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THE INFINITE GROWTH OF SOLUTIONS OF SECOND ORDER LINEAR COMPLEX DIFFERENTIAL EQUATIONS WITH COMPLETELY REGULAR GROWTH COEFFICIENT

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Abstract. In this paper we discuss the classical problem of finding conditions on the entire coefficients $A(z)$ and $B(z)$ guaranteeing that all nontrivial solutions of $f'' + A(z)f' + B(z)f = 0$ are of infinite order. We assume $A(z)$ is an entire function of completely regular growth and $B(z)$ satisfies three different conditions, then we obtain three results respectively. The three conditions are (1) $B(z)$ has a dynamical property with a multiply connected Fatou component, (2) $B(z)$ satisfies $T(r, B) \sim \log M(r, B)$ outside a set of finite logarithmic measure, (3) $B(z)$ is extremal for Denjoy's conjecture.

1. Introduction and main results

In this article, we shall use the basic results of Nevanlinna theory in the complex plane $\mathbb{C}$ and assume the reader is familiar with standard notations, such as $T(r, f), m(r, f), N(r, f)$ and $\delta(a, f)$, for example see [19,34]. Nevanlinna theory plays an important role in the study of complex differential equations, and there appear many results in this areas recent years. In this paper, the order of an entire function f is defined as

$$\rho(f) = \limsup_{r\to+\infty} \frac{\log^+ T(r, f)}{\log r} = \limsup_{r\to+\infty} \frac{\log^+ \log^+ M(r, f)}{\log r},$$

where $\log^+ x = \max\{\log x, 0\}$ and $M(r, f)$ denotes the maximum modulus of f on the circle $|z| = r$.

Our main purpose is to consider the second order linear differential equation

$$f'' + A(z)f' + B(z)f = 0, \tag{1}$$

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where $A(z)$ and $B(z)$ are entire functions. It's well known that all solutions of (1) are entire functions. If $B(z)$ is transcendental and f_1, f_2 are two linearly independent solutions of this equation, then at least one of f_1, f_2 is of infinite order, see [16]. However, there exist some equations of form (1) that have a nontrivial solution of finite order. For example, $f(z) = e^z$ satisfies differential equation $f''+e^{-z}f'-(e^{-z}+1)f = 0$. A natural question is that what conditions on $A(z)$ and $B(z)$ can guarantee that every solution $f(\not\equiv 0)$ of the equation (1) is of infinite order? There have been many results in the literature on this subject, see [16, 19]. For example, we collect some classical results and give the following theorem.

Theorem 1.1. *Let $A(z)$ and $B(z)$ be nonconstant entire functions, satisfying any one of the following additional hypotheses*:

(1) $\rho(A) < \rho(B)$, *see* [13];
(2) *$A(z)$ is a polynomial and $B(z)$ is transcendental, see* [13];
(3) $\rho(B) < \rho(A) \leq \frac{1}{2}$, *see* [15].

Then every nontrivial solution f of the equation (1) *has infinite order.*

This is a hot research object and a lot of works have sprung up, such as [7, 11, 15, 21, 23–25, 31–33]. Our main purpose is continue to study the above question, try to find conditions which $A(z), B(z)$ should satisfy to ensure that nontrivial solution of (1) has infinite order. Since every nontrivial solution of (1) satisfies $\rho(f) \geq \max\{\rho(A), \rho(B)\}$, so we consider the questions under the condition $\max\{\rho(A), \rho(B)\} < \infty$ in the following theorems. At first, if $\rho(r)$ is positive, differentiable for large r and satisfies $\lim_{r\to\infty} \rho(r) = \rho \in (0, \infty)$, $\lim_{r\to\infty} \rho'(r) r \log r = 0$, then $\rho(r)$ is called a proximate order, see [10, Section 2, Chapter 2]. In order to motivate and formulate the result, recall the indicator $h(\theta)$ of an entire function $A(z)$ of order ρ with respect to the proximate order $\rho(r)$ is defined by

$$h(\theta) = \limsup_{r\to\infty} \frac{\log |A(re^{i\theta})|}{r^{\rho(r)}}, \tag{2}$$

