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NIELSEN SPECTRUM OF MAPS ON INFRA-SOLVMANIFOLDS MODELED ON Sol_0^4

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Abstract. The 4-dimensional solvable Lie group Sol_0^4 does not admit a lattice. The purpose of this paper is two-fold. We study poly-crystallographic groups of Sol_0^4, and then we study Nielsen fixed point theory on the spaces modeled on Sol_0^4.

1. Introduction

It is well known that every connected, simply connected nilpotent Lie group admits a lattice, i.e., a discrete cocompact subgroup. The famous Bieberbach's first theorem is available for nilpotent Lie groups as proved by Auslander, see [22]. Hence finding almost crystallographic groups and almost Bieberbach groups is a great concern, because such groups give rise to infra-nilorbifolds and infra-nilmanifolds, and infra-nilmanifolds are almost flat manifolds.

The study of Nielsen fixed point theory on infra-nilmanifolds has been successful because such manifolds allow finite regular coverings by nilmanifolds, and as a result, one can use the averaging formula for Nielsen numbers, [1, 7, 14–16, 19, 20]. Certain solvable Lie groups of type (R) (completely solvable, or supersolvable) allow lattices. The 3-dimensional solvable Lie group Sol is such an example. If this is the case, one can extend the averaging formula for Nielsen numbers and use it to study Nielsen fixed point theory on infra-solvmanifolds of type (R), [2–6, 8, 12, 13, 17, 21].

Among 4-dimensional solvable Lie groups, only Sol_0^4 and $\mathrm{Sol}_0'^4$ do not have a lattice. Hence we cannot study their crystallographic groups nor Bieberbach groups. $\mathrm{Sol}_0'^4$ does not have a compact-form, yet Sol_0^4 has infinitely many compact-forms, see [18]. In particular, the averaging formula is not applicable.

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The purpose of this paper is to study compact-forms of Sol_0^4, and then to study Nielsen fixed point theory on compact spaces modeled on Sol_0^4.

2. Poly-crystallographic groups of $\mathbf{Sol_0^4}$

2.1. The Lie group $\mathbf{Sol_0^4}$

Recall, for example from [18], that $\mathrm{Sol}_0^4 = \mathbb{R}^3 \rtimes_\psi \mathbb{R}$ where

$$\psi(s) = \begin{bmatrix} e^s & 0 & 0 \\ 0 & e^s & 0 \\ 0 & 0 & e^{-2s} \end{bmatrix}.$$

Then it can be embedded in $\mathrm{Aff}(4)$ as

$$\left\{ \begin{bmatrix} \psi(s) & 0 & \mathbf{x} \\ 0 & 1 & s \\ 0 & 0 & 1 \end{bmatrix} \right\} \subset \mathrm{Aff}(4) \subset \mathrm{GL}(5, \mathbb{R}),$$

where $\psi(s) \in \mathrm{GL}(3, \mathbb{R})$, $s \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^3$ is a column vector. Furthermore, it can be seen that

$$\mathrm{Aut}(\mathrm{Sol}_0^4) = \mathrm{GL}(2, \mathbb{R}) \times \mathrm{GL}(1, \mathbb{R}),$$

which is generated by

$$\begin{bmatrix} p_{11} & p_{12} & 0 & 0 & 0 \\ p_{21} & p_{22} & 0 & 0 & 0 \\ 0 & 0 & p_{33} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

A maximal compact subgroup is $\mathrm{O}(2) \times \mathrm{O}(1)$, and its identity component is $\mathrm{SO}(2)$.

Recall also that Sol_0^4 does not admit a discrete cocompact subgroup. So, this is a type (R) (i.e., supersolvable) counter-example to the generalized Bieberbach's first theorem.

Consequently, we shall be interested in classifying discrete cocompact subgroups of $\mathrm{Sol}_0^4 \rtimes (\mathrm{O}(2) \times \mathrm{O}(1))$. We call such a group a *poly-crystallographic group* of Sol_0^4, see [26, Theorem 3] for details of poly-crystallographic groups.

