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ON JORDAN AND JORDAN HIGHER DERIVABLE MAPS OF RINGS

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Abstract. Let $\mathcal{R}$ be a 2-torsion free unital ring containing a non-trivial idempotent. An additive map δ from $\mathcal{R}$ into itself is called a Jordan derivable map at commutative zero point if $\delta(AB + BA) = \delta(A)B + B\delta(A) + A\delta(B) + \delta(B)A$ for all $A, B \in \mathcal{R}$ with $AB = BA = 0$. In this paper, we prove that, under some mild conditions, each Jordan derivable map at commutative zero point has the form $\delta(A) = \psi(A) + CA$ for all $A \in \mathcal{R}$, where ψ is an additive Jordan derivation of $\mathcal{R}$ and C is a central element of $\mathcal{R}$. Then we generalize the result to the case of Jordan higher derivable maps at commutative zero point. These results are also applied to some operator algebras.

1. Introduction

Let $\mathcal{R}$ be a ring (or an algebra). Recall that an additive (or a linear) map δ from $\mathcal{R}$ into itself is called an additive (or a linear) derivation if $\delta(AB) = \delta(A)B + A\delta(B)$ for all $A, B \in \mathcal{R}$. More generally, δ is called an additive (or a linear) Jordan derivation if $\delta(A^2) = \delta(A)A + A\delta(A)$ for all $A \in \mathcal{R}$. Note that, if $\mathcal{R}$ is 2-torsion free, then a Jordan derivation can be equivalently defined as $\delta(AB+BA) = \delta(A)B+B\delta(A)+A\delta(B)+\delta(B)A$ for all $A, B \in \mathcal{R}$. Clearly, every additive (or linear) derivation is an additive (or a linear) Jordan derivation. The converse is not true in general [1]. It is well known that derivations and Jordan derivations are very important both in theory and applications, and so they have been studied intensively. Herstein [6] showed that every additive Jordan derivation from a 2-torsion free prime ring into itself is an additive derivation. Brešar [2] proved that Herstein's result is true for 2-torsion free semiprime rings. Sinclair [11] proved that every continuous linear Jordan derivation on semisimple Banach algebras is a linear derivation. Zhang and Yu [15] showed that every additive Jordan derivation on triangular algebras is an additive derivation.

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In recent years, there have been a number of papers on the study of conditions under which derivations of rings or operator algebras can be completely determined by the action on some elements concerning products. This kind of study was initiated by [3] where the elements satisfying $AB = 0$ were considered. We say that an additive (or a linear) map $\delta : \mathcal{R} \to \mathcal{R}$ is derivable at zero point if $\delta(AB) = \delta(A)B + A\delta(B)$ for all $A, B \in \mathcal{R}$ with $AB = 0$. In [3], Brešar proved that if a unital prime ring $\mathcal{R}$ containing a non-trivial idempotent, then every additive derivable map d at zero point of $\mathcal{R}$ is of the form $d(A) = \lambda A + \delta(A)$ for all $A \in \mathcal{R}$, where λ is a central element of $\mathcal{R}$ and δ is an additive derivation of $\mathcal{R}$. Similarly, an additive (or a linear) map $\delta : \mathcal{R} \to \mathcal{R}$ is called Jordan derivable at zero point if $\delta(AB+BA) = \delta(A)B+B\delta(A)+A\delta(B)+\delta(B)A$ for all $A, B \in \mathcal{R}$ with $AB = 0$. It was proved by Zhao and Zhu [16] that every linear Jordan derivable map at zero point is a linear derivation on triangular algebras. Moreover, an additive (or a linear) map δ from $\mathcal{R}$ into itself is called a Jordan derivable map at commutative zero point if $\delta(AB + BA) = \delta(A)B + B\delta(A) + A\delta(B) + \delta(B)A$ for all $A, B \in \mathcal{R}$ with $AB = BA = 0$. Clearly, every Jordan derivable map at zero point is a Jordan derivable map at commutative zero point. But the converse is, in general, not true. We have the following example.

Example 1.1. Let $\mathcal{R}$ be unital ring. Suppose that $\delta : \mathcal{R} \to \mathcal{R}$ is the identical mapping. Then δ is a Jordan derivable mapping at commutative zero point, that is, δ satisfying $\delta(AB + BA) = \delta(A)B + B\delta(A) + A\delta(B) + \delta(B)A$ for all $A, B \in \mathcal{R}$ with $AB = BA = 0$. But note that

$$\delta(AB + BA) = BA \neq 2BA = \delta(A)B + B\delta(A) + A\delta(B) + \delta(B)A$$

for all $A, B \in \mathcal{R}$ with $AB = 0$, $BA \neq 0$. Hence δ is not a Jordan derivable mapping at zero point.

Example 1.1 means Jordan derivable maps at commutative zero point are more general than Jordan derivable maps at zero point. Thus, a natural and interesting question is how to describe Jordan derivable maps at commutative zero point on rings. The main purpose of the present article is to consider the question.

With the development of derivation and Jordan derivation, higher derivation and Jordan higher derivation have attracted much attention of mathematicians. Let us recall the concepts of higher derivations and Jordan higher derivations. Suppose $\mathbb{N}$ is the set of all non-negative integers and $\varphi = \{\varphi^{(n)}\}_{n\in\mathbb{N}}$ is a family of additive (or linear) maps from $\mathcal{R}$ into itself such that $\varphi^{(0)} = I_{\mathcal{R}}$ is the identical map of $\mathcal{R}$. We say φ is an additive (or a linear) higher derivation if $\varphi^{(n)}(ST) = \sum_{i+j=n} \varphi^{(i)}(S)\varphi^{(j)}(T)$ for all $n \in \mathbb{N}$, $S, T \in \mathcal{R}$; φ is an additive (or a linear) Jordan higher derivation if $\varphi^{(n)}(ST+TS) = \sum_{i+j=n}(\varphi^{(i)}(S)\varphi^{(j)}(T)+\varphi^{(i)}(T)\varphi^{(j)}(S))$ for all $n \in \mathbb{N}$, $S, T \in \mathcal{R}$. The study of higher derivations and Jordan higher derivations is an active subject of research in commutative and noncommutative rings, also in some operator algebras, see for example

[4, 5, 9, 10, 13, 12] and references therein. Similar to the additive (or linear) Jordan derivable map at commutative zero point, we say $\varphi = \{\varphi^{(n)}\}_{n\in\mathbb{N}}$ with $\varphi^{(0)} = I_{\mathcal{R}}$ is an additive (or linear) Jordan higher derivable map at commutative zero point if $\varphi^{(n)}(ST+TS) = \sum_{i+j=n}(\varphi^{(i)}(S)\varphi^{(j)}(T) + \varphi^{(i)}(T)\varphi^{(j)}(S))$ for all $n \in \mathbb{N}$, $S, T \in \mathcal{R}$ with $ST = TS = 0$. So it is also interesting to describe Jordan higher derivable maps at commutative zero point on rings. In this paper, we use a new method to generalize the result of additive Jordan derivable maps at commutative zero point to the case of additive Jordan higher derivable maps at commutative zero point.

