

UNIFIED APOSTOL-KOROBOV TYPE POLYNOMIALS AND RELATED POLYNOMIALS

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ABSTRACT. Korobov type polynomials are introduced and extensively investigated many mathematicians ([1, 8–10, 12–14]). In this work, we define unified Apostol Korobov type polynomials and give some recurrences relations for these polynomials. Further, we consider the q -poly Korobov polynomials and the q -poly-Korobov type Changhee polynomials. We give some explicit relations and identities above mentioned functions.

1. Introduction

As usual, throughout this paper, \mathbb{N} denotes the set natural numbers, \mathbb{N}_0 denotes the set of nonnegative integers, \mathbb{Z} denotes the set of integer numbers, \mathbb{R} denotes the set of real numbers.

The generalized Apostol-Bernoulli polynomials $\mathcal{B}_n^{(\alpha)}(x; \lambda)$ of order α and the generalized Apostol-Euler polynomials $\mathcal{E}_n^{(\alpha)}(x; \lambda)$ of order α are defined by the following generating functions (see, for detail [16–18])

$$(1) \quad \sum_{n=0}^{\infty} \mathcal{B}_n^{(\alpha)}(x; \lambda) \frac{t^n}{n!} = \left(\frac{t}{\lambda e^t - 1} \right)^{\alpha} e^{xt},$$

$$(|t| < 2\pi \text{ when } \lambda = 1, |t| < |\log \lambda| \text{ when } \lambda \neq 1)$$

and

$$(2) \quad \sum_{n=0}^{\infty} \mathcal{E}_n^{(\alpha)}(x; \lambda) \frac{t^n}{n!} = \left(\frac{2}{\lambda e^t + 1} \right)^{\alpha} e^{xt},$$

$$(|t| < \pi \text{ when } \lambda = 1, |t| < |\log(-\lambda)| \text{ when } \lambda \neq 1).$$

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For $\lambda = 1$, we get the Bernoulli polynomials $\mathcal{B}_n^{(\alpha)}(x)$ of order α and the Euler polynomials $\mathcal{E}_n^{(\alpha)}(x)$ of order α .

The Stirling numbers of the second kind $S_2(n, k)$ are defined ([7], [11]) and weight Stirling numbers of the second kind $S_2(n, m, x)$ [4] are defined the following generating functions, respectively

$$(3) \quad \sum_{n=0}^{\infty} S_2(n, k) \frac{t^n}{n!} = \frac{(e^t - 1)^k}{k!}$$

and

$$(4) \quad \sum_{n=0}^{\infty} S_2(n, m, x) \frac{t^n}{n!} = \frac{e^{xt} (e^t - 1)^m}{m!}.$$

The polylogarithm function $Li_k(z)$ ([2], [4], [6, 7, 11]) is defined by

$$(5) \quad Li_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k}, \quad k \in \mathbb{Z}, \quad k \geq 1.$$

This function is convergent for $|z| < 1$, when $k = 1$

$$(6) \quad Li_1(z) = -\log(1 - z).$$

The multi-logarithm ([7], [11]) is defined by

$$(7) \quad Li_{k_1, \dots, k_n}(z) = \sum_{0 < m_1 < \dots < m_n} \frac{z^{m_n}}{m_1^{k_1} \dots m_n^{k_n}}, \quad k_i \geq 1, \quad |z| < 1.$$

From (6), the following equation can be obtain

$$(8) \quad \underbrace{Li_{\underbrace{1, \dots, 1}_{n\text{-times}}}}(z) = \frac{1}{n!} (-\log(1 - z))^n.$$

Kim et al. [8] defined the poly-Bernoulli polynomials

$$(9) \quad \sum_{n=0}^{\infty} B_n^{(k)}(x) \frac{t^n}{n!} = \frac{Li_k(1 - e^{-t})}{e^t - 1} e^{xt},$$

when $k = 1$, $B_n^{(1)}(x) = B_n(x)$.

