the existence of traveling wave and going well with hyperbolic coordinate. It seems quite difficult problem to consider the general solution of (1.2) which is a quasilinear degenerate equation. Here we are interested in the solution of special ansanz.

In Section 2 we present basic observations for (1.1). Some special solutions to (1.2) are discussed in Section 3.

2. Basic observations

Here we present some observations on the equation (1.1).

(1) The equation (1.1) can be rewritten as

$$\frac{1}{2}\partial_\alpha(\partial_\beta u \partial^\beta u)\partial^\alpha u = 0.$$

Therefore any solution to the following eikonal equation becomes a solution of (1.1):

$$u_t^2 - |\nabla u|^2 = c, \tag{2.1}$$

where c is a constant.

The remarkable result in [3] is that linear functions are the only C^2 solutions to (1.5) in $\mathbb{R}^2$. However it is easy to check that this is not the case for equation (1.2). In fact, the function $u(x,t) = f(x \pm t)$, where any C^2 function f, satisfies the eikonal equation (2.1) with $c = 0$.

We consider the problem

$$\begin{aligned} u_x^2 - u_y^2 + n^2 = 0, \\ u(x,\, 0) = f(x), \end{aligned} \tag{2.2}$$

where n is a constant and the variable y is used instead of t to apply characteristic method [21] where t usually denotes a parameter. Considering that (2.2) can be expressed as

$$(u_x,\, -u_y,\, -n^2) \cdot (u_x,\, u_y,\, -1) = 0,$$

we set

$$\frac{dx}{dt} = u_x, \quad \frac{dy}{dt} = -u_y, \quad \frac{du}{dt} = -n^2. \tag{2.3}$$

Then we have

$$\begin{aligned} \frac{d^2x}{dt^2} &= u_{xx}\frac{dx}{dt} + u_{xy}\frac{dy}{dt} = \frac{1}{2}\partial_x(-n^2) = 0, \\ \frac{d^2y}{dt^2} &= -\big(u_{yx}\frac{dx}{dt} + u_{yy}\frac{dy}{dt}\big) = \frac{1}{2}\partial_y(n^2) = 0, \\ \frac{du}{dt} &= u_x\frac{dx}{dt} + u_y\frac{dy}{dt} = -n^2. \end{aligned} \tag{2.4}$$

Write the initial condition parametrically in the form $(x,\, y,\, u) = (s,\, 0,\, h(s))$. The condition implies $\frac{dx}{dt}(0, s) = u_x = f'(s)$. Substituting the expression into the (2.2) leads us to $\frac{dy}{dt}(0,s) = -u_y = \sqrt{(f'(s))^2 + n^2}$. Then we have, taking (2.3) and (2.4) into account,

$$\Big(x(t,\,s),\,y(t,\,s),\,u(t,\,s)\Big) = \Big(f'(s)t + s,\,t\sqrt{(f'(s))^2+n^2},\,f(s) - n^2t\Big).$$

(2) Define $u(x,\,t) = V(X,\,T)$, where $X = \frac{1}{\sqrt{2}}(x+t)$ and $T = \frac{1}{\sqrt{2}}(x-t)$. Then the equation (1.2) transforms to

$$V_T^2 V_{XX} + 2V_X V_T V_{XT} + V_X^2 V_{TT} = 0. \tag{2.5}$$

Recall that 1-Laplacian equation in $\mathbb{R}^2$ reads as

$$u_y^2 u_{xx} - 2u_x u_y u_{xy} + u_x^2 u_{yy} = 0. \tag{2.6}$$

Note that (2.5) has a different sign from (2.6). We also note that infinity Laplacian equation

$$u_x^2 u_{xx} + 2u_x u_y u_{xy} + u_y^2 u_{yy} = 0$$

is invariant under the transformation $X = \frac{1}{\sqrt{2}}(x+y)$ and $Y = \frac{1}{\sqrt{2}}(x-y)$.

It is interesting question to ask if there is a solution of (1.2) which does not satisfy eikonal equation (2.1). In next section we study some special solutions of (1.2) among which we find non-eikonal solutions, for instance, (3.3) and (3.9).

3. Special solutions

It seems rather difficult matter to find the general solution to (1.2). Here we are interested in special solutions of (1.2).

