Bull. Korean Math. Soc. **0** (0), No. 0, pp. 1–0
https://doi.org/10.4134/BKMS.b190570
pISSN: 1015-8634 / eISSN: 2234-3016

ON WELL-POSEDNESS AND BLOW-UP CRITERION FOR THE 2D TROPICAL CLIMATE MODEL

Mulan Zhou

Abstract. In this paper, we consider the Cauchy problem to the tropical climate model. We establish the global regularity for the 2D tropical climate model with generalized nonlocal dissipation of the barotropic mode and obtain a multi-logarithmical vorticity blow-up criterion for the 2D tropical climate model without any dissipation of the barotropic mode.

1. Introduction

In this paper, we investigate the following tropical climate model given by

$$(1)\qquad \begin{cases} \partial_t u + u\cdot\nabla u + \nabla\Pi + \mu\Lambda^{2\alpha}u + \mathrm{div}(v\otimes v) = 0, & x\in\mathbb{R}^2,\ t>0,\\ \partial_t v + u\cdot\nabla v + \nu\Lambda^{2\beta}v + v\cdot\nabla u + \nabla\theta = 0, & x\in\mathbb{R}^2,\ t>0,\\ \partial_t\theta + u\cdot\nabla\theta + \eta\Lambda^{2\gamma}\theta + \mathrm{div}v = 0, & x\in\mathbb{R}^2,\ t>0,\\ \mathrm{div}u = 0, & x\in\mathbb{R}^2,\ t\geq 0,\\ (u,v,\theta)(t,x)|_{t=0} = (u_0,v_0,\theta_0)(x), & x\in\mathbb{R}^2, \end{cases}$$

here $\mu,\nu,\eta,\alpha,\beta,\gamma$ are non-negative parameters, $u=(u_1(t,x),u_2(t,x))$ and $v=(v_1(t,x),v_2(t,x))$ stand for the barotropic mode and the first baroclinic mode of the vector velocity, respectively. $\Pi=\Pi(t,x)$ and $\theta=\theta(t,x)$ denote the scalar pressure and scalar temperature, respectively. $\mathrm{div}u=0$ represents the incompressibility condition. $\Lambda=\sqrt{-\Delta}$ is the Zygmund operator and the fractional power operator $\Lambda^\alpha=(-\Delta)^{\frac{\alpha}{2}}$ with $0<\alpha<2$ is defined in the sense of Cauchy principal value (see e.g. [12, 24, 27, 28]),

$$\Lambda^\alpha u(x) = c_\alpha \lim_{\varepsilon\to 0^+}\int_{\{y\in\mathbb{R}^d:|y|\geq\varepsilon\}}\frac{u(x)-u(x-y)}{|y|^{d+\alpha}}dy,$$

where the dimensional constant $c_\alpha=\frac{\alpha 2^{\alpha-1}\Gamma((d+\alpha)/2)}{\pi^{d/2}\Gamma(1-\alpha/2)}$.

The original version of (1) without any fractional Laplacian terms was derived by Frierson-Majda-Pauluis [10] from the inviscid primitive equations with

Received June 10, 2019; Revised August 18, 2019; Accepted September 5, 2019.
2010 *Mathematics Subject Classification.* Primary 35D35, 76D03.
Key words and phrases. tropical climate model, global regularity, blow-up criterion.
The work is supported by Natural Science Foundation of Anhui Province (No.1908085 QA05).

the aid of performing a Galerkin truncation to the hydrostatic Boussinesq equations, of which the first baroclinic mode had been originally used in some studies of tropical atmosphere. More relevant background on the tropical climate model can be found in [11, 22, 23] and the references therein. From the mathematical point of view, the tropical climate model (1) are significant generalizations of the generalized magnetohydrodynamic (GMHD) equations which model the complex interaction between the fluid dynamic phenomena. In fact, when the temperature $\theta \equiv$ Constant, (1) is reduced to the GMHD equations, namely,

$$(2) \quad \begin{cases} \partial_t u + u\cdot\nabla u + \nabla\Pi + \mu\Lambda^{2\alpha}u - v\cdot\nabla v = 0, & x\in\mathbb{R}^2,\ t>0,\\ \partial_t v + u\cdot\nabla v + \nu\Lambda^{2\beta}v - v\cdot\nabla u = 0, & x\in\mathbb{R}^2,\ t>0,\\ \mathrm{div}u = \mathrm{div}v = 0, & x\in\mathbb{R}^2,\ t\geq 0,\\ (u(0,x),v(0,x)) = (u_0(x),v_0(x)), & x\in\mathbb{R}^2, \end{cases}$$

here $u = u(t,x)$ and $v = v(t,x)$ denote the velocity and the magnetic field of the fluid, respectively. The mathematical studies on the tropical climate model (1) have attracted considerable attention recently from various authors and have motivated a large number of research papers concerning the local well-posedness [19], global small solutions [20, 21], blow-up criterions [26], global regularity and so on. Here, we mainly recall some global regularity results which are more relative with our research in this field. Li-Titi [16] introduced a new quantity to bypass the obstacle caused by the absence of thermal diffusion and proved the global well-posedness of strong solutions for the 2D tropical climate model (1) with $\alpha = \beta = 1$ and $\mu > 0$, $\nu > 0$, $\eta = 0$. Later, Ye [25] obtained the global regularity of a tropical climate model with the very weak dissipation of the barotropic ($\alpha > 0$, $\beta = \gamma = 1$ and $\mu,\nu,\eta > 0$) by the "weakly nonlinear" energy estimates approach and the maximal $L_t^q L_x^p$ regularity for the heat kernel. Dong et al. [6] established the global regularity results for the 2D system (1) without thermal diffusion with $\alpha+\beta = 2$, $\beta\in(1,\frac{3}{2}]$, $\mu,\nu > 0$ and $\alpha = 2$, $\mu > 0$, $\nu = \eta = 0$, respectively. Subsequently, Zhu [31] obtained the global regularity for the 3D tropical climate model with fractional diffusion on barotropic mode ($\alpha \geq \frac{5}{2}$ and $\mu > 0$, $\nu = \eta = 0$). Recently, Dong et al. [7] established the global existence and regularity for the 2D system (1) ($\mu,\nu,\eta > 0$, $\beta = 1$) with the fractional dissipation which are in two very broad ranges, namely,

$$\gamma \geq \frac{4+\alpha-\sqrt{\alpha^2+8\alpha+8}}{2}, \quad \text{if} \quad 0<\alpha<\frac{1}{2}; \quad \gamma\geq 1-\alpha, \quad \text{if} \quad \frac{1}{2}\leq\alpha\leq 1.$$

Dong et al. [5] examined the 2D system (1) with following two cases:

$$\mu = 0, \quad \beta > 1, \quad \beta+\gamma > \frac{3}{2} \quad \text{and} \quad \mu = \eta = 0, \quad \frac{3}{2} < \beta \leq 2,$$

and established the global existence and uniqueness of classical solutions.