Hamahata in [5] defined by the poly-Euler polynomials

$$(10) \quad \sum_{n=0}^{\infty} E_n^{(k)}(x) \frac{t^n}{n!} = \frac{2Li_k(1 - e^{-t})}{t(e^t + 1)} e^{xt},$$

when $k = 1$, $E_n^{(1)}(x) = E_n(x)$.

Kim et al. ([9], [10]) defined the Changhee polynomials and the first kind Korobov polynomials the following generating functions, respectively,

$$(11) \quad \sum_{n=0}^{\infty} Ch_n(x) \frac{t^n}{n!} = \frac{2}{t+2} (1+t)^x$$

and

$$(12) \quad \sum_{n=0}^{\infty} K_n(x | \lambda) \frac{t^n}{n!} = \frac{\lambda t}{(t+1)^\lambda - 1} (1+t)^x$$

when $x = 0$, $Ch_n(0) = Ch_n$ are called the Changhee numbers and $K_n(0 | \lambda) = K_n(\lambda)$ are called the Korobov numbers with $\lambda \in \mathbb{R}$.

The Korobov-type Changhee polynomials [9] are defined the following generating function with $\lambda \in \mathbb{R}$

$$(13) \quad \sum_{n=0}^{\infty} Ch_n(x | \lambda) \frac{t^n}{n!} = \frac{2}{(t+1)^\lambda + 1} (1+t)^x.$$

Note that

$$\lim_{\lambda \rightarrow 1} Ch_n(x | \lambda) = Ch_n(x) \text{ and } \lim_{\lambda \rightarrow 0} Ch_n(x | \lambda) = (x)_n,$$

where

$$(x)_n = x(x-1) \cdots (x-(n-1)),$$

$\lambda \in \mathbb{R}$, Carlitz [3] introduced the degenerate Bernoulli polynomials by means of the following generating functions

$$(14) \quad \sum_{n=0}^{\infty} \mathcal{B}_n(x | \lambda) \frac{t^n}{n!} = \frac{t}{(\lambda t + 1)^{1/\lambda} - 1} (1 + \lambda t)^{x/\lambda}$$

so that

$$\mathcal{B}_n(x | \lambda) = \sum_{m=0}^n \binom{n}{m} \mathcal{B}_m(\lambda) \left(\frac{x}{\lambda}\right)_{n-m} \lambda^{n-m}.$$

From (14), we note that

$$\begin{aligned} \sum_{n=0}^{\infty} \lim_{\lambda \rightarrow 0} \mathcal{B}_n(x | \lambda) \frac{t^n}{n!} &= \lim_{\lambda \rightarrow 0} \frac{t}{(\lambda t + 1)^{1/\lambda} - 1} (1 + \lambda t)^{x/\lambda} \\ &= \frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \end{aligned}$$

where $B_n(x)$ are Bernoulli polynomials.

Ozarslan [1] defined unified form of the Apostol-Bernoulli, Euler and Genocchi polynomials

$$(15) \quad \sum_{n=0}^{\infty} P_{n,\beta}^{(\alpha)}(x; k, a, b) \frac{t^n}{n!} = \left(\frac{2^{1-kt} t^k}{\beta^b e^t - a^b} \right)^\alpha e^{xt},$$

$$k \in \mathbb{Z}, a, b \in \mathbb{R} \setminus \{0\}, \alpha, \beta \in \mathbb{C}.$$

2. Unified Apostol-Korobov type polynomials

In this section, we define unified Apostol-Korobov type polynomials. We investigate these polynomials and give some explicit identities and relations for these polynomials.

Firstly, we define the Apostol-Korobov polynomials $K_{n,\alpha}(x : \lambda)$ and the Apostol-Korobov type Changhee polynomials are defined the following generating functions, respectively

$$(16) \quad \sum_{n=0}^{\infty} K_{n,\alpha}(x : \lambda) \frac{t^n}{n!} = \frac{\lambda t}{\alpha(1+t)^\lambda - 1} (1+t)^x$$

and

$$(17) \quad \sum_{n=0}^{\infty} Ch_{n,\alpha}(x : \lambda) \frac{t^n}{n!} = \frac{2}{\alpha(1+t)^\lambda + 1} (1+t)^x,$$

where $\alpha \in \mathbb{N}_0$.