2.2. Poly-crystallographic groups of $\mathbf{Sol_0^4}$

Let $G := \mathrm{Sol}_0^4 \rtimes \mathrm{SO}(2) = (\mathbb{R}^3 \rtimes_\psi \mathbb{R}) \rtimes \mathrm{SO}(2)$. Here, $\mathrm{SO}(2)$ acts on $\mathbb{R}^3 \rtimes_\psi \mathbb{R}$ as

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} * (\mathbf{x}, s) := \left(\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{x}, s \right).$$

Write

$$r(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then $G = \mathbb{R}^3 \rtimes_{\psi'} (\mathbb{R}\times \mathrm{SO}(2))$, where ψ' is determined by $\psi(s)r(\theta) = r(\theta)\psi(s)$.

Let $\Gamma \subset G$ be a poly-crystallographic group of Sol_0^4. The nilradical of $\mathrm{Sol}_0^4 = \mathbb{R}^3 \rtimes_\psi \mathbb{R}$ is $\mathbb{R}^3$, and $G/\mathbb{R}^3 = \mathbb{R}\times \mathrm{SO}(2)$. Consider the commutative diagram between short exact sequences:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{R}^3 & \longrightarrow & G & \longrightarrow & \mathbb{R}\times \mathrm{SO}(2) & \longrightarrow & 1 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & \Gamma\cap\mathbb{R}^3 & \longrightarrow & \Gamma & \longrightarrow & \Gamma/(\Gamma\cap\mathbb{R}^3) & \longrightarrow & 1
\end{array}$$

By [26, Proposition 5.1], $\Gamma \cap \mathbb{R}^3 = \mathbb{Z}^3$ is a lattice of $\mathbb{R}^3$. Therefore, Γ is of the form $\mathbb{Z}^3 \rtimes K$ where $K := \Gamma/(\Gamma\cap\mathbb{R}^3)$ is a discrete cocompact subgroup of $\mathbb{R}\times \mathrm{SO}(2)$.

Next, we will study discrete cocompact subgroups of $\mathbb{R}\times\mathrm{SO}(2)$. Let K_1 and K_2 denote the projections of K into the two factors, respectively. Then K is a subgroup of $K_1 \times K_2$. In general, a subgroup of a product group need not itself be a product group. We remark that K_1 must be a discrete cocompact subgroup of $\mathbb{R}$, and hence $K_1 \cong \mathbb{Z}$. Note also that K_2 is a subgroup of $\mathrm{SO}(2)$, and hence it is a finite cyclic group or an infinite cyclic group (which is dense in $\mathrm{SO}(2)$).

If K_2 is a finite group then $K' := \mathrm{pr}_2^{-1}(\{1\})$ is a finite index subgroup of K where $\mathrm{pr}_2 : \mathbb{R}\times\mathrm{SO}(2) \to \mathrm{SO}(2)$ is the projection onto the second factor. Then the inverse image Γ' of K' under $\Gamma \to K$ is a finite index subgroup of Γ. We can regard K' as a subgroup of $\mathbb{R}$, and Γ' as a subgroup of $\mathbb{R}^3 \rtimes_\psi \mathbb{R}$. However, this is impossible as $\mathrm{Sol}_0^4 = \mathbb{R}^3 \rtimes_\psi \mathbb{R}$ does not admit such a subgroup (a lattice). Therefore K_2 must be an infinite cyclic subgroup of $\mathrm{SO}(2)$. That is, K_2 is of the form

$$K_2 = \left\{ \mathcal{R}(n\theta_0) = \begin{bmatrix} \cos(n\theta_0) & \sin(n\theta_0) \\ -\sin(n\theta_0) & \cos(n\theta_0) \end{bmatrix} \mid n \in \mathbb{Z} \right\} \subset \mathrm{SO}(2),$$

where θ_0 is an irrational multiple of π.

Let $K_i' = K \cap K_i$ for $i = 1, 2$. Then we can see that

$$K/(K_1' \times K_2') \cong K_1/K_1' \cong K_2/K_2'.$$

Indeed, the kernel of the canonical map $K \to K_i/K_i'$ is $K_1' \times K_2'$. We claim that $K_1' = K_2' = \{1\}$. Assume $K_1' \neq \{1\}$. Then the inverse image Γ_1' of K_1' under $\Gamma \to K$ sits inside Sol_0^4 and it fits the short exact sequence $1 \to \mathbb{Z}^3 \to \Gamma_1' \to K_1' \to 1$, hence it is a lattice of Sol_0^4 which is impossible. Thus $K_1' = \{1\}$. Since as a group $K_2 \cong K_1$, we have $K_2' \cong K_1' = \{1\}$.