However, up to now, the global regularity or finite time singularity for strong solutions of the 2D tropical climate model ($\mu = 0$ and $\nu = \eta = \beta = \gamma = 1$)

with large initial data is still a challenging open problem just like the 2D MHD equations with only magnetic diffusion. Naturally, we ask that how weak dissipation which the u-equations possess to ensure that the system (1) has the global regularity? To answer the problem, we consider the following 2D tropical climate model with a generalized nonlocal velocity dissipation

$$(3)\qquad \begin{cases} \partial_t u + u\cdot\nabla u + \nabla\Pi + \mu\mathcal{L}u + \mathrm{div}(v\otimes v) = 0, \\ \partial_t v + u\cdot\nabla v - \Delta v + v\cdot\nabla u + \nabla\theta = 0, \\ \partial_t\theta + u\cdot\nabla\theta + \Lambda^{2\gamma}\theta + \mathrm{div}v = 0, \\ \mathrm{div}u = 0, \\ (u,v,\theta)(t,x)|_{t=0} = (u_0,v_0,\theta_0)(x), \end{cases}$$

here the generalized dissipative operator $\mathcal{L}$ is defined as follows

$$(4)\qquad \mathcal{L}u(x) = \int_{\mathbb{R}^2} J(y)(u(x)-u(x-y))dy,$$

where the radially symmetric kernel function $J(y) = J(|y|)$ defined on $\mathbb{R}^2\backslash\{0\}$ satisfies that there exists a constant $c \geq 1$ such that

$$(5)\qquad c^{-1}\frac{m(|y|^{-1})}{|y|^2} \leq J(y) \leq c\frac{m(|y|^{-1})}{|y|^2}, \qquad \forall y\in\mathbb{R}^2 : 0<|y|\leq 1;$$

$$(6)\qquad 0 \leq J(y) \leq c\frac{m(|y|^{-1})}{|y|^2}, \qquad \forall y\in\mathbb{R}^2 : |y|>1.$$

Can we find some conditions on m that guarantee that the system (3) is globally well-posed in the smooth category? This is our principal goal of this paper which is motivated by recent work on the nonlocal diffusion equations in [4].

To state our main result, we need to give some assumptions on m.

Let $m : (0,\infty) \to [0,\infty)$ be a radially symmetric non-decreasing smooth function satisfying that there exist two constants $\alpha\in(0,2)$ and $\beta\in[0,\alpha)$ such that

$$(7)\qquad (\alpha-\beta)m(r) \leq rm'(r) \quad \text{for all} \quad 1\leq r<\infty.$$

Now let us state our main results.

Theorem 1.1. *Let $\mu = 1$ and $\frac{2}{3} \leq \gamma \leq 1$. Assume that the operator $\mathcal{L}$ fulfills the conditions (4)-(7) and the initial datas $(u_0,v_0,\theta_0)\in(H^2(\mathbb{R}^2))^3$ and $\mathrm{div}u_0 = 0$. Then the 2D tropical climate model (3) has a unique global smooth solution (u,v,θ) satisfying for any $T>0$*

$$(u,v,\theta)\in C([0,T];(H^2(\mathbb{R}^2)^3))$$

and

$$(u,v,\theta)\in L^2([0,T];H^{2+\kappa}(\mathbb{R}^2)\times H^3(\mathbb{R}^2)\times H^{2+\gamma}(\mathbb{R}^2)).$$

Here and in what follows, we denote $\kappa := \frac{\alpha-\beta}{2}\in(0,1)$.

Remark 1.2. It is worth mentioning that we can take $J(y)\equiv 0$ for all $|y|\geq 1$, then the diffusion operator $\mathcal{L}$ defined by (4) with the kernel conditions (5)-(7) contains a large class of operators, for instance, for $\alpha\in(0,2)$ and $\beta\in[0,\alpha)$,

(1) $m_1(r) = r^\sigma \chi_{[1,\infty)}(r)$ with $\sigma \in [\alpha-\beta, \alpha]$;
(2) $m_2(r) = \frac{r^\alpha}{\ln^\beta(e+r)} \chi_{[1,\infty)}(r)$.

Remark 1.3. In view of the above observation, we only need the information about kernel function $J(y)$ near the origin. Due to this, the generalized dissipative operator $\mathcal{L}$ defined in (4) is weaker than any fractional Laplacian Λ^α, we generalize the result in [25] for the Cauchy problem (3) with $\alpha > 0$ and $\beta = 1$. On the other hand, in the absence of the incompressibility condition $\mathrm{div} v = 0$, we will encounter the obstacle to obtain the key global H^1-estimation for the system (3) by the standard energy method. To overcome this, we will close the H^1-estimation by the well-known logarithmic Sobolev embedding inequality even without the dissipation of the u-equations. Compared with the "weakly nonlinear" energy estimate approach utilized by Ye [25] where the dissipation $\Lambda^{2\alpha} u$ ($\alpha > 0$) plays a crucial role, our method seems more direct.

Remark 1.4. Consider the 2D GMHD equations (2) with $\Lambda^{2\alpha} u$ replaced by $\mathcal{L} u$ (see [9, 29]), where the nonlocal operator $\mathcal{L}$ fulfills the conditions (4)-(7), and following the similar procedure, we can establish the global regularity of 2D GMHD equations (2) with $\beta = 1$ in terms of the initial datas $(u_0(x), v_0(x)) \in H^2(\mathbb{R}^2) \times H^2(\mathbb{R}^2)$.

As mentioned above, the global regularity of the 2D system (3) with $\mu = 0$ remains open. Here, we give a logarithmical blow-up criterion for the two-dimensional case.

Theorem 1.5. *Let $\mu = 0$ and $\frac{2}{3} \le \gamma \le 1$. Given $T > 0$. Assume that $(u_0, v_0, \theta_0) \in (H^s(\mathbb{R}^2))^3$ with $s > 2$ and $\mathrm{div} u_0 = 0$. Let (u, v, θ) be the unique local smooth solution in time interval $[0, T)$ to the Cauchy problem of the 2D tropical climate model (3). If the following condition involving vorticity $w := \mathrm{curl} u$*

$$\int_0^T \frac{||w||_{L^\infty}}{\ln\ln(e^3 + ||w||_{L^\infty}) \ln\ln\ln(e^3 + ||w||_{L^\infty})} ds < \infty, \tag{8}$$

holds true, then the solution $(u(t,x), v(t,x), \theta(t,x))$ can be extended beyond time T.