We consider the following unified form of the Apostol-Korobov polynomials

$$(18) \quad \sum_{n=0}^{\infty} \mathfrak{R}_{n,\beta}(x : k, a, b, \lambda) \frac{t^n}{n!} = \frac{2^{1-k} (\lambda t)^k}{\beta^b (1+t)^\lambda - a^b} (1+t)^x.$$

Remark 2.1. Setting $k = a = b = 1$ and $\alpha = \beta$ in (18), we get

$$\mathfrak{R}_{n,\alpha}(x : k, 1, 1, \lambda) = K_{n,\alpha}(x : \lambda).$$

Remark 2.2. Setting $k = 0, a = -1, b = 1$ and $\alpha = \beta$ in (18), we get

$$\mathfrak{R}_{n,\alpha}(x : 0, -1, 1, \lambda) = Ch_{n,\alpha}(x : \lambda).$$

From (18), we get the following relations

$$(i) \quad \mathfrak{R}_{n,\beta}(x : k, a, b, \lambda) = \sum_{m=0}^n \binom{n}{m} \mathfrak{R}_{m,\beta}(0 : k, a, b, \lambda) (x)_{n-m},$$

$$(ii) \quad \mathfrak{R}_{n,\beta}(x + y : k, a, b, \lambda) = \sum_{m=0}^n \binom{n}{m} \mathfrak{R}_{m,\beta}(x : k, a, b, \lambda) (y)_{n-m}$$

and

$$(iii) \quad \mathfrak{R}_{n,\beta}(x + y : k, a, b, \lambda) = \sum_{m=0}^n \binom{n}{m} \mathfrak{R}_{m,\beta}(0 : k, a, b, \lambda) (x + y)_{n-m}.$$

Theorem 2.3. *The following relation holds true:*

$$(19) \quad \beta^b \sum_{m=0}^l \frac{1}{m! (l-m)!} \mathfrak{R}_{m,\beta}(x : k, a, b, \lambda) (\lambda)_{l-m} - \frac{a^b}{l!} \mathfrak{R}_{l,\beta}(x : k, a, b, \lambda) \\ = 2^{1-k} \lambda^k \frac{(x)_{l-k}}{(l-k)!}.$$

Proof. By using (18), we write by

$$\sum_{n=0}^{\infty} \mathfrak{R}_{n,\beta}(x : k, a, b, \lambda) \frac{t^n}{n!} \left(\beta^b (1+t)^\lambda - a^b \right) = 2^{1-k} (\lambda t)^k (1+t)^x$$

and

$$\begin{aligned} & \sum_{l=0}^{\infty} \left(\beta^b \sum_{m=0}^l \binom{l}{m} \mathfrak{R}_{m,\beta}(x : k, a, b, \lambda) (\lambda)_{l-m} - a^b \mathfrak{R}_{l,\beta}(x : k, a, b, \lambda) \right) \frac{t^l}{l!} \\ &= 2^{1-k} \lambda^k \sum_{l=0}^{\infty} (x)_l \frac{t^{l+k}}{l!} = 2^{1-k} \lambda^k \sum_{l=k}^{\infty} (x)_{l-k} \frac{1}{(l-k)!} t^l. \end{aligned}$$

Comparing the coefficients of t^m on both sides, we have (19). \square

Theorem 2.4. *There is the following relation between unified Apostol-Korobov polynomials and the weighted Stirling numbers of second kind $S(n, m, x)$ and unified Apostol-Bernoulli, Euler and Genocchi polynomials $\mathfrak{R}_{n,\beta}(x : k, a, b, \lambda)$ as*

$$\begin{aligned} & \sum_{m=0}^{n-k} \frac{\mathfrak{R}_{m,\beta}(x : k, a, b, \lambda) S_2(n-k, m)}{(n-k)!} \\ (20) \quad &= \sum_{i=0}^n \frac{1}{n! (n-i)!} \mathfrak{R}_{n-i,\beta}(0 : k, a, b) \lambda^m k! S_2(i, k, x). \end{aligned}$$