Consequently, $K \cong \mathbb{Z}$, and $\Gamma \cong \mathbb{Z}^3 \rtimes_A \mathbb{Z} \subset \mathbb{R}^3 \rtimes (\mathbb{R}\times\mathrm{SO}(2))$ where

$$\mathbb{Z} \longrightarrow \mathbb{R}\times\mathrm{SO}(2), \quad n \longmapsto (ns_0, \mathcal{R}(n\theta_0))$$

for some nonzero $s_0, \theta_0 \in \mathbb{R}$ where θ_0 is an irrational multiple of π. This implies that there exists a matrix $P \in \mathrm{GL}(3,\mathbb{R})$ such that

$$PAP^{-1} = \begin{bmatrix} e^{s_0} & 0 & 0 \\ 0 & e^{s_0} & 0 \\ 0 & 0 & e^{-2s_0} \end{bmatrix} \begin{bmatrix} \cos\theta_0 & \sin\theta_0 & 0 \\ -\sin\theta_0 & \cos\theta_0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \psi(s_0) r(\theta_0).$$

With $\varphi : \mathbb{R} \to \mathrm{GL}(3,\mathbb{R}), t \mapsto \varphi(t) = \psi(ts_0) r(t\theta_0)$, the $\mathbb{Z}$-action on $\mathbb{R}^3$ is $n * \mathbf{x} = \varphi(n)(\mathbf{x}) = PA^nP^{-1}(\mathbf{x})$. The embedding $\mathbb{Z}^3 \rtimes_A \mathbb{Z} \hookrightarrow \mathbb{R}^3 \rtimes_{\psi'} (\mathbb{R} \times \mathrm{SO}(2))$ is given explicitly as

$$(\mathbf{m}, n) \longmapsto (P\mathbf{m}, (ns_0, \mathcal{R}(n\theta_0))).$$

We remark that A has a positive real eigenvalue $a = e^{-2s_0}$ and complex eigenvalues $e^{s_0 \pm i\theta_0}$. By taking $-s_0$ or equivalently by taking A^{-1} if it is necessary, we may assume that $a > 1$.

The converse is known, see for example [10,25]. If $A \in \mathrm{SL}(3,\mathbb{Z})$ has positive real eigenvalue $a \neq 1$ and complex eigenvalues, then the group $\mathbb{Z}^3 \rtimes_A \mathbb{Z}$ can be realized as a discrete cocompact subgroup of $\mathrm{Sol}_0^4 \rtimes \mathrm{SO}(2)$. The Lie group $\mathrm{Sol}_0^4 \rtimes \mathrm{SO}(2)$ acts on $\mathbb{C} \times \mathbb{H}$ by isometries and the complex surface $\Gamma\backslash(\mathbb{C} \times \mathbb{H})$ has finite volume, called an Inoue surface with Sol_0^4-geometry.

It is shown in [18, Theorem 4.3] that there are countably infinite distinct, discrete cocompact subgroups $\Gamma \cong \mathbb{Z}^3 \rtimes \mathbb{Z}$ in $\mathrm{Sol}_0^4 \rtimes \mathrm{SO}(2)$.

Let Γ be any poly-crystallographic group of Sol_0^4. By definition, $\Gamma \subset \mathrm{Sol}_0^4 \rtimes (\mathrm{O}(2) \times \mathrm{O}(1))$. However, by [18, Theorem 4.3] again, we must have $\Gamma \subset \mathrm{Sol}_0^4 \rtimes \mathrm{SO}(2)$. In summary, there are countably infinite many poly-crystallographic groups of Sol_0^4, all of them of the form $\mathbb{Z}^3 \rtimes \mathbb{Z}$. In particular, they are torsion-free.

3. Infra-solvmanifolds modeled on Sol_0^4

Let $\Gamma = \mathbb{Z}^3 \rtimes_A \mathbb{Z}$ be a poly-crystallographic group with an embedding $\imath : \Gamma \hookrightarrow \mathrm{Sol}_0^4 \rtimes \mathrm{SO}(2) = \mathbb{R}^3 \rtimes_{\psi'} (\mathbb{R} \times \mathrm{SO}(2))$ so that its image in $\mathrm{SO}(2)$ is dense. As a result, $\mathrm{Sol}_0^4 \rtimes \mathrm{SO}(2)$ is the *Lie hull* of $\imath(\Gamma)$ in $\mathrm{Sol}_0^4 \rtimes \mathrm{SO}(2)$. The orbit space $M = \Gamma\backslash\mathrm{Sol}_0^4 = \Gamma\backslash(\mathrm{Sol}_0^4 \rtimes \mathrm{SO}(2))/\mathrm{SO}(2)$ is an infra-solvmanifold modeled on the supersolvable Lie group Sol_0^4.