where $\rho(r) \to \rho$ as $r \to \infty$. The function $A(z)$ is said to be completely regular growth (in the sense of Levin and Pfluger) if there exist disks $D(a_k, s_k)$ satisfying

$$\sum_{|a_k|\leq r} s_k = o(r) \tag{3}$$

such that

$$\log |A(re^{i\theta})| = h(\theta) r^{\rho(r)} + o(r^{\rho(r)}), \quad re^{i\theta} \notin \bigcup_k D(a_k, s_k) \tag{4}$$

as $r \to \infty$, uniformly in θ. A union of disks satisfying (3) is called a C_0 set. We refer to Levin's book [22] for a thorough discussion of functions of completely regular growth. There have been some works about the coefficients of (1)

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involving completely regular growth, such as [14, 30]. In [14], the authors got the following result.

Theorem 1.2. *Let $A(z)$ be an entire function of completely regular growth, and let $B(z)$ be any entire function such that $\rho(B) < \rho(A)$. Define $E = \{\theta \in [-\pi, \pi) : h(\theta) \le 0\}$. Then every nonzero solution of (1) satisfies*

$$\rho(f) \ge \max\{\rho(A), \left(21\sqrt{m(E)}\right)^{-1} - 1\},$$

where $\rho(f) = \infty$ if $m(E) = 0$, here $m(E)$ is the Lebesgue measure of E.

From the above theorem, it's easy to see that if $h(\theta) > 0$ for almost every $\theta \in [0, 2\pi)$, then the nonzero solutions of (1) have infinite order. In this paper, we release the restriction on $h(\theta)$, that is, assume that $h(\theta)$ can take negative value for θ in some intervals which are contained in $[0, 2\pi)$. Moreover, we should give some more conditions for $B(z)$, then the order of solutions of (1) are of infinite order.

In our first theorem let $B(z)$ have a dynamical property. A result involving the coefficient of complex differential equation with dynamical property was given in [37]. For the convenience of reading an introduction of dynamical property we used here is given in Section 2.

Theorem 1.3. *Let $A(z)$ be a completely regular growth entire function and the set $E = \{\theta \in [0, 2\pi) : h(\theta) = 0\}$ is of Lebesgue measure zero, and let $B(z)$ be a transcendental entire function with a multiply connected Fatou component such that $\rho(A) \ne \rho(B)$. Then every nontrivial solution of (1) is of infinite order.*

In the second result, we assume $B(z)$ is a transcendental entire function satisfying $T(r, B) \sim \log M(r, B)$ as $r \to \infty$ outside a set of finite logarithmic measure. This method was ever used in [29, Lemma 2.7]. The function $B(z)$ in Theorem 1.4 really exists. For example, entire function has Fejér gaps. Here, $f(z) = \sum_{n=1}^{\infty} a_n z^{\lambda_n}$ is said to have Fejér gaps if $\sum_{n=1}^{\infty} \lambda_n^{-1} < \infty$, see [26]. A result involving Fejér gaps and concerning infinite growth of the solution of the equation (1) was given in [20].

Theorem 1.4. *Let $A(z)$ be a completely regular growth entire function and the set $E = \{\theta \in [0, 2\pi) : h(\theta) = 0\}$ is of Lebesgue measure zero, and let $B(z)$ be a transcendental entire function satisfying $T(r, B) \sim \log M(r, B)$ as $r \to \infty$ outside a set of finite logarithmic measure such that $\rho(A) \ne \rho(B)$. Then every nontrivial solution of (1) is of infinite order.*

In the sequel, we shall give the last result. In 1907 Denjoy [8] raised a conjecture says if $B(z)$ is an entire function with finite order $\rho(B)$ and $B(z)$ has k distinct finite asymptotic values, then $k \le 2\rho(B)$. This conjecture verified by Ahlfors [1] in 1930. If $k = 2\rho(B)$ we call $B(z)$ extremal for Denjoy's conjecture. These functions have been well investigated, for example see Ahlfors [1] and

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Zhang [36]. An example of function extremal for Denjoy's conjecture is

$$B(z)=\int_0^z \frac{\sin t^q}{t^q}dt,$$

where q is a positive integer. In fact, this function is of order q and has $2q$ distinct finite asymptotic values as

$$a_k=e^{\frac{k\pi i}{q}}\int_0^\infty \frac{\sin r^q}{r^q}dr,$$

where $k=1,2,\ldots,2q$, see [36, p. 28] for details. The study of growth of solutions of linear complex differential equation involving function extremal for Denjoy's conjecture appeared in paper [31]. Motivated by this idea, we consider the situation $B(z)$ is extremal for Denjoy's conjecture in (1) and obtain the following result.