This article is organized as follows. Let $\mathcal{R}$ be a 2-torsion free unital ring with a non-trivial idempotent E. In Section 2, we prove that every additive Jordan derivable map $\delta : \mathcal{R} \to \mathcal{R}$ at commutative zero point is of the form $\delta(A) = \psi(A) + CA$ for all $A \in \mathcal{R}$, where ψ is an additive Jordan derivation of $\mathcal{R}$ and C is a central element of $\mathcal{R}$. In Section 3, we generalize the results in Section 2 to the case of additive Jordan higher derivable maps at commutative zero point. In Section 4, the above results are applied to some operator algebras such as triangular algebras and von Neumann algebras.

2. Jordan derivable maps at commutative zero point

In this section, we discuss the additive Jordan derivable maps at commutative zero point on rings.

Theorem 2.1. *Let $\mathcal{R}$ be a 2-torsion free unital ring with a non-trivial idempotent E. Assume that $\mathcal{R}$ satisfies the following two conditions*:

(1) *for $A \in \mathcal{R}$, $A\mathcal{R}(I - E) = \{0\}$ implies $A = 0$,*

(2) *for $A \in \mathcal{R}$, $E\mathcal{R}A = \{0\}$ implies $A = 0$.*

If an additive map $\delta : \mathcal{R} \to \mathcal{R}$ is Jordan derivable at commutative zero point, then there exist an additive Jordan derivation ψ from $\mathcal{R}$ into itself and a central element $C \in \mathcal{R}$ such that $\delta(A) = \psi(A) + CA$ for all $A \in \mathcal{R}$.

Proof of Theorem 2.1. Let $E_1 = E$ and $E_2 = I - E_1$. We denote $\mathcal{R}_{ij} = E_i\mathcal{R}E_j$ for $i, j = 1, 2$. Then $\mathcal{R} = \mathcal{R}_{11} + \mathcal{R}_{12} + \mathcal{R}_{21} + \mathcal{R}_{22}$, and each element $A \in \mathcal{R}$ can be written as $A = A_{11} + A_{12} + A_{21} + A_{22}$, where $A_{ij} \in \mathcal{R}_{ij}$ for $i, j = 1, 2$.

We complete the proof within several claims.

Claim 1. *$\delta(I)$ is in the centre of $\mathcal{R}$.*

Since $E_1(I - E_1) = (I - E_1)E_1 = 0$, we have

$$\begin{aligned} 0 &= \delta(E_1)(I - E_1) + E_1\delta(I - E_1) + \delta(I - E_1)E_1 + (I - E_1)\delta(E_1) \\ &= 2\delta(E_1) - 2\delta(E_1)E_1 - 2E_1\delta(E_1) + E_1\delta(I) + \delta(I)E_1. \end{aligned}$$

Multiplying the above equation from the left and the right by E_1 respectively, we obtain that

$$0 = 2E_1\delta(E_1)E_1 - E_1\delta(I) - E_1\delta(I)E_1$$

and

$$0 = 2E_1\delta(E_1)E_1 - \delta(I)E_1 - E_1\delta(I)E_1,$$

which imply $E_1\delta(I)E_2 = 0$, $E_2\delta(I)E_1 = 0$ and $E_1\delta(I) = \delta(I)E_1$. Similarly, for any $A_{12} \in \mathcal{R}_{12}$, since

$$(E_1 + A_{12})(I - E_1 - A_{12}) = (I - E_1 - A_{12})(E_1 + A_{12}) = 0,$$

we can get $(E_1 + A_{12})\delta(I) = \delta(I)(E_1 + A_{12})$. It follows that

$$E_1\delta(I)A_{12} = A_{12}\delta(I)E_2. \tag{2.1}$$

Now, for any $A_{11} \in \mathcal{R}_{11}$, $B_{12} \in \mathcal{R}_{12}$, by Eq. (2.1), we have

$$E_1\delta(I)A_{11}B_{12} = A_{11}B_{12}\delta(I)E_2 = A_{11}\delta(I)B_{12}.$$

It follows from assumption (1) in the theorem that

$$E_1\delta(I)A_{11} = A_{11}\delta(I)E_1. \tag{2.2}$$

At the same time, for any $A_{22} \in \mathcal{R}_{22}$, $B_{12} \in \mathcal{R}_{12}$, by Eq. (2.1), we get

$$B_{12}\delta(I)A_{22} = E_1\delta(I)B_{12}A_{22} = B_{12}A_{22}\delta(I)E_2.$$

By assumption (2) in the theorem, we have

$$E_2\delta(I)A_{22} = A_{22}\delta(I)E_2. \tag{2.3}$$

Furthermore, for any $A_{21} \in \mathcal{R}_{21}$, $B_{12} \in \mathcal{R}_{12}$, Eq. (2.1) and Eq. (2.3) together imply that

$$E_2\delta(I)A_{21}B_{12} = A_{21}B_{12}\delta(I)E_2 = A_{21}\delta(I)B_{12}.$$

By assumption (1) in the theorem,

$$E_2\delta(I)A_{21} = A_{21}\delta(I)E_1 \tag{2.4}$$

is obtained. Consequently, for any $A \in \mathcal{R}$, by Eq. (2.1)-(2.4), we have

$$\begin{aligned}
&(E_1\delta(I)E_1 + E_2\delta(I)E_2)A \\
=\ &E_1\delta(I)A_{11} + E_1\delta(I)A_{12} + E_2\delta(I)A_{21} + E_2\delta(I)A_{22} \\
=\ &A_{11}\delta(I)E_1 + A_{12}\delta(I)E_2 + A_{21}\delta(I)E_1 + A_{22}\delta(I)E_2 \\
=\ &A(E_1\delta(I)E_1 + E_2\delta(I)E_2).
\end{aligned}$$

This together with the facts $E_1\delta(I)E_2 = 0$ and $E_2\delta(I)E_1 = 0$ imply $\delta(I)A = A\delta(I)$ for all $A \in \mathcal{R}$. Hence $\delta(I)$ is in the centre of $\mathcal{R}$, as desired.

Now, let $T = E_1\delta(E_1)E_2 - E_2\delta(E_1)E_1$ and define an additive map $\phi : \mathcal{R} \to \mathcal{R}$ as follows

$$\phi(A) = \delta(A) - \delta_T(A) - \delta(I)A,$$

where δ_T is the inner derivation implemented by T, that is, $\delta_T(A) = AT - TA$ for all $A \in \mathcal{R}$. Since $\delta(I)$ is in the centre of $\mathcal{R}$, one can easily verify that

$$\phi(AB + BA) = \phi(A)B + B\phi(A) + A\phi(B) + \phi(B)A$$

for $A, B \in \mathcal{R}$ with $AB = BA = 0$.

Claim 2. $\phi(E_1) = \phi(E_2) = 0$.

Since $E_1(I - E_1) = (I - E_1)E_1 = 0$ and Claim 1, we can assert that

$$0 = \delta(E_1) - \delta(E_1)E_1 - E_1\delta(E_1) + \delta(I)E_1,$$

which implies $E_2\delta(E_1)E_2 = 0$ and $E_1\delta(E_1)E_1 - \delta(I)E_1 = 0$. It follows that

$$\begin{aligned}\phi(E_1) &= \delta(E_1) - \delta_T(E_1) - \delta(I)E_1\\ &= E_1\delta(E_1)E_1 + E_2\delta(E_1)E_2 - \delta(I)E_1\\ &= 0.\end{aligned}$$

Then $\phi(E_2) = 0$ can be obtained immediately. The claim is true.