Proof. By replacing t by $e^t - 1$ in (18), we get

$$\begin{aligned} & \sum_{m=0}^{\infty} \mathfrak{R}_{m,\beta}(x : k, a, b, \lambda) \frac{(e^t - 1)^m}{m!} = \frac{2^{1-k} \lambda^k (e^t - 1)^k e^{xt}}{\beta^b e^{t\lambda} - a^b}, \\ & \sum_{m=0}^{\infty} \mathfrak{R}_{m,\beta}(x : k, a, b, \lambda) \sum_{n=m}^{\infty} S_2(n, m) \frac{t^n}{n!} = \frac{2^{1-k} (\lambda t)^k t^{-k}}{\beta^b e^{t\lambda} - a^b} k! \frac{(e^t - 1) e^{xt}}{k!}. \end{aligned}$$

By using (4) and (15) in the last equations, we have

$$\begin{aligned} & \sum_{n=0}^{\infty} \sum_{m=0}^n \mathfrak{R}_{m,\beta}(x : k, a, b, \lambda) S_2(n, m) \frac{t^n}{n!} \\ &= t^{-k} \sum_{n=0}^{\infty} \sum_{i=0}^n \binom{n}{i} \mathfrak{R}_{n-i,\beta}(0 : k, a, b) \lambda^m k! S_2(i, k, x) \frac{t^n}{n!}. \end{aligned}$$

From here, comparing the coefficients of t^m , we have (20). \square

From $\lim_{\mu \rightarrow 0} (1 + \mu t)^{1/\mu} = e^t$, we consider the degenerate function of t which are given by

$$t = \lim_{\mu \rightarrow 0} \log(1 + \mu t)^{1/\mu}.$$

$\frac{\log(1+\mu t)}{\mu}$ is called the degenerate function of t . Now, we consider the degenerate unified Apostol-Korobov type polynomials by

$$(21) \quad \begin{aligned} & \sum_{n=0}^{\infty} \mathfrak{R}_{n,\beta,\mu}(x : k, a, b, \lambda) \frac{t^n}{n!} \\ &= \frac{2^{1-k} \lambda^k \left(\log(1 + \mu t)^{1/\mu} \right)^k}{\beta^b \left[1 + \log(1 + \mu t)^{1/\mu} \right]^\lambda - a^b} \left[1 + \log(1 + \mu t)^{1/\mu} \right]^x, \end{aligned}$$

where $\mu \in \mathbb{R}^+$.

Theorem 2.5. *There is the following relation between the degenerate unified Apostol-Korobov type polynomials and weighted Stirling numbers of second kind $S(n, m, x)$*

$$(22) \quad \begin{aligned} & \sum_{p=0}^{j-k} \mu^p l^p S_2(j-k, p) \sum_{m=0}^{\infty} \mathfrak{R}_{m,\beta,\mu}(x : k, a, b, \lambda) \frac{\lambda^{-m}}{m! (j-k)!} \sum_{l=0}^m \binom{m}{l} (-1)^{m-l} \\ &= \sum_{i=0}^j \binom{j}{i} \mathfrak{R}_{j-i,\beta}(0 : k, a, b) \lambda^{j-i} S_2(i, k, x). \end{aligned}$$

Proof. By using t by $\frac{e^{\mu(e^t-1)}-1}{\mu}$ in (21), we get

$$(23) \quad \sum_{m=0}^{\infty} \mathfrak{R}_{m,\beta,\mu}(x : k, a, b, \lambda) \mu^{-m} \frac{\left(e^{\mu(e^t-1)} - 1 \right)^m}{m!} = \frac{2^{1-k} \lambda^k (e^t - 1)^k e^{tx}}{\beta^b e^{t\lambda} - a^b}.$$