Theorem 1.5. *Let $A(z)$ be a completely regular growth entire function and the set $E=\{\theta\in[0,2\pi):h(\theta)=0\}$ is of Lebesgue measure zero, and let $B(z)$ be an entire function extremal for Denjoy's conjecture such that $\rho(A)\neq\rho(B)$. Then every nontrivial solution of* (1) *is of infinite order.*

In the following, we shall give an example to illustrate the condition for $A(z)$ in the above results do exist. Firstly we introduce the definition so called $SCRG$. An example of completely regular growth function is the exponential sum

$$A(z)=\sum_{k=0}^{n-1}a_k\exp(b_kz),$$

provided $\arg b_k<\arg b_{k+1}<\arg b_k+\pi$ for $0\leq k\leq n-2$ and $\arg b_0<\arg b_{n-1}-\pi$, see details in [28]. In fact, exponential polynomials form an important subclass of functions of completely regular growth. It's well known [27] that the zeros of exponential sums are close to certain rays. Motivated by this we consider the functions satisfying the following condition, which are more general than the exponential polynomials.

Definition 1.6. If $A(z)$ is an entire function satisfying the following items, then we say $A(z)$ has the $SCRG$ (special completely regular growth) property.

(1) Let the rays $\arg z=\theta_j$ be the accumulated lines of zeros of $A(z)$, where $j=1,2,\ldots,m$ and $\theta_1<\theta_2<\cdots<\theta_m<\theta_{m+1}=\theta_1+2\pi$;
(2) Let $h(\theta)$ be the indicator of $A(z)$ in the sector $S(\theta_j,\theta_{j+1})=\{z=re^{i\theta}: r>0,\theta_j<\theta<\theta_{j+1}\}$, $j=1,2,\ldots,m$ and $\rho(r)(\to\rho)$ be a proximate order of $A(z)$;
(3) $\varepsilon(r)=1/\log^N(r)$ for some $N\in\mathbb{N}$, where $\log^N$ denotes the N-th iterate of the logarithm;
(4) $\log|A(re^{i\theta})|=h(\theta)r^\rho+O(r^{\rho(r)}\varepsilon(r))$ for $|\theta-\theta_j|>\varepsilon(r)$, $j=1,2,\ldots,m$.

The $SCRG$ property was first used in [6], in which some functions satisfying this property were given and the complex dynamical properties of entire

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function satisfying $SCRG$ were investigated. By Lemma 2.5, it's easy to see that the functions have $SCRG$ property satisfying the condition for $A(z)$ in Theorems 1.3, 1.4 and 1.5. Inspired by this, we assume the coefficient $A(z)$ of the equation (1) involving the $SCRG$ property and get the following result.

Corollary 1.7. *Let $A(z)$ be an entire function satisfying the SCRG property, and let $B(z)$ be a transcendental entire function satisfying the condition for $B(z)$ in Theorems 1.3, 1.4 and 1.5 respectively. Then every nontrivial solution of (1) is of infinite order.*

2. Preliminary lemmas and auxiliary results

The Lebesgue linear measure of a set $E \subset [0,\infty)$ is $meas(E) = \int_E dt$, and the logarithmic measure of a set $F \subset [1,\infty)$ is $m_l(F) = \int_F \frac{dt}{t}$. The upper and lower logarithmic densities of $F \subset [1,\infty)$ are given by

$$\overline{\log dens} F = \limsup_{r\to\infty} \frac{m_l(F\cap[1,r])}{\log r}$$

and

$$\underline{\log dens} F = \liminf_{r\to\infty} \frac{m_l(F\cap[1,r])}{\log r}$$

respectively. We say F has logarithmic density if $\overline{\log dens}(F) = \underline{\log dens}(F)$.