Claim 3. *For any $A_{ij} \in \mathcal{R}_{ij}$ $(1 \le i, j \le 2)$, the following statements hold.*
(1) $\phi(A_{ii}) \in \mathcal{R}_{ii}$, $i = 1, 2$.
(2) $E_i\phi(A_{ij})E_i = E_j\phi(A_{ij})E_j = 0$, $1 \le i \ne j \le 2$.

For any $A_{11} \in \mathcal{R}_{11}$, since $A_{11}E_2 = E_2A_{11} = 0$, by Claim 2, we have

$$0 = \phi(A_{11}E_2 + E_2A_{11}) = \phi(A_{11})E_2 + E_2\phi(A_{11}),$$

which implies $E_1\phi(A_{11})E_2 = E_2\phi(A_{11})E_1 = E_2\phi(A_{11})E_2 = 0$. Then $\phi(A_{11}) \in \mathcal{R}_{11}$.

Similarly, by considering $E_1A_{22} = A_{22}E_1 = 0$, we can prove $\phi(A_{22}) \in \mathcal{R}_{22}$ for all $A_{22} \in \mathcal{R}_{22}$.

To prove (2) we only check $E_1\phi(A_{12})E_1 = E_2\phi(A_{12})E_2 = 0$. The proof for the other case $\{i = 2, j = 1\}$ is similar. In fact, since $A_{12}A_{12} = 0$ for all $A_{12} \in \mathcal{R}_{12}$, we have

$$0 = \phi(A_{12}A_{12} + A_{12}A_{12}) = 2(\phi(A_{12})A_{12} + A_{12}\phi(A_{12})). \tag{2.5}$$

Moreover, for any $A_{12} \in \mathcal{R}_{12}$, since

$$(E_1 + A_{12})(E_2 - A_{12}) = (E_2 - A_{12})(E_1 + A_{12}) = 0,$$

by Claim 2, we get

$$\begin{aligned}0 &= \phi(E_1 + A_{12})(E_2 - A_{12}) + (E_1 + A_{12})\phi(E_2 - A_{12})\\ &\quad + \phi(E_2 - A_{12})(E_1 + A_{12}) + (E_2 - A_{12})\phi(E_1 + A_{12})\\ &= \phi(A_{12})E_2 - 2\phi(A_{12})A_{12} - E_1\phi(A_{12}) - 2A_{12}\phi(A_{12})\\ &\quad - \phi(A_{12})E_1 + E_2\phi(A_{12}).\end{aligned}$$

Combining with Eq. (2.5), we obtain $\phi(A_{12})E_2 - E_1\phi(A_{12}) - \phi(A_{12})E_1 + E_2\phi(A_{12}) = 0$, which leads to

$$E_1\phi(A_{12})E_1 = E_2\phi(A_{12})E_2 = 0.$$

Claim 4. *Let $A_{ii} \in \mathcal{R}_{ii}$, $B_{ij} \in \mathcal{R}_{ij}$, $1 \le i \ne j \le 2$. Then*
(1) $\phi(A_{ii}B_{ij}) = \phi(A_{ii})B_{ij} + A_{ii}\phi(B_{ij}) + \phi(B_{ij})A_{ii}$;
(2) $\phi(B_{ij}A_{jj}) = \phi(B_{ij})A_{jj} + B_{ij}\phi(A_{jj}) + A_{jj}\phi(B_{ij})$.

For any $A_{11} \in \mathcal{R}_{11}$, $B_{12} \in \mathcal{R}_{12}$, since

$$(A_{11} - A_{11}B_{12})(B_{12} + E_2) = (B_{12} + E_2)(A_{11} - A_{11}B_{12}) = 0,$$

by Claim 2, we have

$$0 = \phi(A_{11} - A_{11}B_{12})(B_{12} + E_2) + (A_{11} - A_{11}B_{12})\phi(B_{12} + E_2)$$

$$\begin{aligned}&+\phi(B_{12}+E_2)(A_{11}-A_{11}B_{12})+(B_{12}+E_2)\phi(A_{11}-A_{11}B_{12})\\ =\ &\phi(A_{11})B_{12}-\phi(A_{11}B_{12})B_{12}-\phi(A_{11}B_{12})E_2+A_{11}\phi(B_{12})-A_{11}B_{12}\phi(B_{12})\\ &+\phi(B_{12})A_{11}-\phi(B_{12})A_{11}B_{12}-B_{12}\phi(A_{11}B_{12})-E_2\phi(A_{11}B_{12}).\end{aligned}$$

Moreover, since $A_{11}B_{12}B_{12}=B_{12}A_{11}B_{12}=0$, we have

$$\phi(A_{11}B_{12})B_{12}+A_{11}B_{12}\phi(B_{12})+\phi(B_{12})A_{11}B_{12}+B_{12}\phi(A_{11}B_{12})=0.$$

Then we can deduce that

$$\phi(A_{11}B_{12})E_2+E_2\phi(A_{11}B_{12})=\phi(A_{11})B_{12}+A_{11}\phi(B_{12})+\phi(B_{12})A_{11}.$$

It follows from Chaim 3(2) that $\phi(A_{11}B_{12})=\phi(A_{11}B_{12})E_2+E_2\phi(A_{11}B_{12})$. So we have

$$\phi(A_{11}B_{12})=\phi(A_{11})B_{12}+A_{11}\phi(B_{12})+\phi(B_{12})A_{11}.$$

Similarly, by considering $(A_{22}-A_{22}B_{21})(E_1+B_{21})=(E_1+B_{21})(A_{22}-A_{22}B_{21})=0$, one can show that

$$\phi(A_{22}B_{21})=\phi(A_{22})B_{21}+A_{22}\phi(B_{21})+\phi(B_{21})A_{22}.$$

For any $A_{12}\in\mathcal{R}_{12}$, $B_{22}\in\mathcal{R}_{22}$, since

$$(B_{22}-A_{12}B_{22})(E_1+A_{12})=(E_1+A_{12})(B_{22}-A_{12}B_{22})=0,$$

by Claim 2, we get

$$\begin{aligned}0=\ &\phi(B_{22}-A_{12}B_{22})(E_1+A_{12})+(B_{22}-A_{12}B_{22})\phi(E_1+A_{12})\\ &+\phi(E_1+A_{12})(B_{22}-A_{12}B_{22})+(E_1+A_{12})\phi(B_{22}-A_{12}B_{22})\\ =\ &-\phi(A_{12}B_{22})E_1-\phi(A_{12}B_{22})A_{12}-A_{12}B_{22}\phi(A_{12})+B_{22}\phi(A_{12})\\ &-\phi(A_{12})A_{12}B_{22}+\phi(A_{12})B_{22}-E_1\phi(A_{12}B_{22})\\ &-A_{12}\phi(A_{12}B_{22})+A_{12}\phi(B_{22}).\end{aligned}$$

Moreover, since $A_{12}B_{22}A_{12}=A_{12}A_{12}B_{22}=0$, we have

$$\phi(A_{12}B_{22})A_{12}+A_{12}B_{22}\phi(A_{12})+\phi(A_{12})A_{12}B_{22}+A_{12}\phi(A_{12}B_{22})=0.$$