In this section, we go one step further to examine such infra-solvmanifolds closely. Similarly, infra-solvmanifolds modeled on Sol_1^4 were studied in [23,24]. Unlike Sol_0^4, Sol_1^4 admits a lattice.

Regarding $\imath$ as inclusion, we have $\Gamma \cap \mathrm{Sol}_0^4 = \mathbb{Z}^3$, and its Lie hull in Sol_0^4 and hence in $\mathrm{Sol}_0^4 \rtimes \mathrm{SO}(2)$ is $\mathbb{R}^3$, which is the nilradical of Sol_0^4. Consider the following commutative diagram between short exact sequences:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{R}^3 & \longrightarrow & \mathrm{Sol}_0^4 \rtimes \mathrm{SO}(2) & \longrightarrow & \mathbb{R} \times \mathrm{SO}(2) & \longrightarrow & 1 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & \mathbb{Z}^3 & \longrightarrow & \Gamma & \longrightarrow & \mathbb{Z} & \longrightarrow & 1 \end{array}$$

where the rightmost vertical map is given by $n \mapsto (ns_0, \mathcal{R}(n\theta_0))$, and hence $\mathbb{Z}$ acts on $\mathbb{R}^3$ via $\varphi(1) = PAP^{-1}$ of Section 2. Furthermore, since $\Gamma = \mathbb{Z}^3 \rtimes_A \mathbb{Z}$, the universal covering projection $\mathrm{Sol}_0^4 \to M$ has a decomposition by covering projections

$$\mathrm{Sol}_0^4 \longrightarrow \bar{M} := \mathbb{Z}^3\backslash\mathrm{Sol}_0^4 \xrightarrow{\backslash\mathbb{Z}} M = \Gamma\backslash\mathrm{Sol}_0^4.$$

On the other hand, the space $\bar{M}$ fits the fibration

$$\mathbb{Z}^3\backslash\mathbb{R}^3 \longrightarrow \bar{M} = \mathbb{Z}^3\backslash(\mathbb{R}^3 \rtimes_\psi \mathbb{R}) \longrightarrow \mathbb{R}$$

with fiber $\mathbb{T}^3$ and base $\mathbb{R}$. The action by the transformation group $\mathbb{Z}$ on the orbit space $\bar{M}$ is fiber preserving. Consequently, we have the fibration

$$\mathbb{Z}^3\backslash\mathbb{R}^3 \longrightarrow \mathbb{Z}\backslash\bar{M} = M \longrightarrow \mathbb{Z}\backslash\mathbb{R}$$

with fiber $\mathbb{T}^3$ and base S^1. This implies that the infra-solvmanifold $M = \Gamma\backslash\mathrm{Sol}_0^4$ is the mapping torus M_φ of the diffeomorphism $\varphi(1) = PAP^{-1} : \mathbb{T}^3 \to \mathbb{T}^3$. Clearly it is diffeomorphic to the mapping torus M_A of the diffeomorphism $A : \mathbb{T}^3 \to \mathbb{T}^3$.

Conversely a mapping torus M_A of $A \in \mathrm{GL}(3, \mathbb{Z})$ has infra-solvmanifold structure modeled on Sol_0^4 if $A \in \mathrm{SL}(3, \mathbb{Z})$ has a positive real eigenvalue and two complex eigenvalues. We refer to [25] for details.