The left side of the equation (23),

$$(24) \quad \begin{aligned} & \sum_{m=0}^{\infty} \mathfrak{R}_{m,\beta,\mu}(x : k, a, b, \lambda) \frac{\mu^{-m}}{m!} \sum_{l=0}^m \binom{m}{l} (-1)^{m-l} e^{\mu l (e^t - 1)} \\ &= \sum_{m=0}^{\infty} \mathfrak{R}_{m,\beta,\mu}(x : k, a, b, \lambda) \frac{\mu^{-m}}{m!} \sum_{l=0}^m \binom{m}{l} (-1)^{m-l} \sum_{p=0}^{\infty} \mu^p l^p \frac{(e^t - 1)^p}{p!} \\ &= \sum_{m=0}^{\infty} \mathfrak{R}_{m,\beta,\mu}(x : k, a, b, \lambda) \frac{\mu^{-m}}{m!} \sum_{l=0}^m \binom{m}{l} (-1)^{m-l} \sum_{p=0}^{\infty} \mu^p l^p \sum_{j=p}^{\infty} S_2(j, p) \frac{t^j}{j!} \\ &= \sum_{j=0}^{\infty} \left(\sum_{p=0}^j \mu^p l^p S_2(j, p) \sum_{m=0}^{\infty} \mathfrak{R}_{m,\beta,\mu}(x : k, a, b, \lambda) \frac{\mu^{-m}}{m!} \sum_{l=0}^m \binom{m}{l} (-1)^{m-l} \right) \frac{t^j}{j!}. \end{aligned}$$

By using (4) and (15) in right side of the equation (23)

$$(25) \quad = t^{-k} \sum_{n=0}^{\infty} \sum_{i=0}^n \binom{n}{i} \mathfrak{R}_{j-i,\beta}(0 : k, a, b) \lambda^{n-i} S_2(i, k, x) \frac{t^n}{n!}.$$

From (24) and (25), comparing the coefficients of t^n , we have (22). \square

3. On the q -poly-Korobov polynomials and the q -poly-Korobov type Changhee polynomials

In this section, we introduce and investigate some properties and explicit relations between the q -poly-Korobov polynomials and the q -poly-Korobov type Changhee polynomials.

We define the q -poly-Korobov polynomials and the q -poly-Korobov type Changhee polynomials the following generating functions, respectively;

$$(26) \quad \sum_{n=0}^{\infty} K_{n,q}^{(k)}(x | \lambda) \frac{t^n}{n!} = \frac{\lambda Li_{[k,q]}(1 - e^{-t})}{(t+1)^\lambda - 1} (1+t)^x$$

and

$$(27) \quad \sum_{n=0}^{\infty} Ch_{n,q}^{(k)}(x | \lambda) \frac{t^n}{n!} = \frac{2Li_{[k,q]}(1 - e^{-t})}{t((t+1)^\lambda + 1)} (1+t)^x,$$

where $q \in \mathbb{R}$, $0 < q \leq 1$ and the poly-logarithm function is defined as

$$(28) \quad Li_{[k,q]}(t) = \sum_{n=1}^{\infty} \frac{t^n}{[n]_q^k}.$$

The polynomials $K_{n,q}^{(k)}(\lambda) := K_{n,q}^{(k)}(0 | \lambda)$ are called the q -poly-Korobov numbers and the polynomial $Ch_{n,q}^{(k)}(\lambda) = Ch_{n,q}^{(k)}(0 | \lambda)$ are called the q -poly-Korobov type Changhee numbers.

The q -numbers and q -factorial are defined by

$$[n]_q = \frac{1 - q^n}{1 - q}, \quad q \neq 1, \quad [n]_q! = [n]_q [n-1]_q \cdots [1]_q,$$

$n \in \mathbb{N}$ and $q \in \mathbb{C}$, respectively where $[0]_q! = 1$.

The first values of the q -polylogarithm function for $k \leq 0$,

$$Li_{[0,q]}(t) = \frac{t}{1-t}, \quad Li_{[-1,q]}(t) = \frac{t}{(1-t)(1-qt)}.$$

The q -polylogarithm function for $k \leq 0$ is a rational functions. For k is a nonnegative integer

$$Li_{[-k,q]}(t) = \frac{1}{(1-q)^k} \sum_{l=0}^k \binom{k}{l} (-1)^l \frac{q^l t}{1 - q^l t}.$$

For $n = 2$ in (8), we get $Li_{[1,1]}(1 - e^{-t}) = \frac{t^2}{2!}$.