So we obtain

$$\phi(A_{12}B_{22})E_1+E_1\phi(A_{12}B_{22})=\phi(A_{12})B_{22}+A_{12}\phi(B_{22})+B_{22}\phi(A_{12}).$$

It follows from Chaim 3(2) that $\phi(A_{12}B_{22})=\phi(A_{12}B_{22})E_1+E_1\phi(A_{12}B_{22})$. Then we can conclude that

$$\phi(A_{12}B_{22})=\phi(A_{12})B_{22}+A_{12}\phi(B_{22})+B_{22}\phi(A_{12}).$$

In a similar fashion, by considering $(B_{11}-A_{21}B_{11})(A_{21}+E_2)=(A_{21}+E_2)(B_{11}-A_{21}B_{11})=0$, one can prove

$$\phi(A_{21}B_{11})=\phi(A_{21})B_{11}+A_{21}\phi(B_{11})+B_{11}\phi(A_{21}).$$

Claim 5. *Let A_{ii}, $B_{ii}\in\mathcal{R}_{ii}$, $i=1,2$. Then $\phi(A_{ii}B_{ii})=\phi(A_{ii})B_{ii}+A_{ii}\phi(B_{ii})$.*

For any $D_{12}\in\mathcal{R}_{12}$, by Claims 3-4, on the one hand

$$\phi(A_{11}B_{11}D_{12})=\phi(A_{11}B_{11})D_{12}+A_{11}B_{11}\phi(D_{12})+\phi(D_{12})A_{11}B_{11};$$

on the other hand

$$\begin{aligned}\phi(A_{11}B_{11}D_{12}) &= \phi(A_{11})B_{11}D_{12} + A_{11}\phi(B_{11}D_{12}) + \phi(B_{11}D_{12})A_{11} \\ &= \phi(A_{11})B_{11}D_{12} + A_{11}(\phi(B_{11})D_{12} + B_{11}\phi(D_{12}) + \phi(d_{12})B_{11}) \\ &\quad + (\phi(B_{11})D_{12} + B_{11}\phi(D_{12}) + \phi(D_{12})B_{11})A_{11} \\ &= \phi(A_{11})B_{11}D_{12} + A_{11}\phi(B_{11})D_{12} \\ &\quad + a_{11}B_{11}\phi(D_{12}) + \phi(D_{12})B_{11}A_{11}.\end{aligned}$$

Comparing these two equations, we obtain $\phi(A_{11}B_{11})D_{12} = (\phi(A_{11})B_{11} + A_{11}\phi(B_{11}))D_{12}$. It follows from assumption (1) in the theorem that $\phi(A_{11}B_{11}) = \phi(A_{11})B_{11} + A_{11}\phi(B_{11})$.

Similarly, one can verify that $\phi(A_{22}B_{22}) = \phi(A_{22})B_{22} + A_{22}\phi(B_{22})$.

Claim 6. *Let $A_{ij} \in \mathcal{R}_{ij}$, $B_{ji} \in \mathcal{R}_{ji}$, $1 \le i \ne j \le 2$. Then $\phi(A_{ij}B_{ji}) = \phi(A_{ij})B_{ji} + A_{ij}\phi(B_{ji})$.*

For any $A_{12} \in \mathcal{R}_{12}$ and $B_{21} \in \mathcal{R}_{21}$, since

$$(E_1 + A_{12} + B_{21} + B_{21}A_{12})(E_2 - A_{12} - B_{21} + A_{12}B_{21}) = 0$$

and

$$(E_2 - A_{12} - B_{21} + A_{12}B_{21})(E_1 + A_{12} + B_{21} + B_{21}A_{12}) = 0,$$

by Claim 3, we have

$$\begin{aligned}0 &= \phi((E_1 + A_{12} + B_{21} + B_{21}A_{12})(E_2 - A_{12} - B_{21} + A_{12}B_{21}) \\ &\quad + (E_2 - A_{12} - B_{21} + A_{12}B_{21})(E_1 + A_{12} + B_{21} + B_{21}A_{12})) \\ &= \phi(E_1 + A_{12} + B_{21} + B_{21}A_{12})(E_2 - A_{12} - B_{21} + A_{12}B_{21}) \\ &\quad + (E_1 + A_{12} + B_{21} + B_{21}A_{12})\phi(E_2 - A_{12} - B_{21} + A_{12}B_{21}) \\ &\quad + \phi(E_2 - A_{12} - B_{21} + A_{12}B_{21})(E_1 + A_{12} + B_{21} + B_{21}A_{12}) \\ &\quad + (E_2 - A_{12} - B_{21} + A_{12}B_{21})\phi(E_1 + A_{12} + B_{21} + B_{21}A_{12}) \\ &= 2(\phi(A_{12}B_{21}) - \phi(A_{12})B_{21} - A_{12}\phi(B_{21})) \\ &\quad + 2(\phi(B_{21}A_{12}) - \phi(B_{21})A_{12} - B_{21}\phi(A_{12})) \\ &\quad + (\phi(B_{21})A_{12}B_{21} + B_{21}\phi(A_{12}B_{21}) + A_{12}B_{21}\phi(B_{21}) \\ &\quad - \phi(B_{21}A_{12})B_{21} - B_{21}A_{12}\phi(B_{21}) - \phi(B_{21})B_{21}A_{12}) \\ &\quad + (\phi(A_{12}B_{21})A_{12} + A_{12}B_{21}\phi(A_{12}) + \phi(A_{12})A_{12}B_{21} \\ &\quad - \phi(A_{12})B_{21}A_{12} - A_{12}\phi(B_{21}a_{12}) - B_{21}A_{12}\phi(A_{12})) \\ &\quad + (\phi(A_{12})E_2 + E_2\phi(A_{12}) - \phi(A_{12})E_1 - E_1\phi(A_{12})) \\ &\quad + (\phi(B_{21})E_2 + E_2\phi(B_{21}) - \phi(B_{21})E_1 - E_1\phi(B_{21})).\end{aligned}$$

By Claims 3-4, the above equation implies that

$$\phi(A_{12}B_{21}) - \phi(A_{12})B_{21} - A_{12}\phi(B_{21}) + \phi(B_{21}A_{12}) - \phi(B_{21})A_{12} - B_{21}\phi(A_{12}) = 0.$$

Then we can deduce that

$$\phi(A_{12}B_{21}) = \phi(A_{12})B_{21} + A_{12}\phi(B_{21})$$

and

$$\phi(B_{21}A_{12}) = \phi(B_{21})A_{12} + B_{21}\phi(A_{12}).$$

The claim is true.

Now, by Claims 4-6, we can infer that $\phi(A^2) = \phi(A)A + A\phi(A)$ for all $A \in \mathcal{R}$, that is, ϕ is an additive Jordan derivation from $\mathcal{R}$ into itself. Let $C = \delta(I)$ and $\psi = \phi + \delta_T$. It is easy to check that ψ is also an additive Jordan derivation from $\mathcal{R}$ into itself. Then, by the definitions of δ, we have $\delta(A) = \psi(A) + CA$ for all $A \in \mathcal{R}$. The proof is finished. □

By [2], every additive Jordan derivation is an additive derivation on semiprime rings. The following result is immediate.