3.1. Solutions in hyperbolic coordinate

Let us transform the equation (1.2) in the coordinate

$$x = \rho\cosh s \quad \text{and} \quad t = \rho\sinh s,$$

where $\rho = \pm\sqrt{|x^2 - t^2|}$ and $s = \frac{1}{2}\log\frac{|x+t|}{|x-t|}$. Note that the infinity Laplacian equation (1.5) in $\mathbb{R}^2$ is written as

$$v_x^2 v_{xx} + 2v_x v_y v_{xy} + v_y^2 v_{yy} = 0. \tag{3.1}$$

For the equation (3.1), G. Aronsson [4] considered the polar coordinate $(r,\,\theta)$ and looked for the solution of the form $v(x,y) = (x^2+y^2)^{\frac{k}{2}} g(\theta)$.

For the simple representation, we will consider the region $-x < t < x$. We are interested in solutions of the form

$$u(x,\,t) = \rho^k f(s) = (x^2 - t^2)^{\frac{k}{2}} f\left(\frac{1}{2}\log\frac{x+t}{x-t}\right),$$

where k is a constant and f is a function of s. Substituting in (1.2), we have the following ODE for $f(s)$

$$(f')^2 f'' - (2k^2 - k) f (f')^2 + k^3(k-1) f^3 = 0, \tag{3.2}$$

where $f' = \frac{df}{ds}$ and $f'' = \frac{d^2 f}{ds^2}$.

• The case of $k = 0$: It is easy to check that $f(s) = as + b$ for constants a, b. Then we have

$$u(x,\, t) = \frac{a}{2} \log \frac{x+t}{x-t} + b. \tag{3.3}$$

• The case of $k = 1$: We have $f'' - f = 0$ or $f' = 0$ which implies that

$$u(x,\, t) = \sqrt{x^2 - t^2} \left(a\sqrt{\frac{x+t}{x-t}} + b\sqrt{\frac{x-t}{x+t}} \right) \quad \text{or} \quad c\sqrt{x^2 - t^2},$$

where a, b, c are constants.

Assume that $f(s) \neq 0$ and $f'(s) \neq 0$ in some interval $\alpha < s < \beta$. Then (3.2) can be rewritten as

$$\frac{f''}{f'} - (2k^2 - k)\frac{f}{f'} + k^3(k-1)\left(\frac{f}{f'}\right)^3 = 0. \tag{3.4}$$

Let $h = f/f'$. We can check

$$h' = 1 - h\frac{f''}{f'} \quad \text{and} \quad f(s) = c\exp\left(\int \frac{1}{h}\, ds\right).$$

Then we have an equation for h from (3.4).

$$\begin{aligned} \frac{dh}{ds} &= k^3(k-1)h^4 - (2k^2 - k)h^2 + 1 \\ &= (k^2h^2 - 1)\left((k^2 - k)h^2 - 1\right). \end{aligned} \tag{3.5}$$

For $k^2 - k > 0$, (3.5) can be rewritten as

$$\left(\frac{k}{kh+1} - \frac{k}{kh-1} + \frac{k-1}{\sqrt{k^2-k}h - 1} - \frac{k-1}{\sqrt{k^2-k}h + 1} \right) \frac{dh}{ds} = 2$$

which can be integrated as

$$\log\left|\frac{kh+1}{kh-1}\right| + \sqrt{\frac{k-1}{k}} \log\left|\frac{\sqrt{k^2-k}h - 1}{\sqrt{k^2-k}h + 1}\right| = 2s.$$

For $k^2 - k < 0$, (3.5) can be integrated as

$$\log\left|\frac{kh+1}{kh-1}\right| + 2\frac{1-k}{\sqrt{k-k^2}} \arctan(\sqrt{k-k^2}h) = 2s.$$

3.2. Self-similar solutions

We study a solution of the form $u(x,t) = t^\alpha h(\frac{x}{t})$. Substituting the ansatz in (1.2), we have