Remark 1.6. Of course, the regularity criteria (8) is suitable for the 2D MHD equations with magnetic diffusion only which improves the classical Beale-Kato-Majda's criterion in [17]. For more logarithmical blow-up criteria, we refer the readers to [1, 8] and the references therein.

Notations. For convenience, we recall some useful quantities which have been mentioned above:

$$\mathrm{div} u := \partial_1 u_1 + \partial_2 u_2 \quad \text{and} \quad \mathrm{curl} u := \partial_1 u_2 - \partial_2 u_1.$$

We denote the tensor product of two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$ by the matric $\mathbf{u} \otimes \mathbf{v}$ with the entries $\mathbf{u}_i \mathbf{v}_j$, $i, j = 1, 2$. For the sake of simplicity, $a \lesssim b$ means that

$a \leq Cb$ for some "harmless" positive constant C which may vary from line to line. We also use the notation $||f_1, \dots, f_n||_p^2 := ||f_1||_{L^p}^2 + \cdots + ||f_n||_{L^p}^2$.

2. Proof of main theorems

In this section, we prove our main results. Our efforts here are focused on proving the global a priori estimates for u, v, θ.

To begin with, we recall the following calculus inequality will be needed later to prove our results.

Lemma 2.1 ([13, 14])**.** *Let $s > 0$ and $p \in (1, \infty)$. Suppose that f, g are two smooth functions such that $f \in L^{p_1} \cap W^{s,p_3}$ and $g \in L^{p_4} \cap W^{s,p_2}$, then we have*

$$||\Lambda^s(fg)||_{L^p} \leq C(||f||_{L^{p_1}}||\Lambda^s g||_{L^{p_2}} + ||\Lambda^s f||_{L^{p_3}}||g||_{L^{p_4}})$$

and

$$||[\Lambda^s, f]g||_{L^p} \leq C(||\nabla f||_{L^{p_1}}||\Lambda^{s-1} g||_{L^{p_2}} + ||\Lambda^s f||_{L^{p_3}}||g||_{L^{p_4}}),$$

with $p_2, p_3 \in (1, \infty)$ such that

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.$$

The following lemma plays an essential role in the proof of Theorem 1.1. For the sake of convenience, we give the complete proof.

Lemma 2.2. *Under the hypotheses of Theorem 1.1, we have the following estimate*

$$\int_{\mathbb{R}^2} u\mathcal{L}u(x)dx \geq C||\Lambda^\kappa u||_{L^2}^2 - C||u||_{L^2}^2, \tag{9}$$

where C is a positive constant depending only α, β and $m(1)$.

Especially, if the kernel function of the nonlocal operator $\mathcal{L}$ is $J(y) = \frac{1}{|y|^{2+\alpha}}$, namely, $m(r) = r^\alpha$, we have the result

$$\int_{\mathbb{R}^2} u\mathcal{L}u(x)dx = c_\alpha^{-1}||\Lambda^{\frac{\alpha}{2}} u||_{L^2}^2. \tag{10}$$

Proof. By the definition of the nonlocal operator $\mathcal{L}$, we have

$$\begin{aligned}
\int_{\mathbb{R}^2} u\mathcal{L}u(x)dx &= \int_{\mathbb{R}^2 \times \mathbb{R}^2} J(y)(u(x) - u(x-y))u(x)dxdy \\
&= \frac{1}{2}\int_{\mathbb{R}^2 \times \mathbb{R}^2} J(y)\Big(|u(x) - u(x-y)|^2 - |u(x-y)|^2 + |u(x)|^2\Big)dxdy \\
&= \frac{1}{2}\int_{\mathbb{R}^2 \times \mathbb{R}^2} J(y)|u(x) - u(x-y)|^2 dxdy \\
&= \frac{1}{2}\int_{\mathbb{R}^2} J(y)||u(x) - u(x-y)||_{L^2}^2 dy \\
&= \frac{1}{2}\int_{\mathbb{R}^2} J(y)||\mathcal{F}(u(\cdot) - u(\cdot - y))||_{L^2}^2 dy
\end{aligned}$$

$$= \frac{1}{2}\int_{\mathbb{R}^2\times\mathbb{R}^2} J(y)|1-e^{-i\xi\cdot y}|^2|\hat{u}(\xi)|^2 d\xi dy$$

$$= \int_{\mathbb{R}^2\times\mathbb{R}^2} (1-\cos(\xi\cdot y))J(y)|\hat{u}(\xi)|^2 d\xi dy$$

$$= \int_{\mathbb{R}^2} F(\xi)|\hat{u}(\xi)|^2 d\xi, \tag{11}$$

where

$$F(\xi) := \int_{\mathbb{R}^2} (1-\cos(\xi\cdot y))J(y)dy. \tag{12}$$

Obviously, we have

$$\int_{\mathbb{R}^2} u\mathcal{L}u(x)dx \geq 0. \tag{13}$$

Next, we estimate the term $F(\xi)$.

Considering the assumption (7), by direct computation, it is easy to deduce that $f(r) = r^{2\kappa}m(r^{-1})$ is non-increasing in $r \in (0,1]$ for $\kappa \in (0,1)$, namely,

$$m(|y|^{-1}) \geq \frac{m(1)}{|y|^{2\kappa}} \quad \text{for all} \quad 0 < |y| \leq 1. \tag{14}$$

Then, from the assumption (6) and (14), we have

$$\begin{aligned} F(\xi) &= \int_{\mathbb{R}^2} (1-\cos(\xi\cdot y))J(y)dy \\ &\geq \int_{0<|y|\leq 1} (1-\cos(\xi\cdot y))J(y)dy \\ &\geq c^{-1}m(1)\int_{0<|y|\leq 1} (1-\cos(\xi\cdot y))\frac{1}{|y|^{2+2\kappa}}dy \\ &\geq c^{-1}m(1)\Big(|\xi|^{2\kappa} - \int_{|y|\geq 1}(1-\cos(\xi\cdot y))\frac{1}{|y|^{2+2\kappa}}dy\Big) \\ &\geq c^{-1}m(1)\Big(|\xi|^{2\kappa} - 2\int_1^\infty \frac{1}{r^{1+2\kappa}}dr\Big) \\ &= C|\xi|^{2\kappa} - C, \end{aligned} \tag{15}$$

where we have used the fact [12]

$$|\xi|^\delta = c_\delta \text{P.V.}\int_{\mathbb{R}^2} \frac{1-\cos(\xi\cdot y)}{|y|^{2+\delta}}dy \quad \text{for all} \quad \delta \in (0,2).$$

Consequently, from (11) we have

$$\begin{aligned} \int_{\mathbb{R}^2} u\mathcal{L}u(x)dx &= \int_{\mathbb{R}^2} F(\xi)|\hat{u}(\xi)|^2 d\xi \\ &\geq \int_{\mathbb{R}^2} \Big(C|\xi|^{2\kappa} - C\Big)|\hat{u}(\xi)|^2 d\xi \end{aligned}$$

$$= C||\Lambda^\kappa u||_{L^2}^2 - C||u||_{L^2}^2. \tag{16}$$

This completes the proof of Lemma 2.2. □

Next, we establish some global a priori estimates for the system (3) with $\mu \geq 0$ which will be both crucial in the proof of Theorem 1.1 and Theorem 1.5.