Theorem 2.2. *Let $\mathcal{R}$ be a 2-torsion free semiprime unital ring with a non-trivial idempotent E. Assume that $\mathcal{R}$ satisfies the following two conditions*:

(1) *for $A \in \mathcal{R}$, $A\mathcal{R}(I-E) = \{0\}$ implies $A = 0$,*

(2) *for $A \in \mathcal{R}$, $E\mathcal{R}A = \{0\}$ implies $A = 0$.*

Suppose that additive map $\delta : \mathcal{R} \to \mathcal{R}$ is Jordan derivable at commutative zero point. Then there exist an additive derivation ψ from $\mathcal{R}$ into itself and a central element $C \in \mathcal{R}$ such that $\delta(A) = \psi(A) + CA$ for all $A \in \mathcal{R}$.

Because a prime ring (i.e., $\mathcal{R}$ satisfies that, for any $A, B \in \mathcal{R}$, $A\mathcal{R}B=\{0\}$ implies $A = 0$ or $B = 0$) satisfies the hypotheses of Theorem 2.2 if it contains a non-trivial idempotent, we have

Theorem 2.3. *Let $\mathcal{R}$ be a 2-torsion free unital prime ring with a non-trivial idempotent. Suppose that additive map $\delta : \mathcal{R} \to \mathcal{R}$ is Jordan derivable at commutative zero point. Then there exist an additive derivation ψ from $\mathcal{R}$ into itself and a central element $C \in \mathcal{R}$ such that $\delta(A) = \psi(A) + CA$ for all $A \in \mathcal{R}$.*

3. Jordan higher derivable maps at commutative zero point

In this section, we discuss the additive Jordan higher derivable maps at commutative zero point on rings.

Theorem 3.1. *Let $\mathcal{R}$ be a 2-torsion free unital ring with a non-trivial idempotent E. Assume that $\mathcal{R}$ satisfies the following two conditions*:

(1) *for $A \in \mathcal{R}$, $A\mathcal{R}(I-E) = \{0\}$ implies $A = 0$,*

(2) *for $A \in \mathcal{R}$, $E\mathcal{R}A = \{0\}$ implies $A = 0$.*

If $\delta = \{\delta^{(n)}\}_{n\in\mathbb{N}}$ from $\mathcal{R}$ into itself is an additive Jordan higher derivable map at commutative zero point, then there exist an additive Jordan higher derivation $\phi = \{\phi^{(n)}\}_{n\in\mathbb{N}}$ from $\mathcal{R}$ into itself and a sequence of central elements $C^{(n)} \in \mathcal{R}$ such that $\delta^{(n)}(A) = \phi^{(n)}(A) + C^{(n)}A$ for all $n \geq 1$, $A \in \mathcal{R}$.

Proof of Theorem 3.1. We will use the same symbols as that in Section 2 and complete the proof by checking two claims.

Claim 1. *$\delta^{(n)}(I)$ is in the centre of $\mathcal{R}$ for all $n \in \mathbb{N}$.*

Let us make induction for the index n. By Theorem 2.1, we have $\delta^{(1)}(A) = \psi^{(1)}(A) + \delta^{(1)}(I)A$ for all $A \in \mathcal{R}$, where $\psi^{(1)}$ is an additive Jordan derivation from $\mathcal{R}$ into itself and $\delta^{(1)}(I)$ is a central element in $\mathcal{R}$. This implies $E_1\delta^{(1)}(E_1)E_1 = \delta^{(1)}(I)E_1 = E_1\delta^{(1)}(I)$ and $(I-E_1)\delta^{(1)}(E_1)(I-E_1) = 0$. So we assume that $E_1\delta^{(k)}(E_1)E_1 = \delta^{(k)}(I)E_1 = E_1\delta^{(k)}(I)$, $(I-E_1)\delta^{(k)}(E_1)(I-E_1) = 0$ and $\delta^{(k)}(I)$ is in the centre of $\mathcal{R}$ for all $k < n$. Since $E_1(I-E_1) = (I-E_1)E_1 = 0$, we have

$$
\begin{aligned}
0 &= \sum_{i+j=n} (\delta^{(i)}(E_1)\delta^{(j)}(I-E_1) + \delta^{(i)}(I-E_1)\delta^{(j)}(E_1)) \\
&= E_1\delta^{(n)}(I-E_1) + \delta^{(n)}(I-E_1)E_1 + \delta^{(n)}(E_1)(I-E_1) + (I-E_1)\delta^{(n)}(E_1) \\
&\quad + \sum_{i+j=n, i,j\neq\{0,n\}} (\delta^{(i)}(E_1)\delta^{(j)}(I-E_1) + \delta^{(i)}(I-E_1)\delta^{(j)}(E_1)) \\
&= 2\delta^{(n)}(E_1) - 2\delta^{(n)}(E_1)E_1 - 2E_1\delta^{(n)}(E_1) + E_1\delta^{(n)}(I) + \delta^{(n)}(I)E_1 \\
&\quad + \sum_{i+j=n, i,j\neq\{0,n\}} (\delta^{(i)}(E_1)\delta^{(j)}(I-E_1) + \delta^{(i)}(I-E_1)\delta^{(j)}(E_1)).
\end{aligned}
$$

Multiplying the above equation from the left and the right by E_1 respectively, we obtain that

$$
\begin{aligned}
0 &= 2E_1\delta^{(n)}(E_1)E_1 - E_1\delta^{(n)}(I) - E_1\delta^{(n)}(I)E_1 \\
&\quad - \sum_{i+j=n, i,j\neq\{0,n\}} (E_1\delta^{(i)}(E_1)\delta^{(j)}(I-E_1) + E_1\delta^{(i)}(I-E_1)\delta^{(j)}(E_1))
\end{aligned} \tag{3.1}
$$

and

$$
\begin{aligned}
0 &= 2E_1\delta^{(n)}(E_1)E_1 - E_1\delta^{(n)}(I)E_1 - \delta^{(n)}(I)E_1 \\
&\quad - \sum_{i+j=n, i,j\neq\{0,n\}} (\delta^{(i)}(E_1)\delta^{(j)}(I-E_1)E_1 + \delta^{(i)}(I-E_1)\delta^{(j)}(E_1)E_1).
\end{aligned} \tag{3.2}
$$

Set

$$\triangle = \sum_{i+j=n, i,j\neq\{0,n\}} (E_1\delta^{(i)}(E_1)\delta^{(j)}(I-E_1) + E_1\delta^{(i)}(I-E_1)\delta^{(j)}(E_1))$$

and

$$\nabla = \sum_{i+j=n, i,j\neq\{0,n\}} (\delta^{(i)}(E_1)\delta^{(j)}(I-E_1)E_1 + \delta^{(i)}(I-E_1)\delta^{(j)}(E_1)E_1)$$

in Eq. (3.1)-(3.2). It is clear that $E_1 \triangle E_1 = E_1 \nabla E_1$ and $(I - E_1)\triangle(I - E_1) = (I - E_1)\nabla(I - E_1)$. At the same time, by the induction hypothesis, we have

$$\begin{aligned} E_1\triangle(I-E_1) &= \sum_{i+j=n, i,j\neq\{0,n\}} \Big(E_1\delta^{(i)}(E_1)\delta^{(j)}(I-E_1)(I-E_1) \\ &\qquad + E_1\delta^{(i)}(I-E_1)\delta^{(j)}(E_1)(I-E_1)\Big) \\ &= \sum_{i+j=n, i,j\neq\{0,n\}} (E_1\delta^{(i)}(E_1)\delta^{(j)}(I-E_1)(I-E_1) \\ &= \sum_{i+j=n, i,j\neq\{0,n\}} \Big(E_1\delta^{(i)}(E_1)\delta^{(j)}(I)(I-E_1) \\ &\qquad - E_1\delta^{(i)}(E_1)\delta^{(j)}(E_1)(I-E_1)\Big) \\ &= \sum_{i+j=n, i,j\neq\{0,n\}} \Big(E_1\delta^{(j)}(E_1)E_1\delta^{(i)}(E_1)(I-E_1) \\ &\qquad - E_1\delta^{(i)}(E_1)E_1\delta^{(j)}(E_1)(I-E_1)\Big) \\ &= 0. \end{aligned}$$