The proofs of our results highly rely on the estimation of logarithmic derivatives, which is due to Gundersen [12].

Lemma 2.1 ([12])**.** *Let f be a transcendental meromorphic function of finite order $\rho(f)$. Let $\varepsilon > 0$ be a given real constant, and let k and j be two integers such that $k > j \geq 0$. Then there exists a set $E \subset (1,\infty)$ with $m_l(E) < \infty$ such that for all z satisfying $|z| \notin (E\cup[0,1])$, we have*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq |z|^{(k-j)(\rho(f)-1+\varepsilon)}. \tag{5}$$

The following result which was proved by Zheng [38] is crucial to the proof of Theorem 1.2. Set $M_c(r,a,f) = \max\{|f(z)| : |z-a| = r|\}$, $L_c(r,a,f) = \min\{|f(z)| : |z-a| = r\}$. When $a = 0$, we simply write $M(r,f), L(r,f)$ for $M_c(r,0,f), L_c(r,a,f)$ respectively.

Lemma 2.2 ([38, Corollary 1])**.** *Let $f(z)$ be a transcendental meromorphic function with at most finitely many poles. If $J(f)$ has only bounded components, then for any complex number a, there exist a constant $0 < d < 1$ and two sequences $\{r_n\}$ and $\{R_n\}$ of positive numbers with $r_n \to \infty$ and $R_n/r_n \to \infty (n \to \infty)$ such that*

$$M_c(r,a,f)^d \leq L_c(r,a,f), \quad r \in G, \tag{6}$$

where $G = \bigcup_{n=1}^{\infty}\{r : r_n < r < R_n\}$.

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Remark 2.3. In the above lemma, particularly we have $M(r,f)^d \leq L(r,f),\ r \in G$. Obviously, the set G has infinite logarithmic measure.

In the below, in order to explain the assumption of Theorem 1.3 we give some introduction of complex dynamics, see [5] for example. The Fatou set $F(f)$ of a transcendental entire function f is the subset of the plane $\mathbb{C}$ where the iterates f^n of f form a normal family. The complement of $F(f)$ in $\mathbb{C}$ is called the Julia set $J(f)$ of f. The set $F(f)$ is completely invariant under f in the sense that $z \in F(f)$ if and only if $f(z) \in F(f)$. Therefore, if U is a component of $F(f)$, a so-called Fatou component, then there exists, for some $n = 0, 1, 2, \ldots$, a Fatou component U_n such that $f^n(U) \subset U_n$. If, for some $p \geq 1$, we have $U_p = U_0 = U$, then we say that U is a periodic component of period p, assuming p to be the minimal. If U_n is not eventually periodic, then U is a wandering domain of f. Although some entire functions with only simply connected Fatou component, such as Eremenko-Lyubich class function [9], there are many examples of entire function with multiply connected Fatou components. The first such function was constructed by Baker [2], who proved later [4] that this function has a multiply connected Fatou component that is a wandering domain. Moreover, Baker showed [3] that this is not a special property of this example: if U is any multiply connected Fatou component of a transcendental entire function f, then U is wandering domain which called Baker wandering domain. It has the following properties: (1) each U_n is bounded and multiply connected; (2) there exists $N \in \mathbb{N}$ such that U_n and 0 lie in a bounded complementary component of U_{n+1} for $n \geq N$; (3) $\mathrm{dis}(U_n, 0) \to \infty$ as $n \to \infty$. Therefore, if transcendental entire function f has a multiply connected Fatou component, then $J(f)$ has only bounded component.