Proposition 2.3. *The solution (u, v, θ) to the Cauchy problem (3) with $\mu \geq 0$ fulfills for any $t > 0$*

$$||u(t), v(t), \theta(t)||_2^2 + 2\int_0^t ||\nabla v, \Lambda^\gamma \theta||_2^2 d\tau \leq ||u_0, v_0, \theta_0||_2^2. \tag{17}$$

Proof. Taking the L^2 inner product of Eqs. $(3)_1$-$(3)_3$ with u, v and θ, respectively, then using the fact $\text{div} u = 0$, we obtain

$$\frac{1}{2}\frac{d}{dt}||u, v, \theta||_2^2 + \mu \int_{\mathbb{R}^2} u\mathcal{L}u dx + ||\nabla v, \Lambda^\gamma \theta||_2^2 = 0, \tag{18}$$

where we have used the following cancellation identities

$$\int_{\mathbb{R}^2} \text{div}(v \otimes v)u + (v \cdot \nabla)u \cdot v dx = 0 \quad \text{and} \quad \int_{\mathbb{R}^2} \nabla\theta \cdot v + \text{div} v\theta dx = 0.$$

In fact, integrating by parts gives

$$\begin{aligned}
&\int_{\mathbb{R}^2} \text{div}(v \otimes v)u + (v \cdot \nabla)u \cdot v dx \\
= &\sum_{i,j=1}^{2} \int_{\mathbb{R}^2} \partial_i(v_i v_j)u_j + v_i \partial_i u_j v_j dx \\
= &\sum_{i,j=1}^{2} \Big(-\int_{\mathbb{R}^2} v_i v_j \partial_i u_j dx + \int_{\mathbb{R}^2} v_i \partial_i u_j v_j dx \Big) = 0
\end{aligned}$$

and

$$\begin{aligned}
\int_{\mathbb{R}^2} \nabla\theta \cdot v + \text{div} v\theta dx &= \sum_{i=1}^{2} \int_{\mathbb{R}^2} \partial_i \theta v_i + \partial_i v_i \theta dx \\
&= \sum_{i=1}^{2} \Big(-\int_{\mathbb{R}^2} \theta \partial_i v_i dx + \int_{\mathbb{R}^2} \partial_i v_i \theta dx \Big) = 0.
\end{aligned}$$

Combining (13) and (18), we deduce the desired result (17). □

Proposition 2.4. *The solution (u, v, θ) to the Cauchy problem (3) with $\mu \geq 0$ fulfills for any $t > 0$ and $\frac{2}{2\gamma - 1} \leq q \leq \frac{2}{1-\gamma}$*

$$||\nabla u, \nabla v, \nabla\theta||_2^2 + ||w, \nabla v||_q^2 + \int_0^t (||\Delta v, \Lambda^{1+\gamma}\theta||_2^2 + ||\Delta v||_q^2) ds \leq e^{C(e+t)}, \tag{19}$$

here $C > e$ is a constant depending only on initial value.

Proof. Taking the L^2 inner product of Eq. $(3)_1$ with $-\Delta u$, using the divergence-free condition $\mathrm{div} u = 0$, applying Hölder's and Young's inequalities gives that

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}||\nabla u||_2^2 &\le \int_{\mathbb{R}^2} \mathrm{div}(v\otimes v)\cdot \Delta u dx \\
&= -\int_{\mathbb{R}^2} \Delta(v\otimes v)\cdot \nabla u dx \\
&\le C||v||_\infty^2||\nabla u||_2^2 + \varepsilon||\Delta v||_2^2,
\end{aligned}
\tag{20}
$$

where we have used the fact

$$
\int_{\mathbb{R}^2} (u\cdot\nabla)u\cdot\Delta u dx = 0 \quad \text{from [30]} \quad \text{and} \quad \mu\int_{\mathbb{R}^2} \nabla u \mathcal{L}\nabla u dx \ge 0.
$$

Taking the L^2 inner product of Eq. $(3)_2$ with $-\Delta v$ yields that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}||\nabla v||_2^2 + ||\Delta v||_2^2 \\
&= \int_{\mathbb{R}^2} (u\cdot\nabla v + v\cdot\nabla u)\cdot\Delta v dx + \int_{\mathbb{R}^2} \nabla\theta\cdot\Delta v dx \\
&\le C||\nabla v||_4^2||\nabla u||_2 + C||v||_\infty||\nabla u||_2||\Delta v||_2 + \int_{\mathbb{R}^2}\nabla\theta\cdot\Delta v dx \\
&\le C(||\nabla v||_2^2 + ||v||_\infty^2)||\nabla u||_2^2 + \varepsilon||\Delta v||_2^2 + \int_{\mathbb{R}^2}\nabla\theta\cdot\Delta v dx,
\end{aligned}
\tag{21}
$$

where we have used the following interpolation inequality for $2 < r < \infty$

$$
||f||_{L^r(\mathbb{R}^d)} \lesssim ||f||_{L^2(\mathbb{R}^d)}^{\vartheta}||\nabla f||_{L^2(\mathbb{R}^d)}^{1-\vartheta} \quad \text{with} \quad \vartheta = 1 - \frac{d}{2} + \frac{d}{r}.
$$

Similarly, taking the L^2 inner product of Eq. $(3)_3$ with $-\Delta\theta$, one has

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}||\nabla\theta||_2^2 + ||\Lambda^{1+\gamma}\theta||_2^2 \\
&= \int_{\mathbb{R}^2} (u\cdot\nabla\theta + \mathrm{div} v)\cdot\Delta\theta dx \\
&\le ||\nabla u||_q||\nabla\theta||_{\frac{2q}{q-1}}^2 - \int_{\mathbb{R}^2}\nabla\theta\cdot\Delta v dx \\
&\le C||w||_q||\Lambda^\gamma\theta||_2^{2(\gamma-\frac{1}{q})}||\Lambda^{1+\gamma}\theta||_2^{2(1-\gamma+\frac{1}{q})} - \int_{\mathbb{R}^2}\nabla\theta\cdot\Delta v dx \\
&\le C(1+||w||_q^2)||\Lambda^\gamma\theta||_2^2 + \varepsilon||\Lambda^{1+\gamma}\theta||_2^2 - \int_{\mathbb{R}^2}\nabla\theta\cdot\Delta v dx,
\end{aligned}
\tag{22}
$$

where we have used the fact $q \ge \frac{2}{2\gamma-1}$.