(3.6)

$$t^{3\alpha-4}\Big((1-y^2)h' + \alpha yh\Big)^2 h''$$
$$+\, t^{3\alpha-4}(yh' - \alpha h)\Big(2(\alpha-1)(1-y^2)(h')^2 + 3(\alpha^2-\alpha)yhh' - (\alpha^3-\alpha^2)h^2\Big) = 0,$$

where we denote $y = x/t$.
• The case of $\alpha = 0$: We have

$$(h')^2\big((1-y^2)h'' - 2yh'\big) = 0.$$

Then we obtain $h'(y) = \pm\frac{1}{y^2-1}$ from which we derive

$$h(y) = \pm\frac{1}{2}\log\frac{|y-1|}{|y+1|}.$$

Then we have

$$u(x,t) = \pm\frac{1}{2}\log\frac{|x-t|}{|x+t|},$$

which is already known in (3.3). For infinity Laplacian equation (3.1), we have $h'(y) = \pm\frac{1}{y^2+1}$ from which we derive

$$v(x,\, y) = \arctan\left(\frac{y}{x}\right).$$

• The case of $\alpha = 1$: We have $\Big((1-y^2)h' + yh\Big)^2 h'' = 0$. Then we derive $h(y) = c\sqrt{|y^2-1|}$ which implies

$$u(x,t) = c\sqrt{|x^2-t^2|}.$$

Assume $h'(y) \neq 0$ and let $H = h/h'$. Dividing the equation (3.6) by $(h')^3$ and considering $\frac{h''}{h'} = \frac{1-H'}{H}$, we derive the first order ODE for H.

$$\frac{dH}{dy} = 1 + \frac{H(\alpha H - y)\big((\alpha^3-\alpha^2)H^2 - 3(\alpha^2-\alpha)yH - 2(\alpha-1)(1-y^2)\big)}{(1-y^2+\alpha yH)^2}.$$

In particular, we have for $\alpha = -1$

$$\frac{dH}{dy} = 1 + \frac{H(H+y)\big(2H^2 + 6yH - 4(1-y^2)\big)}{(1-y^2-yH)^2},$$

which can be rewritten as, with the notation $F(y) = H(y) + y$,

$$\frac{dF}{dy} = 2 + \frac{F(F-y)(2F^2+2yF-4)}{(yF-1)^2}.$$

The further analysis of the above ODEs will be considered later.

3.3. Separation of variables

We substitute $u(x,t) = f(x) + g(t)$ in (1.2). Then we have

$$\left(\frac{df}{dx}\right)^2\frac{d^2f}{dx^2} + \left(\frac{dg}{dt}\right)^2\frac{d^2g}{dt^2} = 0,$$

which leads us to

$$u(x,t) = c\left(|x|^{\frac{4}{3}} - |t|^{\frac{4}{3}}\right), \tag{3.7}$$

where c is a constant. Note that the function (3.7) was presented as $C^{1,\frac{1}{3}}(\mathbb{R}^2)$ singular solution of (3.1) in [3].

We substitute $u(x,t) = f(x) + g(t)$ in (1.6) with $p = 1$ to obtain

$$\left(1+\left(\frac{df}{dx}\right)^2\right)\frac{d^2f}{dx^2} - \left(1-\left(\frac{dg}{dt}\right)^2\right)\frac{d^2g}{dt^2} = 0,$$

which implies

$$\left(1+\left(\frac{df}{dx}\right)^2\right)\frac{d^2f}{dx^2} = c \quad \text{and} \quad \left(1-\left(\frac{dg}{dt}\right)^2\right)\frac{d^2g}{dt^2} = c \tag{3.8}$$

for some constant c. Following the idea in [17], we have from (3.8)

$$f'(x)^3 + 3f'(x) - 3cx = 0 \quad \text{and} \quad g'(t)^3 - 3g'(t) + 3ct = 0.$$

Let us consider the case of $c = 1$. By Cardanno's formulae, we have

$$\begin{aligned} f'(x) &= 2^{-\frac{1}{3}}\left((\sqrt{9x^2+4}+3x)^{\frac{1}{3}} - (\sqrt{9x^2+4}-3x)^{\frac{1}{3}}\right), \\ g'(t) &= 2^{-\frac{1}{3}}\left((\sqrt{9t^2-4}-3t)^{\frac{1}{3}} - (\sqrt{9t^2-4}+3t)^{\frac{1}{3}}\right). \end{aligned} \tag{3.9}$$