4. Nielsen spectrum of maps on the mapping torus M_A

The infra-solvmanifold $\Gamma\backslash\mathrm{Sol}_0^4$ with fundamental group $\Gamma = \mathbb{Z}^3 \rtimes_A \mathbb{Z}$ is diffeomorphic to the mapping torus M_A. In this section, we shall study the set of Reidemeister numbers of all endomorphisms on Γ, and the sets of Lefschetz numbers and Nielsen numbers of all self-maps on M_A

Lemma 4.1. *If $A \in \mathrm{GL}(n, \mathbb{Z})$ does not have an eigenvalue 1, then the subgroup $\mathbb{Z}^n$ of the group $\Gamma = \mathbb{Z}^n \rtimes_A \mathbb{Z}$ is a fully invariant subgroup.*

Proof. Since the quotient group $\Gamma/\mathbb{Z}^n = \mathbb{Z}$ is abelian, we have $[\Gamma, \Gamma] \subset \mathbb{Z}^n$. Since the group Γ can be presented as

$$\Gamma = \langle x_1, \ldots, x_n, t \mid [x_i, x_j] = 1,\ t x_i t^{-1} = \theta(x_i)\rangle,$$

it follows that $[\Gamma, \Gamma]$ is the image of the homomorphism $I - A : \mathbb{Z}^n \to \mathbb{Z}^n$. Hence the index of $[\Gamma, \Gamma]$ in $\mathbb{Z}^n$ equals $|\det(I - A)|$. By the assumption on A, $\det(I - A) \neq 0$. Thus, $[\Gamma, \Gamma]$ has finite index in $\mathbb{Z}^n$.

Now we shall show that $\mathbb{Z}^n$ is a fully invariant subgroup of Γ. Let ϕ be an endomorphism of Γ and let $\mathbf{m} \in \mathbb{Z}^n$. Then $\phi(\mathbf{m}) = (\mathbf{m}', k) \in \mathbb{Z}^n \rtimes_\theta \mathbb{Z}$ for some $\mathbf{m}' \in \mathbb{Z}^n$ and $k \in \mathbb{Z}$. Since $[\Gamma, \Gamma]$ has finite index in $\mathbb{Z}^n$, $\mathbf{m}^\ell \in [\Gamma, \Gamma]$ for some $\ell \in \mathbb{Z}$, and since $[\Gamma, \Gamma]$ is a fully invariant subgroup of Γ, $\phi(\mathbf{m}^\ell) \in [\Gamma, \Gamma]$. Observe that

$$\phi(\mathbf{m}^\ell) = \phi(\mathbf{m})^\ell = (\mathbf{m}', k)^\ell = (\mathbf{m}'', k\ell) \in [\Gamma, \Gamma]$$

for some $\mathbf{m}'' \in \mathbb{Z}^n$. This implies that $k = 0$, hence $\phi(\mathbf{m}) = (\mathbf{m}', 0) \in \mathbb{Z}^n$. Consequently, $\mathbb{Z}^n$ is a fully invariant subgroup of Γ. □

Let $f : M_A \to M_A$ be a self-map, inducing an endomorphism $\phi : \Gamma \to \Gamma$ on the group Γ of covering transformations of the universal cover $\mathrm{Sol}_0^4 \to M_A$. By Lemma 4.1, $\Gamma \cap \mathrm{Sol}_0^4 = \mathbb{Z}^3$ is a fully invariant subgroup of Γ, and hence we obtain the following commutative diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z}^3 & \longrightarrow & \Gamma & \longrightarrow & \mathbb{Z} & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \phi'} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \bar{\phi}} & & \\ 1 & \longrightarrow & \mathbb{Z}^3 & \longrightarrow & \Gamma & \longrightarrow & \mathbb{Z} & \longrightarrow & 1 \end{array} \tag{4.1}$$

This commutative diagram induces the identity

$$\phi' \circ \varphi(1) = \varphi(\bar{\phi}(1)) \circ \phi'.$$

With a suitable basis for $\mathbb{Z}^3$, if B is the matrix of ϕ' and $\bar{\phi}(1) = k$, then the above identity means that

$$A^k B = BA. \tag{4.2}$$

Moreover, from the commutative diagram (4.1) we may assume that f is a fiber-preserving map so that the following diagram is commutative:

$$\begin{array}{ccccc} \mathbb{T}^3 & \longrightarrow & M_A & \longrightarrow & S^1 \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \bar{f}} \\ \mathbb{T}^3 & \longrightarrow & M_A & \longrightarrow & S^1 \end{array} \tag{4.3}$$