Summing up (20)-(22), absorbing the ε-terms and integrating in time, we obtain

$$
||(\nabla u, \nabla v, \nabla\theta)(t)||_2^2 + \int_0^t ||(\Delta v, \Lambda^{1+\gamma}\theta)(s)||_2^2 ds \le ||\nabla u_0, \nabla v_0, \nabla\theta_0||_2^2
$$

$$+C\int_0^t\Big[(||\nabla v||_2^2+||v||_\infty^2)||\nabla u||_2^2+(1+||w||_q^2)||\Lambda^\gamma\theta||_2^2\Big]ds. \tag{23}$$

Taking operator curl to Eq. $(3)_1$, we obtain the following vorticity equation

$$\partial_t w+u\cdot\nabla w+\mu\mathcal{L}w=-\mathrm{curldiv}(v\otimes v). \tag{24}$$

Multiplying (24) by $|w|^{q-2}w$ and integrating the resulting equation over $\mathbb{R}^2$ yields

$$\begin{aligned}\frac{1}{q}\frac{d}{dt}||w||_q^q &\le ||\mathrm{curldiv}(v\otimes v)||_q||w||_q^{q-1}\\ &\lesssim ||v||_\infty||\Delta v||_q||w||_q^{q-1}\\ &\le \Big(\varepsilon||\Delta v||_q^2+C||v||_\infty^2||w||_q^2\Big)||w||_q^{q-2},\end{aligned} \tag{25}$$

here we used the fact $\int_{\mathbb{R}^2}|w|^{q-2}w\mathcal{L}w(x)dx\ge 0$. Indeed, using the definition of the nonlocal operator $\mathcal{L}$ and non-negative symmetry properties of J again, by the change of variable $y=-\tilde{y}$ and $x=\tilde{x}-\tilde{y}$ orderly, we have

$$\begin{aligned}&\int_{\mathbb{R}^2}|w|^{q-2}w\mathcal{L}w(x)dx\\ =&\int_{\mathbb{R}^2\times\mathbb{R}^2}J(y)|w(x)|^{q-2}w(x)(w(x)-w(x-y))dxdy\\ =&\int_{\mathbb{R}^2\times\mathbb{R}^2}J(\tilde{y})|w(\tilde{x}-\tilde{y})|^{q-2}w(\tilde{x}-\tilde{y})(w(\tilde{x}-\tilde{y})-w(\tilde{x}))d\tilde{x}d\tilde{y}\\ =&\frac{1}{2}\int_{\mathbb{R}^2\times\mathbb{R}^2}J(y)f(x,y)dxdy\ge 0,\end{aligned}$$

since we notice that

$$\begin{aligned}f(x,y)&:=(|w(x)|^{q-2}w(x)-|w(x-y)|^{q-2}w(x-y))(w(x)-w(x-y))\\ &\ge\frac{4(q-1)}{q^2}\Big||w(x)|^{\frac{q-2}{2}}w(x)-|w(x-y)|^{\frac{q-2}{2}}w(x-y)\Big|^2\end{aligned}$$

holds for all $x,y\in\mathbb{R}^2$ and $q\in[2,\infty)$.

Applying operator Δ to Eq. $(3)_2$ and by Duhamel's Principle, one has

$$\begin{aligned}\Delta v(t,x)&=e^{t\Delta}\Delta v_0-\int_0^t e^{(t-s)\Delta}\Delta(u\cdot\nabla v+v\cdot\nabla u+\nabla\theta)ds\\ &=K_t(x)*\Delta v_0-A(u\cdot\nabla v+v\cdot\nabla u+\nabla\theta)(t,x),\end{aligned}$$

where we denote the heat kernel by $K_t(x)=(4\pi t)^{-\frac{d}{2}}e^{-\frac{|x|^2}{4t}}$ and define the operator A by

$$Af(t,x):=\int_0^t e^{(t-s)\Delta}\Delta f(s,\cdot)ds.$$

With the aid of the maximal $L^p_tL^q_x$ regularity for the heat kernel (see [18]), we obtain

$$
\begin{aligned}
(26)\qquad & \int_0^t ||\Delta v(s)||_q^2 ds \\
\le\ & \int_0^t ||K_t(x)*\Delta v_0 - A(u\cdot\nabla v + v\cdot\nabla u + \nabla\theta)(s,x)||_q^2 ds \\
\le\ & ||\Delta v_0||_2 \int_0^t s^{\frac{2}{q}-1} ds + \int_0^t ||A(u\cdot\nabla v + v\cdot\nabla u + \nabla\theta)(s,x)||_q^2 ds \\
\le\ & Ct^{\frac{2}{q}} + C\int_0^t ||u\cdot\nabla v + v\cdot\nabla u + \nabla\theta||_q^2 ds \\
\le\ & C(e+t) + C\int_0^t (||u||_q^2||\nabla v||_\infty^2 + ||v||_\infty^2||\nabla u||_q^2 + ||\nabla\theta||_q^2) ds \\
\le\ & C(e+t) + C\int_0^t (||v||_\infty^2 + ||\nabla v||_2^2)||w||_q^2 ds + \frac{C}{2}\int_0^t ||\Lambda^{1+\gamma}\theta||_2^2 ds \\
& + \frac{1}{2}\int_0^t ||\Delta v||_q^2 ds,
\end{aligned}
$$

where we have used the following Gagliardo-Nirenberg inequalities

$$
||u||_q \lesssim ||u||_2^\sigma ||\nabla u||_q^{1-\sigma}, \quad ||\nabla v||_\infty \lesssim ||\nabla v||_2^{1-\sigma}||\Delta v||_q^\sigma \quad \text{with} \quad \sigma = \frac{q}{2(q-1)}
$$

and

$$
||\nabla\theta||_q \lesssim ||\Lambda^\gamma\theta||_2^{\frac{2}{q}+\gamma-1}||\Lambda^{1+\gamma}\theta||_2^{2-\frac{2}{q}-\gamma} \quad \text{with} \quad 2 < q \le \frac{2}{1-\gamma}.
$$

From (26), one has

$$
\begin{aligned}
\int_0^t ||\Delta v(s)||_q^2 ds \lesssim\ & (e+t) + \int_0^t (||v||_\infty^2 + ||\nabla v||_2^2)||w||_q^2 ds \\
(27)\qquad & + \int_0^t ||\Lambda^{1+\gamma}\theta||_2^2 ds.
\end{aligned}
$$