3.4. Hodograph method

Let $\phi = u_x$ and $\psi = u_t$. Then we can rewrite (1.6) as

$$\begin{aligned} &\phi_t - \psi_x = 0, \\ &(\frac{1}{p} - \psi^2)\psi_t + \phi\psi(\phi_t + \psi_x) - (\frac{1}{p} + \phi^2)\phi_x = 0. \end{aligned} \tag{3.10}$$

Applying hodograph transformation [8]

$$\phi_x = Jt_\psi, \quad \phi_t = -Jx_\psi, \quad \psi_x = -Jt_\phi, \quad \psi_t = Jx_\phi,$$

where $J = \phi_x\psi_t - \phi_t\psi_x$, we have from (3.10)

$$\begin{aligned} &x_\psi - t_\phi = 0, \\ &(\frac{1}{p} - \psi^2)x_\phi - \phi\psi(x_\psi + t_\phi) - (\frac{1}{p} + \phi^2)t_\psi = 0. \end{aligned}$$

The first equation implies that a potential function $U = U(\phi, \psi)$ exists such that the relations

$$x = U_\phi \quad \text{and} \quad t = U_\psi$$

hold, and the second equation then takes the form

$$(\frac{1}{p} - \psi^2)U_{\phi\phi} - 2\phi\psi U_{\phi\psi} - (\frac{1}{p} + \phi^2)U_{\psi\psi} = 0. \tag{3.11}$$

Every solution $U(\phi, \psi)$ defined in a certain region of the (ϕ, ψ)-plane leads to (x, t) as functions of ϕ and ψ. Then x and t can be introduced as new variables provided that the Jacobian

$$J = U_{\phi\phi}U_{\psi\psi} - U_{\phi\psi}^2 = x_\phi t_\psi - x_\psi t_\phi$$

does not vanish.

When $p = \infty$, the equation (3.11) becomes

$$\psi^2 U_{\phi\phi} + 2\phi\psi U_{\phi\psi} + \phi^2 U_{\psi\psi} = 0. \tag{3.12}$$

The equation (3.12) is parabolic one in the sense of the definition in [8]. In fact, we define

$$U(\phi,\, \psi) = V(\xi,\, \eta),$$

where $\xi = \phi$, $\eta = \phi^2 - \psi^2$. Then the equation (3.12) transforms to

$$2\eta V_\eta - (\xi^2 - \eta) V_{\xi\xi} = 0. \tag{3.13}$$

We consider a solution to (3.13) of the form $V(\xi, \eta) = V(\frac{\xi^2}{\eta})$. Then we have an ODE

$$2z(z-1)\frac{d^2V}{dz^2} + (2z-1)\frac{dV}{dz} = 0,$$

where $z = \xi^2/\eta$. Integrating the above ODE, we have

$$V(z) = \begin{cases} \arccos(1-2z) & \text{for } z(z-1) < 0, \\ \log|z - \frac{1}{2} + \sqrt{z^2 - z}\,| & \text{for } z(z-1) > 0. \end{cases}$$

References

[1] S. N. Armstrong and C. K. Smart, *A finite difference approach to the infinity Laplace equation and tug-of-war games*, Trans. Amer. Math. Soc. **364** (2012), no. 2, 595–636. https://doi.org/10.1090/S0002-9947-2011-05289-X

[2] G. Aronsson, *Extension of functions satisfying Lipschitz conditions*, Ark. Mat. **6** (1967), 551–561 (1967). https://doi.org/10.1007/BF02591928

[3] ———, *On the partial differential equation* ${u_x}^2 u_{xx} + 2u_x u_y u_{xy} + {u_y}^2 u_{yy} = 0$, Ark. Mat. **7** (1968), 395–425. https://doi.org/10.1007/BF02590989

[4] ———, *On certain singular solutions of the partial differential equation* $u_x^2 u_{xx} + 2u_x u_y u_{xy} + u_y^2 u_{yy} = 0$, Manuscripta Math. **47** (1984), no. 1-3, 133–151. https://doi.org/10.1007/BF01174590

[5] G. Aronsson, M. G. Crandall, and P. Juutinen, *A tour of the theory of absolutely minimizing functions*, Bull. Amer. Math. Soc. (N.S.) **41** (2004), no. 4, 439–505. https://doi.org/10.1090/S0273-0979-04-01035-3