Lemma 2.4 (Phragmén-Lindelöf principle, [17, Theorem 7.3])**.** *Let $f(z)$ be an analytic function of $z = re^{i\theta}$, regular in a region D between rays making a sector π/α at the origin and on the straight lines themselves. Suppose that $|f(z)| \leq M$ on the lines and as $r \to \infty$, $f(z) = O(e^{r^\beta})$, where $\beta < \alpha$ uniformly. Then $|f(z)| \leq M$ throughout D.*

Lemma 2.5 ([22, p. 115, Corollary])**.** *If the zeros of entire function $A(z)$ of proximate order $\rho(r)$ are regular distribution for the index $\rho(r)$, and if the density of the set of zeros within some sectors $S(\alpha,\beta)$ is equal to zero, then the indicator function within this sector is a ρ-trigonometric function, i.e.,*

$$h(\theta) = a\cos\rho\theta + b\sin\rho\theta, \tag{7}$$

where $\alpha \leq \theta \leq \beta$, a and b are constants. If, however, inside this sector there are no zeros of the function, then for $\alpha < \theta < \beta$ there exists the limit

$$h(\theta) = \lim_{r\to\infty} \frac{\log|A(re^{\theta})|}{r^{\rho(r)}}, \tag{8}$$

where the variable tends to the limit uniformly when $\alpha + \varepsilon \leq \theta \leq \beta - \varepsilon$ for any given $\varepsilon > 0$.

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The following result due to Gundersen [13, Theorem 3] shows the asymptotic properties of finite order solutions of the equation (1).

Lemma 2.6. *Let $A(z)$ and $B(z)(\not\equiv 0)$ be two entire functions such that for real constants $\alpha, \beta, \theta_1, \theta_2$, where $\alpha > 0, \beta > 0$ and $\theta_1 < \theta_2$,*

$$|A(z)| \geq \exp\{(1+o(1))\alpha|z|^{\beta}\}$$

and

$$|B(z)| \leq \exp\{o(1)|z|^{\beta}\}$$

as $z \to \infty$ in $\overline{S}(\theta_1, \theta_2) = \{z : \theta_1 \leq \arg z \leq \theta_2\}$. Let $\varepsilon > 0$ be a given small constant, and let $\overline{S}(\theta_1 + \varepsilon, \theta_2 - \varepsilon) = \{z : \theta_1 + \varepsilon \leq \arg z \leq \theta_2 - \varepsilon\}$. If f is a nontrivial solution of (1) with $\rho(f) < \infty$, then the following conclusions hold.

(1) *There exists a constant $b(\neq 0)$ such that $f(z) \to b$ as $z \to \infty$ in $\overline{S}(\theta_1 + \varepsilon, \theta_2 - \varepsilon)$. Furthermore,*

$$|f(z) - b| \leq \exp\{-(1+o(1))\alpha|z|^{\beta}\}$$

as $z \to \infty$ in $\overline{S}(\theta_1 + \varepsilon, \theta_2 - \varepsilon)$.

(2) *For each integer $k \geq 1$,*

$$|f^{(k)}(z)| \leq \exp\{-(1+o(1))\alpha|z|^{\beta}\}$$

as $z \to \infty$ in $\overline{S}(\theta_1 + \varepsilon, \theta_2 - \varepsilon)$.

Lemma 2.7 ([18, Lemma 2.2])**.** *Let $\varphi(r)$ be a non-decreasing, continuous function on $\mathbb{R}^+$. Suppose that*

$$\limsup_{r\to\infty} \frac{\log \varphi(r)}{\log r} > \alpha > 0$$

and set $G = \{r \in \mathbb{R}^+ : \varphi(r) \geq r^{\alpha}\}$. Then we have $\overline{\log dens}(G) > 0$.

3. Proof of Theorems

3.1. Proof of Theorem 1.3

The case $\rho(A) < \rho(B)$ has been proved by Gundersen [13, Theorem 2], thus we assume $\rho(A) > \rho(B)$. Suppose that there is a nontrivial solution f of (1) with finite order. Set $E^* = \{\theta \in [0, 2\pi) : h(\theta) \leq 0\}$. We divide into two cases on basis of $meas(E^*) = 0$ or $meas(E^*) > 0$.