4.1. Converse of (4.1)

Given endomorphisms $\phi' : \mathbb{Z}^3 \to \mathbb{Z}^3$ and $\bar{\phi} : \mathbb{Z} \to \mathbb{Z}$, we want to know if there is an endomorphism $\phi : \Gamma \to \Gamma$ fitting the commutative diagram (4.1). Assume (4.1) is given. Then

$$\begin{aligned} \phi(\mathbf{m}, n) &= \phi\left((\mathbf{m}, 0)(\mathbf{0}, n)\right) \\ &= (\phi'(\mathbf{m}), 0)(\xi(n), \bar{\phi}(n)) \\ &= (\phi'(\mathbf{m}) + \xi(n), \bar{\phi}(n)) \end{aligned} \tag{4.4}$$

for $(\mathbf{m}, n) \in \Gamma$, where $\xi : \mathbb{Z} \to \mathbb{Z}^3$. Since ϕ is an endomorphism, one can show that ξ is a crossed homomorphism, i.e.,

$$\xi(m + n) = \xi(m) + \theta(\bar{\phi}(m))\xi(n) = \xi(m) + A^{km}\xi(n). \tag{4.5}$$

Such a crossed homomorphism is determined by the image $\xi(1)$.

In summary, we see that with ϕ' and $\bar{\phi}$, simply by choosing $\xi(1) \in \mathbb{Z}^3$ we can define an endomorphism ϕ so that the commutative diagram (4.1) is obtained.

4.2. Analysis of (4.2)

Recall

$$PAP^{-1}=\begin{bmatrix} e^{s_0} & 0 & 0\\ 0 & e^{s_0} & 0\\ 0 & 0 & e^{-2s_0}\end{bmatrix}\begin{bmatrix} \cos\theta_0 & \sin\theta_0 & 0\\ -\sin\theta_0 & \cos\theta_0 & 0\\ 0 & 0 & 1\end{bmatrix} =: \begin{bmatrix} E & \mathbf{0}\\ \mathbf{0} & e^{-2s_0}\end{bmatrix}\begin{bmatrix} R & \mathbf{0}\\ \mathbf{0} & 1\end{bmatrix}$$

and let

$$PBP^{-1}=\begin{bmatrix} a & b & x\\ c & d & y\\ u & v & w\end{bmatrix} =: \begin{bmatrix} \mathcal{B} & \mathbf{x}\\ \mathbf{u} & w\end{bmatrix},$$

where $\mathbf{x},\mathbf{u}$ are 2-dimensional column vector and row vector respectively. If $k=1$, the identity (4.2) yields that

$$PBP^{-1}=\begin{bmatrix} a & b & 0\\ -b & a & 0\\ 0 & 0 & w\end{bmatrix}.$$

Assume $k\neq 1$. Denoting $\mathcal{N}:=ER$, by (4.2) we have $\mathcal{N}^k\mathcal{B}=\mathcal{B}\mathcal{N}$. Since $\mathcal{N}$ has complex eigenvalues $e^{s_0\pm i\theta_0}$, it can be diagonalized with its eigenvalues on the diagonal entries. It follows that $\mathcal{B}=0$ since the modulus of our eigenvalues are not roots of unity. We can show further that B itself is the zero matrix.

4.3. The Reidemeister number $R(f)$ of f

First we determine the Reidemeister number $R(f)$ of f. By definition, $R(f)$ is just the Reidemeister number $R(\phi)$ of the endomorphism ϕ, which is the cardinality of the Reidemeister set $\mathcal{R}[\phi]$.

Recall from [6, Sect. 1] or [9, Lemma 3.1] that the commutative diagram (4.1) induces a 6-term exact sequence between fixed point groups and Reidemeister sets:

$$1\longrightarrow \mathrm{fix}(\phi')\longrightarrow \mathrm{fix}(\phi)\longrightarrow \mathrm{fix}(\bar\phi)\longrightarrow \mathcal{R}[\phi']\longrightarrow \mathcal{R}[\phi]\longrightarrow \mathcal{R}[\bar\phi]\longrightarrow 1.$$

It is clear that $\mathrm{fix}(\bar\phi)=\mathbb{Z}$ if $k=1$, and $\mathrm{fix}(\bar\phi)=\{0\}$ if $k\neq 1$. It is well-known that $R(\bar\phi)=\sigma(1-k)$ and $R(\phi')=\sigma(\det(I-M))$, where $\sigma:\mathbb{R}\to\mathbb{R}\cup\{\infty\}$ is defined by $\sigma(0)=\infty$ and $\sigma(x)=|x|$ for all $x\neq 0$.