From (26) and (27), for $k = q = 1$, we have

$$K_{l,1}^{(1)}(x | \lambda) = \frac{1}{2} l K_{l-1}^{(k)}(x | \lambda) \text{ and } Ch_{l,1}^{(1)}(x | \lambda) = \frac{1}{2} l Ch_{l-1}^{(k)}(x | \lambda), \quad l \geq 1.$$

Theorem 3.1. *The following relations holds true:*

$$K_{n,q}^{(k)}(x | \lambda) = \sum_{m=0}^n \binom{n}{m} K_{m,q}^{(k)}(\lambda)(x)_{n-m},$$

$$(i) \quad Ch_{n,q}^{(k)}(x | \lambda) = \sum_{m=0}^n \binom{n}{m} Ch_{m,q}^{(k)}(\lambda)(x)_{n-m},$$

$$K_{n,q}^{(k)}(x + y | \lambda) = \sum_{m=0}^n \binom{n}{m} K_{m,q}^{(k)}(x | \lambda)(y)_{n-m}$$

and

$$(ii) \quad Ch_{n,q}^{(k)}(x + y | \lambda) = \sum_{m=0}^n \binom{n}{m} Ch_{m,q}^{(k)}(x | \lambda)(y)_{n-m}.$$

Theorem 3.2. *There are the following relationships for the q -poly-Korobov polynomials and the q -poly-Korobov type Changhee polynomials as*

$$(29) \quad \begin{aligned} & K_{m,q}^{(k)}(x + \lambda | \lambda) - K_{m,q}^{(k)}(x | \lambda) \\ &= \lambda \sum_{j=0}^m \binom{m}{j} \sum_{n=0}^{j-1} \frac{(-1)^{n+1+j} (n+1)!}{[n+1]_q^k} S_2(j, n+1)(x)_{m-j} \end{aligned}$$

and

$$(30) \quad \begin{aligned} & m \left(Ch_{m-1,q}^{(k)}(x + \lambda | \lambda) + Ch_{m-1,q}^{(k)}(x | \lambda) \right) \\ &= 2 \sum_{j=0}^m \binom{m}{j} \sum_{n=0}^{j-1} \frac{(-1)^{n+1+j} (n+1)!}{[n+1]_q^k} S_2(j, n+1)(x)_{m-j}. \end{aligned}$$

Proof. By using (3), (26) and (28), we write as

$$\begin{aligned} & \sum_{m=0}^{\infty} \left(K_{m,q}^{(k)}(x + \lambda | \lambda) - K_{m,q}^{(k)}(x | \lambda) \right) \frac{t^m}{m!} \\ &= \lambda Li_{k,q}(1 - e^{-t})(1+t)^x \\ &= \lambda \sum_{n=0}^{\infty} \frac{(-1)^{n+1} (n+1)! (e^{-t} - 1)^{n+1}}{[n+1]_q^k (n+1)!} (1+t)^x \\ &= \lambda \sum_{n=0}^{\infty} \frac{(-1)^{n+1} (n+1)!}{[n+1]_q^k} \sum_{j=n+1}^{\infty} S_2(j, n+1) \frac{(-t)^j}{j!} \sum_{l=0}^{\infty} (x)_l \frac{t^l}{l!} \\ &= \lambda \sum_{m=0}^{\infty} \left(\sum_{j=0}^m \binom{m}{j} \sum_{n=0}^{j-1} \frac{(-1)^{n+1+j} (n+1)!}{[n+1]_q^k} S_2(j, n+1)(x)_{m-j} \right) \frac{t^m}{m!}. \end{aligned}$$

Comparing the coefficients of both sides, we have (29).

Since the proof of (30) is similar to (29), we omit it. \square

From (29) and (30), we have the following Corollary 3.3.