Similarly, we can verify that $E_1\nabla(I - E_1) = 0$, $(I - E_1)\triangle E_1 = 0$ and $(I - E_1)\nabla E_1 = 0$. It follows that $\triangle = \nabla$, and hence $E_1\delta^{(n)}(I) = \delta^{(n)}(I)E_1$ by Eq. (3.1)-(3.2). The rest is similar to the proof of Claim 1 in Theorem 2.1. We can get $\delta^{(n)}(I)A = A\delta^{(n)}(I)$ for all $A \in \mathcal{R}$. Hence $\delta^{(n)}(I)$ is in the centre of $\mathcal{R}$, as desired.

Now, for any $n \geq 1$, we define an additive map $\phi^{(n)} : \mathcal{R} \to \mathcal{R}$ as follows

$$\phi^{(n)}(A) = \delta^{(n)}(A) - \delta^{(n)}(I)A$$

and let $\varphi^{(0)}$ is the identical map of $\mathcal{R}$. Since $\delta^{(n)}(I)$ is in the centre of $\mathcal{R}$ for all $n \in \mathbb{N}$, one can easily verify that

$$\phi^{(n)}(AB + BA) = \sum_{i+j=n} \phi^{(i)}(A)\phi^{(j)}(B) + \phi^{(i)}(B)\phi^{(j)}(A)$$

for all $A, B \in \mathcal{R}$ with $AB = BA = 0$.

Claim 2. $\phi = \{\phi^{(n)}\}_{n\in\mathbb{N}}$ *is an additive Jordan higher derivation.*

Let $d^{(1)} = \phi^{(1)}$ and $d^{(n)} = n\phi^{(n)} - \sum_{k=0}^{n-2} d^{(k+1)}\phi^{(n-1-k)}$, $n \geq 2$. We shall show $d^{(n)}$ is an additive Jordan derivation for all $n \geq 1$ which in turn yields $\phi = \{\phi^{(n)}\}_{n\in\mathbb{N}}$ is an additive Jordan higher derivation by [8, Theorem 2.5] in the case of Jordan derivations and Jordan higher derivations].

We use induction. It follows from Theorem 2.1 that $\phi^{(1)}$ is an additive Jordan derivation, hence $d^{(1)}$ is an additive Jordan derivation. We may assume that $d^{(k)}$ are additive Jordan derivations for all $k < n$. For any $A, B \in \mathcal{R}$ with

$AB = BA = 0$, it follows from the induction hypothesis that

$$\begin{aligned}
d^{(n)}(AB+BA) &= n\phi^{(n)}(AB+BA) - \sum_{k=0}^{n-2} d^{(k+1)}(\phi^{(n-1-k)}(AB+BA)) \\
&= n\sum_{k=0}^{n}(\phi^{(k)}(A)\phi^{(n-k)}(B) + \phi^{(k)}(B)\phi^{(n-k)}(A)) \\
&\quad - \sum_{k=0}^{n-2} d^{(k+1)}\Big(\sum_{i=0}^{n-1-k}(\phi^{(i)}(A)\phi^{(n-1-k-i)}(B) \\
&\qquad + \phi^{(i)}(B)\phi^{(n-1-k-i)}(A))\Big) \\
&= \sum_{k=0}^{n} k(\phi^{(k)}(A)\phi^{(n-k)}(B) + \phi^{(k)}(B)\phi^{(n-k)}(A)) \\
&\quad - \sum_{k=0}^{n-2}\sum_{i=0}^{n-1-k}\Big(d^{(k+1)}(\phi^{(i)}(A))\phi^{(n-1-k-i)}(B) \\
&\qquad + \phi^{(i)}(B)d^{(k+1)}(\phi^{(n-1-k-i)}(A))\Big) \\
&\quad + \sum_{k=0}^{n}(n-k)(\phi^{(k)}(A)\phi^{(n-k)}(B) + \phi^{(k)}(B)\phi^{(n-k)}(A)) \\
&\quad - \sum_{k=0}^{n-2}\sum_{i=0}^{n-1-k}\Big(\phi^{(i)}(A)d^{(k+1)}(\phi^{(n-1-k-i)}(B) \\
&\qquad + d^{(k+1)}(\phi^{(i)}(B))\phi^{(n-1-k-i)}(A))\Big).
\end{aligned}$$

Noting that

$$\begin{aligned}
&\sum_{k=0}^{n} k(\phi^{(k)}(A)\phi^{(n-k)}(B) + \phi^{(k)}(B)\phi^{(n-k)}(A)) \\
&\quad - \sum_{k=0}^{n-2}\sum_{i=0}^{n-1-k}\Big(d^{(k+1)}(\phi^{(i)}(A))\phi^{(n-1-k-i)}(B) \\
&\qquad + \phi^{(i)}(B)d^{(k+1)}(\phi^{(n-1-k-i)}(A))\Big) \\
= &\sum_{k=0}^{n} k(\phi^{(k)}(A)\phi^{(n-k)}(B) + \phi^{(n-k)}(B)\phi^{(k)}(A)) \\
&\quad - \sum_{k=0}^{n-2}\sum_{i=0}^{n-1-k}\Big(d^{(k+1)}(\phi^{(i)}(A))\phi^{(n-1-k-i)}(B)) \\
&\qquad + \phi^{(n-1-k-i)}(B)d^{(k+1)}(\phi^{(i)}(A))\Big)
\end{aligned}$$