Dividing $||w||_q^{q-2}$ on both sides of (25), integrating in time and combing (27) yields

$$
(28)\qquad ||w(t)||_q^2 \lesssim (e+t) + \varepsilon\int_0^t ||\Lambda^{1+\gamma}\theta||_2^2 ds + \int_0^t (||v||_\infty^2 + ||\nabla v||_2^2)||w||_q^2 ds.
$$

Summing up (23) and (27)-(28), absorbing the ε-terms, we arrive at

$$
(29)\qquad M(t) \le C(e+t) + C\int_0^t (||\nabla v||_2^2 + ||v||_\infty^2 + ||\Lambda^\gamma\theta||_2^2)M(s) ds,
$$

where

$$
M(t) := 1 + ||\nabla u, \nabla v, \nabla\theta||_2^2 + ||w||_q^2 + \int_0^t (||\Delta v, \Lambda^{1+\gamma}\theta||_2^2 + ||\Delta v||_q^2) ds,
$$

which follows from (17) and Gronwall's inequality that for any $t \in [0,T]$

$$M(t) \leq \; C(e+t)\exp\Big\{C\int_0^t ||v(\tau)||_\infty^2 d\tau\Big\}. \tag{30}$$

Applying operator ∇ to Eq. $(3)_2$ and multiplying the resulting equation by $|\nabla v|^{q-2}\nabla v$, integrating over $\mathbb{R}^2$ and using Hölder's inequality, we get

$$\begin{aligned}
&\frac{1}{q}\frac{d}{dt}||\nabla v||_q^q + (q-1)|||\nabla v||^{\frac{q-2}{2}}|\nabla^2 v|||_2^2\\
&= \int_{\mathbb{R}^2}\nabla(-u\cdot\nabla v - v\cdot\nabla u - \nabla\theta)|\nabla v|^{q-2}\nabla v dx\\
&\leq (q-1)||u\cdot\nabla v + v\cdot\nabla u + \nabla\theta||_q|||\nabla v|^{\frac{q-2}{2}}||_{\frac{2q}{q-2}}|||\nabla v|^{\frac{q-2}{2}}|\nabla^2 v|||_2\\
&\leq \frac{q-1}{2}|||v|^{\frac{q-2}{2}}|\nabla^2 v|||_2^2 + C\Big(||u||_q^2||\nabla v||_\infty^2 + ||v||_\infty^2||\nabla u||_q^2 + ||\nabla\theta||_q^2\Big)||\nabla v||_q^{q-2},
\end{aligned}$$

from which, going along the lines of proof of (26), one deduces from (30)

$$\begin{aligned}
||\nabla v(t)||_q^2 &\leq C + C\int_0^t\Big(||u||_q^2||\nabla v||_\infty^2 + ||v||_\infty^2||w||_q^2 + ||\nabla\theta||_q^2\Big)ds\\
&\lesssim (e+t) + \int_0^t\Big[(||\Lambda^{1+\gamma}\theta||_2^2 + ||\Delta v||_q^2 + (||v||_\infty^2 + ||\nabla v||_2^2)||w||_q^2\Big]ds\\
&\lesssim (e+t)\exp\Big\{C\int_0^t ||v(\tau)||_\infty^2 d\tau\Big\}. \qquad (31)
\end{aligned}$$

Due to the following logarithmic Sobolev embedding inequality (see [2,15])

$$||v(s)||_{L^\infty} \leq C\Big(1 + ||\nabla v(s)||_{L^2}\ln^{\frac{1}{2}}(1 + ||\nabla v(s)||_{L^m})\Big) \quad \text{with} \quad 2 < m < \infty,$$

we obtain from (31) that

$$\begin{aligned}
\int_0^t ||v(s)||_{L^\infty}^2 ds &\lesssim t + \int_0^t ||\nabla v(s)||_{L^2}^2 \ln\Big(C(e+s)\exp\Big\{C\int_0^s ||v(\tau)||_\infty^2 d\tau\Big\}\Big)ds\\
&\lesssim t + [\ln C + \ln(e+t)]\int_0^t ||\nabla v(s)||_{L^2}^2 ds\\
&\quad + \int_0^t ||\nabla v(s)||_{L^2}^2\int_0^s ||v(\tau)||_{L^\infty}^2 d\tau ds\\
&\lesssim (e+t) + \int_0^t ||\nabla v(s)||_{L^2}^2\int_0^s ||v(\tau)||_{L^\infty}^2 d\tau ds. \qquad (32)
\end{aligned}$$

It follows from Gronwall' inequality and (17) that

$$\int_0^t ||v(s)||_{L^\infty}^2 ds \leq C(e+t), \tag{33}$$

which in turn from (30) and (31) leads to (19). Thus, Proposition 2.4 is finished. □

Proposition 2.5. *The solution* (u, v, θ) *to the Cauchy problem* (3) *with* $\mu \geq 0$ *fulfills for any* $t > 0$ *and* $\frac{2}{1-\gamma} \leq p < \infty$

$$||w, \nabla v, \nabla\theta||_p^2 + \int_0^t ||\Delta v||_p^2 ds \leq e^{C(e+t)}, \tag{34}$$

here $C > e$ *is a constant depending only on initial value.*

Proof. Applying operator ∇ to Eq. $(3)_3$ and multiplying the resulting equation by $|\nabla\theta|^{p-2}\nabla\theta$ with $\frac{2}{1-\gamma} \leq p < \infty$, integrating over $\mathbb{R}^2$ yields

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}||\nabla\theta||_p^p + C_2||\nabla\theta||^p_{\frac{p}{1-\gamma}} = &-\int_{\mathbb{R}^2} \nabla u : \nabla\theta \cdot |\nabla\theta|^{p-2}\nabla\theta dx \\ &-\int_{\mathbb{R}^2} \nabla \text{div} v \cdot |\nabla\theta|^{p-2}\nabla\theta dx = H_1 + H_2,\end{aligned} \tag{35}$$

where we have used the pointwise inequality (see [3]) and Sobolev embedding inequality

$$\int_{\mathbb{R}^2} \Lambda^{2\gamma}\theta \cdot |\nabla\theta|^{p-2}\nabla\theta dx \geq \ C_1\int_{\mathbb{R}^2} (\Lambda^\gamma|\nabla\theta|^{\frac{p}{2}})^2 dx \geq C_2||\nabla\theta||^p_{\frac{p}{1-\gamma}}.$$