Corollary 3.3. *There is the following relations between the q -poly-Korobov polynomials and the q -poly-Korobov type Changhee polynomials*

$$\begin{aligned} & 2 \left[K_{m,q}^{(k)}(x + \lambda | \lambda) - K_{m,q}^{(k)}(x | \lambda) \right] \\ &= \lambda m \left[Ch_{m-1,q}^{(k)}(x + \lambda | \lambda) + Ch_{m-1,q}^{(k)}(x | \lambda) \right]. \end{aligned}$$

Theorem 3.4. *The following relation for the q -poly-Korobov polynomial holds true:*

$$\begin{aligned} & \left(\sum_{m=0}^n \binom{n}{m} K_{m,q}^{(k)}(x | \lambda) (x)_{n-m} - K_{n,q}^{(k)}(x | \lambda) \right) \\ (31) \quad &= \lambda \sum_{j=0}^m \binom{m}{j} \sum_{n=0}^{j-1} \frac{(-1)^{n+1+j} (n+1)!}{[n+1]_q^k} S_2(j, n+1) (x)_{m-j}. \end{aligned}$$

Proof. By using (3) and (26), we write as

$$\sum_{n=0}^{\infty} K_{n,q}^{(k)}(x | \lambda) \frac{t^n}{n!} \left((1+t)^\lambda - 1 \right) = \lambda Li_{k,q}(1 - e^{-t}) (1+t)^x$$

and

$$\begin{aligned} & \sum_{m=0}^{\infty} K_{m,q}^{(k)}(x | \lambda) \frac{t^m}{m!} \sum_{l=0}^{\infty} (x)_l \frac{t^l}{l!} - \sum_{n=0}^{\infty} K_{n,q}^{(k)}(x | \lambda) \frac{t^n}{n!} \\ &= \lambda \sum_{m=0}^{\infty} \left(\sum_{j=0}^m \binom{m}{j} \sum_{n=0}^{j-1} \frac{(-1)^{n+1+j} (n+1)!}{[n+1]_q^k} S_2(j, n+1) (x)_{m-j} \right) \frac{t^m}{m!}. \end{aligned}$$

By using Cauchy product and comparing the coefficients of $\frac{t^n}{n!}$, we have (31). \square

From (29) and (31), we have the following Corollary 3.5.

Corollary 3.5. *There are another relations for the q -poly-Korobov polynomials as*

$$\begin{aligned} & \left(K_{m,q}^{(k)}(x + \lambda | \lambda) - K_{m,q}^{(k)}(x | \lambda) \right) \\ &= \sum_{n=0}^m \binom{m}{n} K_{n,q}^{(k)}(x | \lambda) (\lambda)_{m-n} - K_{m,q}^{(k)}(x | \lambda). \end{aligned}$$

Theorem 3.6. *The following relation holds true:*

$$\left(\sum_{n=0}^m n \binom{m}{n} Ch_{n-1,q}^{(k)}(x | \lambda) (x)_{m-n} + m Ch_{m-1,q}^{(k)}(x | \lambda) \right)$$

$$(32) \quad = 2 \sum_{j=0}^m \binom{m}{j} \sum_{n=0}^{j-1} \frac{(-1)^{n+1+j} (n+1)!}{[n+1]_q^k} S_2(j, n+1)(x)_{m-j}.$$

From (30) and (32), we have the following Corollary.

Corollary 3.7. *There is the another relation for the q -poly-Korobov type Changhee polynomial:*

$$\begin{aligned} & m \left(Ch_{m-1,q}^{(k)}(x + \lambda | \lambda) + Ch_{m-1,q}^{(k)}(x | \lambda) \right) \\ &= \sum_{n=0}^m n \binom{m}{n} Ch_{n-1,q}^{(k)}(x | \lambda) (x)_{m-n} + m Ch_{m-1,q}^{(k)}(x | \lambda). \end{aligned}$$

Theorem 3.8. *There is the following relation between the q -poly-Korobov polynomials and the Bernoulli polynomials*

$$(33) \quad \sum_{n=0}^{\infty} K_{n,q}^{(k)}(x | \lambda) S_2(m, n) (-1)^m = \sum_{l=0}^m \binom{m}{l} \frac{\lambda^{m-l} (-1)^l l!}{[l+1]_q^k} B_{m-l} \left(\frac{x}{\lambda} \right).$$