[6] D. P. Chi, G. Choi, and J. Chang, *Asymptotic behavior of harmonic maps and exponentially harmonic functions*, J. Korean Math. Soc. **39** (2002), no. 5, 731–743. https://doi.org/10.4134/JKMS.2002.39.5.731

[7] Y.-J. Chiang and Y.-H. Yang, *Exponential wave maps*, J. Geom. Phys. **57** (2007), no. 12, 2521–2532. https://doi.org/10.1016/j.geomphys.2007.09.003

[8] R. Courant and D. Hilbert, *Methods of Mathematical Physics. Vol. I*, Interscience Publishers, Inc., New York, NY, 1953.

[9] J. Eells and L. Lemaire, *Another report on harmonic maps*, Bull. London Math. Soc. **20** (1988), no. 5, 385–524. https://doi.org/10.1112/blms/20.5.385

[10] L. C. Evans, *The 1-Laplacian, the ∞-Laplacian and differential games*, in Perspectives in nonlinear partial differential equations, 245–254, Contemp. Math., 446, Amer. Math. Soc., Providence, RI, 2007. https://doi.org/10.1090/conm/446/08634

[11] L. C. Evans and O. Savin, *$C^{1,\alpha}$ regularity for infinity harmonic functions in two dimensions*, Calc. Var. Partial Differential Equations **32** (2008), no. 3, 325–347. https://doi.org/10.1007/s00526-007-0143-4

[12] L. C. Evans and Y. Yu, *Various properties of solutions of the infinity-Laplacian equation*, Comm. Partial Differential Equations **30** (2005), no. 7-9, 1401–1428. https://doi.org/10.1080/03605300500258956
[13] M. C. Hong, *Liouville theorems for exponentially harmonic functions on Riemannian manifolds*, Manuscripta Math. **77** (1992), no. 1, 41–46. https://doi.org/10.1007/BF02567042
[14] H. Huh, *Global solutions of the exponential wave equation with small initial data*, Bull. Korean Math. Soc. **50** (2013), no. 3, 811–821. https://doi.org/10.4134/BKMS.2013.50.3.811
[15] A. D. Kanfon, A. Füzfa, and D. Lambert, *Some examples of exponentially harmonic maps*, J. Phys. A **35** (2002), no. 35, 7629–7639. https://doi.org/10.1088/0305-4470/35/35/307
[16] H. Koch, Y. R.-Y. Zhang, and Y. Zhou, *Some sharp Sobolev regularity for inhomogeneous infinity Laplace equation in plane*, J. Math. Pures Appl. (9) **132** (2019), 483–521. https://doi.org/10.1016/j.matpur.2019.02.007
[17] P. Lewintan, *The "wrong minimal surface equation" does not have the Bernstein property*, Analysis (Munich) **31** (2011), no. 4, 299–303. https://doi.org/10.1524/anly.2011.1137
[18] E. Lindgren, *On the regularity of solutions of the inhomogeneous infinity Laplace equation*, Proc. Amer. Math. Soc. **142** (2014), no. 1, 277–288. https://doi.org/10.1090/S0002-9939-2013-12180-5
[19] J. M. Overduin and P. S. Wesson, *Kaluza-Klein gravity*, Phys. Rep. **283** (1997), no. 5-6, 303–378. https://doi.org/10.1016/S0370-1573(96)00046-4
[20] Y. Peres, O. Schramm, S. Sheffield, and D. B. Wilson, *Tug-of-war and the infinity Laplacian*, J. Amer. Math. Soc. **22** (2009), no. 1, 167–210. https://doi.org/10.1090/S0894-0347-08-00606-1
[21] Y. Pinchover and J. Rubinstein, *An Introduction to Partial Differential Equations*, Cambridge University Press, Cambridge, 2005. https://doi.org/10.1017/CBO9780511801228
[22] C.-H. Wei and N.-A. Lai, *Global existence of smooth solutions to exponential wave maps in FLRW spacetimes*, Pacific J. Math. **289** (2017), no. 2, 489–509. https://doi.org/10.2140/pjm.2017.289.489

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