Next, by Hölder's inequality, one has

$$\begin{aligned}H_1 &\leq C||\nabla u||_2||\nabla\theta||_{2p}^p \\ &\leq C||w||_2||\nabla\theta||_p^{\frac{(2\gamma-1)p}{2\gamma}}||\nabla\theta||^{\frac{p}{2\gamma}}_{\frac{p}{1-\gamma}} \\ &\leq \frac{C_2}{4}||\nabla\theta||^p_{\frac{p}{1-\gamma}} + C||w||_2^{\frac{2\gamma}{2\gamma-1}}||\nabla\theta||_p^p\end{aligned} \tag{36}$$

and

$$\begin{aligned}H_2 &\leq ||\Delta v||_q||\nabla\theta||^{p-1}_{\frac{q(p-1)}{q-1}} \quad \text{with} \quad q = \frac{2}{1-\gamma} \\ &\leq ||\Delta v||_q||\nabla\theta||_p^{p-1-\frac{p-q}{q\gamma}}||\nabla\theta||^{\frac{p-q}{q\gamma}}_{\frac{p}{1-\gamma}} \\ &\leq \frac{C_2}{4}||\nabla\theta||^p_{\frac{p}{1-\gamma}} + ||\Delta v||_q^{\frac{pq\gamma}{pq\gamma-p+q}}||\nabla\theta||_p^{\frac{p(pq\gamma-q\gamma-p+q)}{pq\gamma-p+q}} \\ &\leq \frac{C_2}{4}||\nabla\theta||^p_{\frac{p}{1-\gamma}} + C\Big(1+||\Delta v||_q^2\Big)(1+||\nabla\theta||_p^p),\end{aligned} \tag{37}$$

here we notice that $q = \frac{2}{1-\gamma}$ implies that $\frac{pq\gamma}{pq\gamma-p+q} < 2$.

Inserting (36) and (37) into (35), by Gronwall's inequality and using (19), we obtain

$$||\nabla\theta||_p \leq \ e^{C(e+t)}.$$

With the crucial boundedness at our hand, coming back to (25), invoking the same procedure again, we can obtain the desired (34). Here we omit the details. □

It is worth mentioning that until here we have not used the information about the generalized dissipative operator $\mathcal{L}$.

Proof of Theorem 1.1. Applying Δ to both sides of $(3)_1$, $(3)_2$ and $(3)_3$, taking the L^2 inner product of the resulting equations with Δu, Δv and $\Delta\theta$, respectively, then adding them together yields that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}||\Delta u,\Delta v,\Delta\theta||_2^2+C||\Lambda^{2+\kappa}u||_2^2+||\Lambda^3 v,\Lambda^{2+\gamma}\theta||_2^2\\ \leq\ & -\underbrace{\int_{\mathbb{R}^2}\Delta(u\cdot\nabla u)\cdot\Delta u dx}_{I_1}-\underbrace{\int_{\mathbb{R}^2}\Delta\mathrm{div}(v\otimes v)\cdot\Delta u dx}_{I_2}\\ &-\underbrace{\int_{\mathbb{R}^2}\Delta(u\cdot\nabla v+v\cdot\nabla u)\cdot\Delta v dx}_{I_3}-\underbrace{\int_{\mathbb{R}^2}\Delta(u\cdot\nabla\theta)\cdot\Delta\theta dx}_{I_4}+C||\Delta u||_2^2,\end{aligned}\tag{38}$$

where we have used the facts

$$\int_{\mathbb{R}^2}\Delta\nabla\theta\cdot\Delta v dx+\int_{\mathbb{R}^2}\Delta\mathrm{div}v\cdot\Delta\theta dx=0$$

and

$$\int_{\mathbb{R}^2}\Delta u\mathcal{L}\Delta u dx\geq\ C||\Lambda^{2+\kappa}u||_2^2-C||\Delta u||_2^2.$$

Next, we need to estimate the above four terms one by one as follows (see Lemma 2.1).

For the first term I_1, we only consider the case when $\kappa\in(0,\frac{1}{2})$ since it is easy to deal with the case when $\kappa\in[\frac{1}{2},1)$.

$$\begin{aligned}I_1&\leq C||\nabla u||_p||\Delta u||^2_{\frac{2p}{p-1}}\\ &\leq C||\Delta u||_2^{2-\frac{2}{p\kappa}}||\Lambda^{2+\kappa}u||_2^{\frac{2}{p\kappa}}\\ &\leq\frac{C}{2}||\Lambda^{2+\kappa}u||_2^2+C||\Delta u||_2^2,\end{aligned}\tag{39}$$

where we have used the fact $||\nabla u||_{L^p}\lesssim||w||_{L^p}$ with $p>\frac{1}{\kappa}>2$.

$$I_2\leq C||v||_\infty||\Lambda^3 v||_2||\Delta u||_2\leq C||v||_\infty^2||\Delta u||_2^2+\epsilon||\Lambda^3 v||_2^2,\tag{40}$$

$$\begin{aligned}I_3&\leq C(||\nabla u||_4||\nabla v||_4+||u||_\infty||\Delta v||_2+||v||_\infty||\Delta u||_2)||\Lambda^3 v||_2\\ &\leq C(||v||_\infty^2+||w||_4^2)||\Delta u,\Delta v||_2^2+\epsilon||\Lambda^3 v||_2^2,\end{aligned}\tag{41}$$

$$\begin{aligned}I_4&\leq||[\Delta,u\cdot\nabla]\theta||_{\frac{2}{2-\gamma}}||\Delta\theta||_{\frac{2}{\gamma}}\\ &\leq C(||\nabla u||_2||\Delta\theta||_{\frac{2}{1-\gamma}}+||\nabla\theta||_{\frac{2}{1-\gamma}}||\Delta u||_2)||\Delta\theta||_{\frac{2}{\gamma}}\\ &\leq C||\Delta\theta||_2^2+C||\Lambda^{1+\gamma}\theta||_2^2||\Delta u||_2^2+\epsilon||\Lambda^{2+\gamma}\theta||_2^2.\end{aligned}\tag{42}$$

Combining the above estimates to (38), one has

$$\begin{aligned}&\frac{d}{dt}||\Delta u,\Delta v,\Delta\theta||_2^2+||\Lambda^{2+\kappa}u,\Lambda^3 v,\Lambda^{2+\gamma}\theta||_2^2\\ \lesssim&\Big(1+||w||_4^2+||\Lambda^{1+\gamma}\theta||_2^2+||v||_\infty^2\Big)||\Delta u,\Delta v,\Delta\theta||_2^2,\end{aligned}$$

which follows from Gronwall's inequality that the desired estimates of Theorem 1.1. Thus, we complete the proof of Theorem 1.1. □