Proof. Replacing t by $e^{-t} - 1$ in (26), we get

$$\sum_{n=0}^{\infty} K_{n,q}^{(k)}(x | \lambda) \frac{(e^{-t} - 1)^n}{n!} = \frac{\lambda e^{-tx}}{e^{-t\lambda} - 1} Li_{k,q}(-t)$$

and

$$\begin{aligned} & \sum_{n=0}^{\infty} K_{n,q}^{(k)}(x | \lambda) \sum_{m=n}^{\infty} S_2(m, n) (-1)^m \frac{t^m}{m!} \\ &= - \sum_{m=0}^{\infty} B_m \left(\frac{x}{\lambda} \right) (-\lambda)^m \frac{t^m}{m!} \sum_{l=0}^{\infty} \frac{(-1)^{l+1} l! t^l}{[l+1]_q^k l!}. \end{aligned}$$

By using Cauchy product and comparing the coefficients, we have (33). \square

Theorem 3.9. *There is the following relations between the q -poly-Korobov type Changhee polynomials and the Euler polynomials:*

$$(34) \quad \begin{aligned} & \sum_{n=0}^m n Ch_{n-1,q}^{(k)}(x | \lambda) S_2(m, n) (-1)^m \\ &= -m \sum_{l=0}^{m-1} \binom{m-1}{l} \frac{(-\lambda)^{m-l-1} l! (-1)^l}{[l+1]_q^k} E_{m-1-l} \left(\frac{x}{\lambda} \right). \end{aligned}$$

Proof. Using t by $e^{-t} - 1$ in (27), we get

$$\begin{aligned} & \sum_{n=0}^{\infty} Ch_{n,q}^{(k)}(x | \lambda) \frac{(e^{-t} - 1)^{n+1}}{n!} = \frac{2e^{-tx}}{e^{-t\lambda} + 1} Li_{k,q}(-t), \\ & \sum_{n=0}^{\infty} n Ch_{n-1,q}^{(k)}(x | \lambda) \sum_{m=n}^{\infty} S_2(m, n) (-1)^m \frac{t^m}{m!} \end{aligned}$$

$$= -t \sum_{r=0}^{\infty} E_r \left(\frac{x}{\lambda} \right) (-\lambda)^r \frac{t^r}{r!} \sum_{l=0}^{\infty} \frac{l! (-1)^l t^l}{[l+1]_q^k l!}$$

and

$$\begin{aligned} & \sum_{m=0}^{\infty} \sum_{n=0}^m n Ch_{n-1,q}^{(k)}(x | \lambda) S_2(m, n) (-1)^m \frac{t^m}{m!} \\ &= - \sum_{m=0}^{\infty} m \sum_{l=0}^{m-1} \binom{m-1}{l} E_{m-1-l} \left(\frac{x}{\lambda} \right) (-\lambda)^{m-l-1} \frac{l! (-1)^l t^m}{[l+1]_q^k m!}. \end{aligned}$$

Comparing the coefficients, we have (34). \square

4. Conclusion

In recent years, many mathematicians studied Korobov polynomials and Korobov type Changhee polynomials. The classical Bernoulli polynomials and Euler polynomials are studied many mathematicians ([1–18]). Srivastava [16], Srivastava et al. in ([17, 18]) introduced and investigated basic properties of these polynomials. Carlitz ([3], [4]) introduced degenerate Bernoulli polynomials and weighted degenerate Bernoulli polynomials. Bayad et al. [2] and Hamahata [4], Imatomi et al. [7], Kim et al. ([8–10, 12]) considered and investigated poly-Bernoulli and poly-Euler polynomials. Kruchinin [14] introduced Korobov polynomials. Kim et al. [12] introduced Korobov type polynomials and Korobov-type Changhee polynomials.

In this work, we define the unified form of the Apostol-Korobov, Apostol-Korobov type Changhee polynomials. We give explicit relations for these polynomials. Further, we define the q -poly-Korobov polynomials and the q -poly-Korobov type Changhee polynomials. We prove some relations between these polynomials and the Bernoulli polynomials and the Euler polynomials.

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