Case 1. Assume that $meas(E^*) = 0$, then the indicator of $A(z)$ satisfies $h(\theta) > 0$ for every $\theta \in [0, 2\pi) \setminus E^*$. We give the details following the idea from [31] for the convenience of reading. By (4), we have

$$\log |A(re^{i\theta})| = h(\theta)r^{\rho(r)} + o(r^{\rho(r)})$$

for $z = re^{i\theta}$ satisfying $\theta \in [0, 2\pi) \setminus E^*$ and outside a C_0 set, where $\rho(r) \to \rho(A)$ as $r \to \infty$. Then for any given $\delta \in (0, \frac{\pi}{4\rho(A)})$ and $\eta \in (0, \frac{\rho(A)-\rho(B)}{4})$, we have

$$|A(z)| \geq \exp\{(1+o(1))\alpha|z|^{\rho(A)-\eta}\}, \tag{9}$$

$$|B(z)| \le \exp\{|z|^{\rho(B)+\eta}\} \le \exp\{|z|^{\rho(A)-2\eta}\} \le \exp\{o(1)|z|^{\rho(A)-\eta}\} \tag{10}$$

as $z = re^{i\theta}(\to \infty)$ satisfying $\theta \in [0, 2\pi) \setminus E^*$ and outside a C_0 set, where α is a positive constant depending on δ. Then by Lemma 2.6, there exist corresponding constants $b_j \neq 0$ such that

$$|f(z) - b_j| \le \exp\{-(1 + o(1))\alpha|z|^{\rho(A)-\eta}\} \tag{11}$$

as $z = re^{i\theta}(\to \infty)$ satisfying $\theta \in [0, 2\pi) \setminus E^*$ and outside a C_0 set. Then $f(z)$ is bounded in the whole complex plane by the Phragmén-Lindelöf principle. Then f is a constant in the complex plane by Liouville's theorem. Obviously, this is a contradiction.

Case 2. Assume $meas(E^*) > 0$, then there exist some sectors in which the indicator of $A(z)$ satisfying $h(\theta) < 0$. We can choose a ray $\arg z = \theta^*$ in these sectors such that $h(\theta^*) < 0$.

By Lemma 2.1, there exists a set $E \subset (1, \infty)$ with finite logarithmic measure such that for all z satisfying $|z| = r \notin E \cup [0, 1]$,

$$\left|\frac{f^{(k)}(z)}{f(z)}\right| \le |z|^{2\rho(f)},\ k = 1, 2. \tag{12}$$

Then, by Lemma 2.2, there exists a sequence $\{z_n = r_n e^{i\theta^*}\}$ satisfying $r_n \in G \setminus (E \cup [0, 1])$ with $r_n \to \infty$ as $n \to \infty$ such that

$$\begin{aligned} M(r_n, B)^d < L(r_n, B) \le |B(z_n)| &\le \left|\frac{f''(z_n)}{f(z_n)}\right| + |A(z_n)| \left|\frac{f'(z_n)}{f(z_n)}\right| \\ &\le (1 + o(1)) r_n^{2\rho(f)}, \end{aligned} \tag{13}$$

where $d \in (0, 1)$. Since $B(z)$ is a nonconstant entire function, by [34, Theorem 1.4] we have

$$dT(r_n, B) \le d \log M(r_n, B) \le 2\rho(f) \log r_n + o(1) \tag{14}$$

as r_n sufficiently large. Since $B(z)$ is transcendental, we have $\lim_{r_n\to\infty} \frac{T(r_n,B)}{\log r_n} = \infty$. Thus, we get a contradiction from (14).

Remark 3.1. The situation of Case 1 can really happen, see Theorem 1.5 and the following content in paper [6].

3.2. Proof of Theorem 1.4

As the similar arguments in Subsection 3.1, we only need to consider the situation $\rho(A) > \rho(B)$. Suppose that there is a nontrivial solution f of (1) with finite order. We treat two cases on basis of $meas(E^*) = 0$ or $meas(E^*) > 0$.

Case 1. Assume that $meas(E^*) = 0$, then the indicator of $A(z)$ satisfies $h(\theta) > 0$ for every $\theta \in [0, 2\pi) \setminus E^*$. The arguments are similar as Case 1 in Subsection 3.1.