It is immediate that if $k=1$, then $R(\bar\phi)=\infty$, and hence from the exact sequence we have $R(\phi)=\infty$. In particular, we have

$$R(\phi)=\infty=R(\phi')\cdot R(\bar\phi).$$

On the other hand, if $k\neq 1$ we have a short exact sequence of sets $1\to\mathcal{R}[\phi']\to\mathcal{R}[\phi]\to\mathcal{R}[\bar\phi]\to 1$ with the finite quotient set $\mathcal{R}[\bar\phi]$ of $|1-k|$ elements, and

$$R(\phi)=R(\phi')\cdot R(\bar\phi)=|1-k|\cdot R(\phi'),$$

hence $R(\phi)=\infty$ if and only if $R(\phi')=\infty$ if and only if $\det(I-M)=0$ if and only if M has an eigenvalue 1.

Proposition 4.2. *Let $f : M \to M$ be a self-map on an infra-solvmanifold $M = \Gamma\backslash\mathrm{Sol}_0^4$ modeled on Sol_0^4. Then the Reidemeister number of f is*

$$R(f) = \sigma(1-k).$$

In particular, $R(f) = \infty$ if and only if $k = 1$. The Reidemeister spectrum of M is $\mathrm{Spec}_R(M) = \mathbb{N} \cup \{\infty\}$.

4.4. The Nielsen number $N(f)$ of f

Because the supersolvable Lie group Sol_0^4 does not admit a lattice, the infra-solvmanifold $M_A = \Gamma\backslash\mathrm{Sol}_0^4$ cannot be finitely covered by a solvmanifold. Therefore, in calculating the Nielsen numbers of self-maps on M_A, one can not use the averaging formula for Nielsen numbers, [17]. However, since M_A has fibration structure, we may use the fibration technique employed in Jiang's book, [11].

A fibration $E \to B$ with fiber F is *orientable* if for every element $[\omega] \in \pi_1(B)$ the induced map $\tau_\omega : F \to F$ on F is homotopic to the identity. We note that our fiber bundle M_A over S^1 with fiber $\mathbb{T}^3$ is non-orientable. The action of $\pi_1(S^1)$ on the fiber $\mathbb{T}^3$ induces an action on $\mathbb{Z}^3 = \pi_1(\mathbb{T}^3)$ which can be represented by conjugation of $\mathbb{Z}^3$ by elements of $\mathbb{Z} = \pi_1(S^1)$. Since the induced $\mathbb{Z}$-action on $\mathbb{Z}^3$ is given by A, it shows that our fibration is not orientable.

Assume we are given the commutative diagram (4.3). Note that $N(\bar{f}) = |L(\bar{f})| = |1-k|$. If $k = 1$, then $\bar{f}$ and hence f are homotopic to fixed point free maps, hence $N(f) = 0$. That is, every fixed point class of f has index 0, hence $L(f) = 0$.

Now we shall consider the case where $k \neq 1$. We may assume $\bar{f} : S^1 \to S^1$ is given by $z \mapsto z^k$. The set of fixed points of $\bar{f}$ is

$$\mathrm{Fix}(\bar{f}) = \{z_\ell := e^{2\pi i \frac{\ell}{k-1}} \mid \ell = 0, 1, \ldots, |k-2|\},$$

and every fixed point class of $\bar{f}$ contains a single fixed point, each of the same index ± 1. Henceforth, we will identify the set of fixed point classes of $\bar{f}$, $\mathrm{FPC}(\bar{f})$, with $\mathrm{Fix}(\bar{f})$.

For $z_\ell = z_1^\ell \in \mathrm{Fix}(\bar{f})$, let $f_\ell : \mathbb{T}_\ell^3 \to \mathbb{T}_\ell^3$ be the restriction of $f : M_A \to M_A$ to the fiber $\mathbb{T}_\ell^3$ over z_ℓ. There is a sequence of maps:

$$\mathrm{FPC}(f_\ell) \xrightarrow{i_{\mathrm{FPC}}} \mathrm{FPC}(f) \xrightarrow{\pi_{\mathrm{FPC}}} \mathrm{FPC}(\bar{f}).$$