$$\begin{aligned}
&\overset{k+i=r}{=} n(\phi^{(n)}(A)B + B\phi^{(n)}(A)) \\
&\quad + \sum_{k=0}^{n-1} k(\phi^{(k)}(A)\phi^{(n-k)}(B) + \phi^{(n-k)}(B)\phi^{(k)}(A)) \\
&\quad - \sum_{r=0}^{n-1} \sum_{k=0, k\neq n-1}^{r} \Big(d^{(k+1)}(\phi^{(r-k)}(A))\phi^{(n-1-r)}(B)) \\
&\qquad\qquad + \phi^{(n-1-r)}(B) d^{(k+1)}(\phi^{(r-k)}(A))\Big) \\
&= n(\phi^{(n)}(A)B + B\phi^{(n)}(A)) \\
&\quad + \sum_{k=0}^{n-1} k(\phi^{(k)}(A)\phi^{(n-k)}(B) + \phi^{(n-k)}(B)\phi^{(k)}(A)) \\
&\quad - \sum_{k=0}^{n-2} d^{(k+1)}(\phi^{(n-1-k)}(A))B + Bd^{(k+1)}(\phi^{(n-1-k)}(A)) \\
&\quad - \sum_{r=0}^{n-2} \sum_{k=0}^{r} \Big(d^{(k+1)}(\phi^{(r-k)}(A))\phi^{(n-1-r)}(B)) \\
&\qquad\qquad + \phi^{(n-1-r)}(B) d^{(k+1)}(\phi^{(r-k)}(A))\Big)
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{k=0}^{n-1} k(\phi^{(k)}(A)\phi^{(n-k)}(B) + \phi^{(n-k)}(B)\phi^{(k)}(A)) \\
&- \sum_{r=0}^{n-2} \sum_{k=0}^{r} (d^{(k+1)}(\phi^{(r-k)}(A))\phi^{(n-1-r)}(B) + \phi^{(n-1-r)}(B) d^{(k+1)}(\phi^{(r-k)}(A))) \\
= &\sum_{k=0}^{n-1} k(\phi^{(k)}(A)\phi^{(n-k)}(B) + \phi^{(n-k)}(B)\phi^{(k)}(A)) \\
&- \sum_{r=0}^{n-2} \sum_{j=0}^{r} (d^{(j+1)}(\phi^{(r-j)}(A))\phi^{(n-1-r)}(B) + \phi^{(n-1-r)}(B) d^{(j+1)}(\phi^{(r-j)}(A))) \\
= &\sum_{k=0}^{n-1} k(\phi^{(k)}(A)\phi^{(n-k)}(B) + \phi^{(n-k)}(B)\phi^{(k)}(A)) \\
&- \sum_{k=1}^{n-1} \sum_{j=0}^{k-1} (d^{(j+1)}(\phi^{(k-1-j)}(A))\phi^{(n-k)}(B) + \phi^{(n-k)}(B) d^{(j+1)}(\phi^{(k-1-j)}(A))) \\
= &\sum_{k=1}^{n-1} [(k\phi^{(k)}(A) - \sum_{j=0}^{k-1} d^{(j+1)}(\phi^{(k-1-j)}(A)))\phi^{(n-k)}(B)
\end{aligned}$$

$$+ \phi^{(n-k)}(B)(k\phi^{(k)}(A) - \sum_{j=0}^{k-1} d^{(j+1)}(\phi^{(k-1-j)}(A)))]$$

$$= \sum_{k=1}^{n-1}[(k\phi^{(k)}(A) - \sum_{j=0}^{k-2} d^{(j+1)}(\phi^{(k-1-j)}(A)) - d^{(k)}(A))\phi^{(n-k)}(B)$$

$$+ \phi^{(n-k)}(B)(k\phi^{(k)}(A) - \sum_{j=0}^{k-2} d^{(j+1)}(\phi^{(k-1-j)}(A)) - d^{(k)}(A))]$$

$$= 0,$$

we obtain

$$\begin{aligned}
&\sum_{k=0}^{n} k(\phi^{(k)}(A)\phi^{(n-k)}(B) + \phi^{(k)}(B)\phi^{(n-k)}(A)) \\
&- \sum_{k=0}^{n-2}\sum_{i=0}^{n-1-k} \Big(d^{(k+1)}(\phi^{(i)}(A))\phi^{(n-1-k-i)}(B)) \\
&\qquad + \phi^{(i)}(B)d^{(k+1)}(\phi^{(n-1-k-i)}(A))\Big) \\
= &\; n(\phi^{(n)}(A)B + B\phi^{(n)}(A)) \\
&- \sum_{k=0}^{n-2} d^{(k+1)}(\phi^{(n-1-k)}(A))B + Bd^{(k+1)}(\phi^{(n-1-k)}(A)) \\
= &\; (n\phi^{(n)}(A) - \sum_{k=0}^{n-2} d^{(k+1)}(\phi^{(n-1-k)}(A)))B \\
&+ B(n\phi^{(n)}(A) - \sum_{k=0}^{n-2} d^{(k+1)}(\phi^{(n-1-k)}(A))) \\
= &\; d^{(n)}(A)B + Bd^{(n)}(A).
\end{aligned}$$

Similarly, we can get

$$\begin{aligned}
&\sum_{k=0}^{n}(n-k)(\phi^{(k)}(A)\phi^{(n-k)}(B) + \phi^{(k)}(B)\phi^{(n-k)}(A)) \\
&- \sum_{k=0}^{n-2}\sum_{i=0}^{n-1-k} (\phi^{(i)}(A)d^{(k+1)}(\phi^{(n-1-k-i)}(B) + d^{(k+1)}(\phi^{(i)}(B))\phi^{(n-1-k-i)}(A)) \\
= &\; Ad^{(n)}(B) + d^{(n)}(B)A.
\end{aligned}$$

Hence $d^{(n)}(AB + BA) = d^{(n)}(A)B + Bd^{(n)}(A) + Ad^{(n)}(B) + d^{(n)}(B)A$ for all $A, B \in \mathcal{R}$ with $AB = BA = 0$, which means $d^{(n)}$ is an additive Jordan derivable map at commutative zero point. By Theorem 2.1, we get $d^{(n)}(A) = \varphi^{(n)}(A) + d^{(n)}(I)A$, where $\varphi^{(n)}$ is an additive Jordan derivation. Since $d^{(n)} =$

$n\phi^{(n)} - \sum_{k=0}^{n-2} d^{(k+1)}\phi^{(n-1-k)}$ and $\phi^{(n)}(I) = 0$, we have $d^{(n)}(I) = 0$. Then $d^{(n)}$ is an additive Jordan derivation. Consequently, $\phi = \{\phi^{(n)}\}_{n\in\mathbb{N}}$ is an additive Jordan higher derivation. Let $C^{(n)} = \delta^{(n)}(I)$, then we have $\delta^{(n)}(A) = \phi^{(n)}(A) + C^{(n)}A$ for all $n \geq 1$, $A \in \mathcal{R}$. The proof is finished. □

By [12], every additive Jordan higher derivation is an additive higher derivation on semiprime rings. The following result is immediate.

Theorem 3.2. *Let $\mathcal{R}$ be a 2-torsion free unital semiprime ring with a non-trivial idempotent E. Assume that $\mathcal{R}$ satisfies the following two conditions*:
(1) *for $A \in \mathcal{R}$, $A\mathcal{R}(I - E) = \{0\}$ implies $A = 0$,*
(2) *for $A \in \mathcal{R}$, $E\mathcal{R}A = \{0\}$ implies $A = 0$.*
If $\delta = \{\delta^{(n)}\}_{n\in\mathbb{N}}$ from $\mathcal{R}$ into itself is an additive Jordan higher derivable map at commutative zero point, then there exists an additive higher derivation $\phi = \{\phi^{(n)}\}_{n\in\mathbb{N}}$ from $\mathcal{R}$ into itself and a sequence of central elements $C^{(n)} \in \mathcal{R}$ such that $\delta^{(n)}(A) = \phi^{(n)}(A) + C^{(n)}A$ for all $n \geq 1$, $A \in \mathcal{R}$.