Case 2. Assume $meas(E^*) > 0$, then there exist some sectors in which the indicator of $A(z)$ satisfying $h(\theta) < 0$. Hence, there must exist an interval $I_A \in [0, 2\pi)$ such that $h(\theta) < 0$ for all $\theta \in I_A$. By Lemma 2.1, there exists a

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set $E_1 \subset (1, \infty)$ with finite logarithmic measure such that for all z satisfying $|z| = r \notin E_1 \cup [0, 1]$, (12) holds. For given $0 < c < 1$, set

$$I_B(r) = \{\theta \in [0, 2\pi) : \log |B(re^{i\theta})| \leq c \log M(r, B)\} \tag{15}$$

and denote its Lebesgue measure by $meas(I_B(r))$. It follows from the definition of proximate function $m(r, B)$ that

$$\begin{aligned} T(r, B) &= m(r, B) \\ &\leq \left(\frac{2\pi - meas(I_B(r))}{2\pi}\right) \log M(r, B) + c\frac{meas(I_B(r))}{2\pi} \log M(r, B). \end{aligned} \tag{16}$$

Therefore, $T(r, B) \sim \log M(r, B)$ outside a set E_2 of finite linear measure implies that $meas(I_B(r)) \to 0$ as $r(\notin E_2) \to \infty$. Combining (4), (12) with (15), it leads to

$$\begin{aligned} M(r, B)^c \leq |B(re^{i\theta})| &\leq \left|\frac{f''(re^{i\theta})}{f(re^{i\theta})}\right| + |A(re^{i\theta})| \left|\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right| \\ &\leq (1 + o(1))r^{2\rho(f)} \end{aligned} \tag{17}$$

for $r(\notin E_1 \cup E_2 \cup [0, 1])$ sufficiently large and $\theta \in I_A \setminus I_B(r)$. Since $B(z)$ is transcendental, we obtain a contradiction.

3.3. Proof of Theorem 1.5

We recall some properties of entire functions that are extremal for Denjoy's conjecture and the definition of Borel direction as follows.

Definition 3.1. Let $B(z)$ be an entire (meromorphic) function in $\mathbb{C}$ with $0 < \mu(B) < \infty$. A ray $\arg z \in [0, 2\pi)$ from the origin is called a Borel direction of order $\geq \mu(B)$ of B, if for any positive number ε and for any complex number $a \in \mathbb{C} \cup \{\infty\}$, possibly with two exceptions, the following inequality holds

$$\limsup_{r\to\infty} \frac{\log n(S(\theta - \varepsilon, \theta + \varepsilon, r), a, B)}{\log r} \geq \mu(B),$$

where $n(S(\theta - \varepsilon, \theta + \varepsilon, r), a, B)$ denotes the number of zeros, counting the multiplicities, of $B - a$ in the region $S(\theta - \varepsilon, \theta + \varepsilon, r) = \{z : \theta - \varepsilon < \arg z < \theta + \varepsilon, |z| < r\}$.

The definition of Borel direction of order $\rho(B)$ is only need to replace $\geq \mu(B)$ with $= \rho(B)$, see [36, p. 78].

Lemma 3.2 ([36, Theorem 4.11])**.** *Let $B(z)$ be an entire function extremal for Denjoy's conjecture. Then, for any $\theta \in [0, 2\pi)$, either $\arg z = \theta$ is a Borel direction of order $\rho(B)$ of $B(z)$ or there exists a constant $\delta (0 < \delta < \pi/4)$, such that*

$$\lim_{|z|=r\to\infty} \frac{\log \log |B(z)|}{\log r} = \rho(B) \tag{18}$$

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for all $z \in S(\theta-\delta,\theta+\delta)\setminus E$, E *denotes a subset of* $S(\theta-\delta,\theta+\delta)$, *and satisfies*

$$\lim_{r\to\infty} meas(S(\theta-\delta,\theta+\delta;r,\infty)\cap E)=0,$$

where $S(\theta-\delta,\theta+\delta;r,\infty)=\{z:\theta-\delta<\arg z<\theta+\delta, 0<|r|<+\infty\}$.

Lemma 3.3 ([35, Lemma 1])**.** *Let* $B(z)$ *be an entire function of order* $\rho(B)\in(0,\infty)$, *and let* $S(\varphi_1,\varphi_2)=\{z:\varphi_1<\arg z<\varphi_2\}$ *be a sector with* $\varphi_2-\varphi_1<\frac{\pi}{\rho(B)}$. *If there exists a Borel direction of order* $\rho(B)$ *of* $B(z)$ *in* $S(\varphi_1,\varphi_2)$, *then for at least one of the two rays* $L_j=\{z:\arg z=\varphi_j, j=1,2\}$, *say* L_2, *we have*

$$\limsup_{r\to\infty}\frac{\log\log|B(re^{i\varphi_2})|}{\log r}=\rho(B). \tag{19}$$

Proof. As the similar arguments in Subsection 3.1, we only need to consider the situation $\rho(A)>\rho(B)$. Suppose that there is a nontrivial solution f of (1) with finite order. We treat two cases.