Let $\tilde{f} : \mathrm{Sol}_0^4 \to \mathrm{Sol}_0^4$ be a (fixed) lifting of f. Then every lifting of f is of the form $\gamma \circ \tilde{f}$ for some covering transformation $\gamma \in \Gamma$. Recall that every fixed point class of f is of the form $\mathbb{F}_\gamma = p\left(\mathrm{Fix}\left(\gamma \circ \tilde{f}\right)\right)$. For $\gamma \in \Gamma$, we denote by $\mu(\gamma)$ the conjugation by γ, and by $\bar{\gamma} \in \mathbb{Z}$ the image of γ under the projection

$\Gamma \to \mathbb{Z}$. From the commutative diagram (4.1), we have a commutative diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z}^3 & \longrightarrow & \Gamma & \stackrel{\pi}{\longrightarrow} & \mathbb{Z} & \longrightarrow & 1 \\ & & \Big\downarrow{\scriptstyle \mu(\gamma)\circ\phi'} & & \Big\downarrow{\scriptstyle \mu(\gamma)\circ\phi} & & \Big\downarrow{\scriptstyle \mu(\bar\gamma)\circ\bar\phi} & & \\ 1 & \longrightarrow & \mathbb{Z}^3 & \longrightarrow & \Gamma & \longrightarrow & \mathbb{Z} & \longrightarrow & 1 \end{array} \tag{4.6}$$

By [11, Proposition III.1.5], we know that

$$\# i_{\mathrm{FPC}}(\mathbb{F}_\gamma) = [\mathrm{fix}(\mu(\bar\gamma)\circ\bar\phi) : \pi(\mathrm{fix}(\mu(\gamma)\circ\phi))].$$

In our case, $\mu(\bar\gamma)\circ\bar\phi = \bar\phi$ is the multiplication by the integer $k \neq 1$. Hence $\mathrm{fix}(\bar\phi) = \{0\}$. This proves that $i_{\mathrm{FPC}} : \mathrm{FPC}(f_\ell) \to \mathrm{FPC}(f)$ is injective for all $z_\ell \in \mathrm{Fix}(\bar f)$. Moreover, we can see that $\pi_{\mathrm{FPC}} : \mathrm{FPC}(f) \to \mathrm{FPC}(\bar f) = \mathrm{Fix}(\bar f)$ is surjective. For any $z_\ell \in \mathrm{Fix}(\bar f)$, when $k \neq 1$, we previously observed that the matrix induced by f_ℓ is a zero matrix. This implies that f_ℓ is homotopic to a constant map, hence has a unique (essential) fixed point class $\mathbb{F}'_\ell$, so $L(f_\ell) = N(f_\ell) = 1$ is independent of z_ℓ. This yields a fixed point class $\mathbb{F}_\ell$ of f which is mapped to the fixed point class z_ℓ. Furthermore,

$$\mathrm{index}(\mathbb{F}_\ell) = \mathrm{index}(z_\ell)\cdot \mathrm{index}(\mathbb{F}'_\ell).$$

In summary, we have shown that when $k \neq 1$,

$$\mathrm{FPC}(f) = \bigcup_{z_\ell \in \mathrm{Fix}(\bar f)} \mathrm{FPC}(f_\ell).$$

Furthermore, we have

$$\begin{aligned} L(f) &= \sum_{z_\ell \in \mathrm{Fix}(\bar f)} \mathrm{index}(\mathbb{F}_\ell) = \sum_{z_\ell \in \mathrm{Fix}(\bar f)} \mathrm{index}(z_\ell) = L(\bar f), \\ N(f) &= \sum_{z_\ell \in \mathrm{Fix}(\bar f)} N(f_\ell) = \sum_{z_\ell \in \mathrm{Fix}(\bar f)} 1 = N(\bar f), \\ |L(f)| &= N(f). \end{aligned}$$

Theorem 4.3. *Let $f : M \to M$ be a self-map on an infra-solvmanifold $M = \Gamma\backslash\mathrm{Sol}_0^4$ modeled on Sol_0^4. Then the Lefschetz number and the Nielsen number of f are*

$$L(f) = 1 - k,\ \ N(f) = |1 - k|.$$

In particular, $L(f) = 0$ if and only if $N(f) = 0$ if and only if $k = 1$. The Lefschetz spectrum and the Nielsen spectrum of M are $\mathrm{Spec}_L(M) = \mathbb{Z}$ and $\mathrm{Spec}_N(M) = \mathbb{N}\cup\{0\}$.

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