Because a prime ring satisfies the hypotheses of Theorem 3.2 if it contains a non-trivial idempotent, we have

Theorem 3.3. *Let $\mathcal{R}$ be a 2-torsion free unital prime ring with a non-trivial idempotent E. If $\delta = \{\delta^{(n)}\}_{n\in\mathbb{N}}$ from $\mathcal{R}$ into itself is an additive Jordan higher derivable map at commutative zero point, then there exist an additive higher derivation $\phi = \{\phi^{(n)}\}_{n\in\mathbb{N}}$ from $\mathcal{R}$ into itself and a sequence of central elements $C^{(n)} \in \mathcal{R}$ such that $\delta^{(n)}(A) = \phi^{(n)}(A) + C^{(n)}A$ for all $n \geq 1$, $A \in \mathcal{R}$.*

4. Applications

In this section, as application to operator algebra theory, we obtain some results in triangular algebras and von Neumann algebras.

Let $\mathcal{A}$ and $\mathcal{B}$ be unital algebras over real or complex field and $\mathcal{M}$ be a $(\mathcal{A}, \mathcal{B})$-bimodule, which is faithful as a left $\mathcal{A}$-module and also as a right $\mathcal{B}$-module. Recall that a left $\mathcal{A}$-module is faithful if for any $a \in \mathcal{A}$, $a\mathcal{M} = \{0\}$ implies that $a = 0$, and right $\mathcal{B}$-module is faithful if for any $b \in \mathcal{B}$, $\mathcal{M}b = \{0\}$ implies that $b = 0$. The algebra

$$\mathcal{T} = Tri(\mathcal{A}, \mathcal{M}, \mathcal{B}) = \left\{ \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} : a \in \mathcal{A}, \quad m \in \mathcal{M}, \quad b \in \mathcal{B} \right\}$$

under the usual matrix operations is called a triangular algebra. Let $I_{\mathcal{A}}$, $I_{\mathcal{B}}$ be the units of $\mathcal{A}$, $\mathcal{B}$, respectively, then $I = I_{\mathcal{A}} \oplus I_{\mathcal{B}}$ is the unit of $\mathcal{T}$.

It is not difficult to verify the triangular algebras satisfy the hypotheses of Theorem 3.1. Since additive Jordan higher derivations are additive higher derivations on triangular algebras [14], the following result is immediate from Theorem 3.1.

Corollary 4.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital algebras over real or complex field and $\mathcal{M}$ be a $(\mathcal{A}, \mathcal{B})$-bimodule, which is faithful as a left $\mathcal{A}$-module and also as a* *right $\mathcal{B}$-module. Let $\mathcal{T} = Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be a triangular algebra. Suppose that $\delta = \{\delta^{(n)}\}_{n\in\mathbb{N}}$ from $\mathcal{T}$ into itself is an additive Jordan higher derivable map at commutative zero point, then there exist an additive higher derivation $\phi = \{\phi^{(n)}\}_{n\in\mathbb{N}}$ from $\mathcal{M}$ into itself and a sequence of central elements $C^{(n)} \in \mathcal{R}$ such that $\delta^{(n)}(A) = \phi^{(n)}(A) + C^{(n)}A$ for all $n \geq 1$, $A \in \mathcal{T}$.*

Recall that a von Neumann algebra $\mathcal{M}$ is a weakly closed, self-adjoint algebra of operators on a Hilbert space H containing the identity I. $\mathcal{Z}_{\mathcal{M}}$ =$\{Z \in \mathcal{M} : ZM = MZ$ for all $M \in \mathcal{M}\}$ is called the centre of $\mathcal{M}$. A von Neumann algebra is called factor if its centre only contains the scalar operators. A projection P is called a central abelian projection if $P \in \mathcal{Z}_{\mathcal{M}}$ and $P\mathcal{M}P$ is abelian. We denote $\overline{A}$ be the central carrier of A, which is the smallest central projection satisfying $PA = A$. It is well known that $\overline{A}$ is the projection whose range is the closed linear span of $\{\mathcal{M}A(h) : h \in H\}$. For each self-adjoint operator $R \in \mathcal{M}$, the core of R denoted by $\underline{R}$ is $\sup\{A \in \mathcal{Z}_{\mathcal{M}} : A = A^*,\ A \leq R\}$. If $P \in \mathcal{M}$ is a projection and $\underline{P} = 0$, we call P a core-free projection. It is easy to verify that $\underline{P} = 0$ if and only if $\overline{I-P} = I$. By [7, Lemma 4], we can say that $\mathcal{M}$ is a von Neumann algebra with no central abelian projections if and only if it has a projection $P \in \mathcal{M}$ such that $\underline{P} = 0$ and $\overline{P} = I$. It is well known that von Neumann algebras are semiprime and factor von Neumann algebras are prime. Applying the results of Theorem 3.2 and Theorem 3.3 to von Neumann algebras, we have the following two corollaries.

Corollary 4.2. *Let $\mathcal{M}$ be a von Neumann algebra without central abelian projections. Suppose that $\delta = \{\delta^{(n)}\}_{n\in\mathbb{N}}$ from $\mathcal{M}$ into itself is an additive Jordan higher derivable map at commutative zero point, then there exist an additive higher derivation $\phi = \{\phi^{(n)}\}_{n\in\mathbb{N}}$ from $\mathcal{M}$ into itself and a sequence of central elements $C^{(n)} \in \mathcal{R}$ such that $\delta^{(n)}(A) = \phi^{(n)}(A) + C^{(n)}A$ for all $n \geq 1$, $A \in \mathcal{M}$.*

Proof. We only need to show that $\mathcal{M}$ satisfies the assumption (1)-(2) in Theorem 3.2. Since $\mathcal{M}$ is a von Neumann algebra with no central abelian projections, there exists a projection P in $\mathcal{M}$ with $\underline{P} = 0$ and $\overline{P} = I$. By the definitions of core and central carrier, $I - P$ is also a core-free projection and $\overline{I-P} = I$. If $A\mathcal{M}(I-P) = \{0\}$, we can check that $A = 0$. In fact, Since $\overline{I-P} = I$, which means $\{\mathcal{M}(I-P)(h) : h \in H\}$ is dense in H, we get $A = 0$. So the assumption (1) in Theorem 3.2 is satisfied. If $P\mathcal{M}A = \{0\}$, then $A^*\mathcal{M}P = 0$. It follows from $\overline{P} = I$ that $A^* = 0$. So $A = 0$. This implies that the assumption (2) in Theorem 3.2 is also satisfied. □

Corollary 4.3. *Let $\mathcal{M}$ be a factor von Neumann algebra without central abelian projections. Suppose that $\delta = \{\delta^{(n)}\}_{n\in\mathbb{N}}$ from $\mathcal{M}$ into itself is an additive Jordan higher derivable map at commutative zero point, then there exist an additive higher derivation $\phi = \{\phi^{(n)}\}_{n\in\mathbb{N}}$ from $\mathcal{M}$ into itself and a sequence of scalars $\lambda^{(n)}$ such that $\delta^{(n)}(A) = \phi^{(n)}(A) + \lambda^{(n)}A$ for all $n \geq 1$, $A \in \mathcal{M}$.*

Before concluding this paper, we pose a question. How about the similar Jordan derivable maps of generalized matrix algebras?

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