Case 1. Assume that $meas(E^*)=0$, then the indicator of $A(z)$ satisfies $h(\theta)>0$ for every $\theta\in[0,2\pi)\setminus E^*$. The arguments are similar as Case 1 in Subsection 3.1.

Case 2. Assume $meas(E^*)>0$, then there exist some sectors in which the indicator of $A(z)$ satisfying $h(\theta)<0$. Hence, there must exist an interval $I_A\in[0,2\pi)$ such that $h(\theta)<0$ for all $\theta\in I_A$. We choose a ray $\arg z=\theta^*$ such that $\theta^*\in I_A$. By Lemma 2.1, there exists a set $E_1\subset(1,\infty)$ with finite logarithmic measure such that for all z satisfying $|z|=r\notin E_1\cup[0,1]$, (12) holds.

Subcase 2.1. Suppose the ray $\arg z=\theta^*$ is a Borel direction of order $\rho(B)$ of $B(z)$. Give η sufficiently small such that the interval $(\theta^*-\eta,\theta^*+\eta)\subset I_A$ and $2\eta<\frac{\pi}{\rho(A)}$. Choose $\varphi_1\in(\theta^*-\eta,\theta^*)$ and $\varphi_2\in(\theta^*,\theta^*+\eta)$, then $\varphi_2-\varphi_1<\frac{\pi}{\rho(A)}$. By Lemma 3.3, at least one of two rays $L_1:\arg z=\varphi_1$ and $L_2:\arg z=\varphi_2$, say L_2, satisfies (19). Combining (4) with (12), it leads to

$$|B(re^{i\varphi_2})|\le\left|\frac{f''(re^{i\varphi_2})}{f(re^{i\varphi_2})}\right|+|A(re^{i\varphi_2})|\left|\frac{f'(re^{i\varphi_2})}{f(re^{i\varphi_2})}\right|\le(1+o(1))r^{2\rho(f)} \tag{20}$$

for $r\notin E_1$ sufficiently large. In view of Lemma 2.7 and (19), there exists a set $G\subset\mathbb{R}$ with infinite logarithmic measure such that $|B(re^{i\theta})|>\exp\{r^{\rho(B)-\epsilon}\}$ for $r\in G,\theta\in[0,2\pi)$ and sufficiently small ϵ. Combining this with (20), we obtain

$$\exp\{r^{\rho(B)-\epsilon}\}<|B(re^{i\varphi_2})|\le(1+o(1))r^{2\rho(f)} \tag{21}$$

for $r\in G\setminus E_1$. This is impossible.

Subcase 2.2. Suppose the ray $\arg z=\theta^*$ is not a Borel direction of order $\rho(B)$ of $B(z)$. Take a sufficiently small positive constant δ such that $(\theta^*-\delta,\theta^*+\delta)\subset I_A$. By Lemma 3.2, we have

$$\lim_{|z|=r\to\infty}\frac{\log\log|B(z)|}{\log r}=\rho(B) \tag{22}$$

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for all $z \in S(\theta^* - \delta, \theta^* + \delta) \setminus E$, E denotes a subset of $S(\theta - \delta, \theta + \delta)$, and satisfies $\lim_{r\to\infty} meas(S(\theta - \delta, \theta + \delta; r, \infty) \cap E) = 0$. Similar as in Subcase 2.1 we have

$$\exp\{r^{\rho(B)-\epsilon}\} < |B(re^{i\theta})| \leq (1 + o(1))r^{2\rho(f)} \tag{23}$$

for $z = re^{i\theta} \in S(\theta^* - \delta, \theta^* + \delta) \setminus E$ and $r \in G \setminus E_1$. This is also impossible. Thus, we complete the proof. □

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