Proof of Theorem 1.5. Applying Λ^s to both sides of $(3)_1$, $(3)_2$ and $(3)_3$, taking the L^2 inner product of the resulting equations with $\Lambda^s u$, $\Lambda^s v$ and $\Lambda^s\theta$, respectively, then adding them together yields that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}||\Lambda^s u,\Lambda^s v,\Lambda^s\theta||_2^2+||\Lambda^{s+1}v,\Lambda^{s+\gamma}\theta||_2^2\\ =&-\int_{\mathbb{R}^2}\Lambda^s(u\cdot\nabla u)\cdot\Lambda^s u dx-\int_{\mathbb{R}^2}\Lambda^s\mathrm{div}(v\otimes v)\cdot\Lambda^s u dx\\ &-\int_{\mathbb{R}^2}\Lambda^s(u\cdot\nabla v+v\cdot\nabla u+\nabla\theta)\cdot\Lambda^s v dx-\int_{\mathbb{R}^2}\Lambda^s(u\cdot\nabla\theta+\mathrm{div}v)\cdot\Lambda^s\theta dx\\ =&\underbrace{-\int_{\mathbb{R}^2}[\Lambda^s,u\cdot\nabla]u\cdot\Lambda^s u dx-\int_{\mathbb{R}^2}[\Lambda^s,u\cdot\nabla]v\cdot\Lambda^s v dx}_{\mathcal{I}_1}\underbrace{-\int_{\mathbb{R}^2}[\Lambda^s,u\cdot\nabla]\theta\cdot\Lambda^s\theta dx}_{\mathcal{I}_2}\\ &\underbrace{-\int_{\mathbb{R}^2}[\Lambda^s,v\cdot\nabla]v\cdot\Lambda^s u dx-\int_{\mathbb{R}^2}[\Lambda^s,v\cdot\nabla]u\cdot\Lambda^s v dx}_{\mathcal{I}_3}\underbrace{-\int_{\mathbb{R}^2}[\Lambda^s,\mathrm{div}v]v\cdot\Lambda^s u dx}_{\mathcal{I}_4},\end{aligned}\tag{43}$$

where we have used the fact

$$\begin{aligned}&\int_{\mathbb{R}^2}\Lambda^s\mathrm{div}(v\otimes v)\cdot\Lambda^s u dx+\int_{\mathbb{R}^2}\Lambda^s(v\cdot\nabla u)\cdot\Lambda^s v dx\\ =&\int_{\mathbb{R}^2}\Lambda^s(v\cdot\nabla v)\cdot\Lambda^s u dx+\int_{\mathbb{R}^2}\Lambda^s(\mathrm{div}v v)\cdot\Lambda^s u dx+\int_{\mathbb{R}^2}\Lambda^s(v\cdot\nabla u)\cdot\Lambda^s v dx\\ =&\int_{\mathbb{R}^2}[\Lambda^s,v\cdot\nabla]v\cdot\Lambda^s u dx+\int_{\mathbb{R}^2}v\cdot\nabla\Lambda^s v\cdot\Lambda^s u dx+\int_{\mathbb{R}^2}\Lambda^s(\mathrm{div}v v)\cdot\Lambda^s u dx\\ &+\int_{\mathbb{R}^2}[\Lambda^s,v\cdot\nabla]u\cdot\Lambda^s v dx+\int_{\mathbb{R}^2}v\cdot\nabla\Lambda^s u\cdot\Lambda^s v dx\\ =&\int_{\mathbb{R}^2}[\Lambda^s,v\cdot\nabla]v\cdot\Lambda^s u dx+\int_{\mathbb{R}^2}[\Lambda^s,v\cdot\nabla]u\cdot\Lambda^s v dx+\int_{\mathbb{R}^2}[\Lambda^s,\mathrm{div}v]v\cdot\Lambda^s u dx.\end{aligned}$$

Next, we need to estimate the above four terms one by one as follows (see Lemma 2.1)

$$\begin{aligned}&\mathcal{I}_1+\mathcal{I}_3\le C||\nabla u,\nabla v||_\infty||\Lambda^s u,\Lambda^s v||_2^2,\\ &\quad\mathcal{I}_2\le C(||\nabla u||_2||\Lambda^s\theta||_{\frac{2}{1-\gamma}}+||\nabla\theta||_{\frac{2}{1-\gamma}}||\Lambda^s u||_2)||\Lambda^s\theta||_{\frac{2}{\gamma}}\end{aligned}\tag{44}$$

$$\leq C||\Lambda^s\theta||_2^2 + C||\Lambda^{1+\gamma}\theta||_2^2||\Lambda^s u||_2^2 + \epsilon||\Lambda^{s+\gamma}\theta||_2^2, \tag{45}$$

$$\mathcal{I}_4 \leq C||\nabla \mathrm{div} v||_p||\Lambda^{s-1}v||_{\frac{2p}{p-2}}||\Lambda^s u||_2 + ||v||_\infty||\Lambda^s u||_2||\Lambda^{s+1}v||_2$$
$$\leq C(||v||_\infty^2 + ||\Delta v||_p^2)||\Lambda^s u, v, \Lambda^s v||_2^2 + \epsilon||\Lambda^{s+1}v||_2^2 \quad \text{for} \quad p \in (2,\infty). \tag{46}$$

Remarking (18), combining the above estimates to (43), and absorbing the ϵ-terms, one has

$$\frac{d}{dt}||u,v,\theta||_{H^s}^2 + ||\Lambda^{s+1}v, \Lambda^{s+1}\theta||_2^2$$
$$\lesssim \Big(1 + ||\nabla u, \nabla v||_\infty + ||\Lambda^{1+\gamma}\theta||_2^2 + ||\Delta v||_p^2 + ||v||_\infty^2\Big)||u,v,\theta||_{H^s}^2$$
$$\lesssim \Big(1 + ||\Delta v||_p^2 + ||\Lambda^{1+\gamma}\theta||_2^2 + ||w||_{L^\infty}\ln(e + ||u||_{H^s})\Big)||u,v,\theta||_{H^s}^2, \tag{47}$$

where we have used the Sobolev extrapolation inequality with logarithmic correction which was firstly introduced in [2]

$$||\nabla u||_{L^\infty(\mathbb{R}^2)} \leq C\Big(1 + ||u||_{L^2(\mathbb{R}^2)} + ||w||_{L^\infty(\mathbb{R}^2)}\ln(e + ||u||_{H^\varrho(\mathbb{R}^2)})\Big), \quad \varrho > 2.$$

From (47) and (19), we have

$$\ln(e^3 + ||u,v,\theta||_{H^s})$$
$$\leq C\Big(e + t + \int_0^t (||\Delta v||_p^2 + ||\Lambda^{1+\gamma}\theta||_2^2)d\tau\Big)\exp\Big\{C\int_0^t ||w||_{L^\infty}d\tau\Big\}$$
$$\leq C\exp\Big\{C\Big(e + t + \int_0^t ||w||_\infty d\tau\Big)\Big\},$$

which implies that

$$\ln\ln(e^3 + ||u,v,\theta||_{H^s}) \leq C(e+t) + C\int_0^t ||w||_{L^\infty}d\tau. \tag{48}$$

We set $F(t) := \ln\ln(e^3 + ||w||_{L^\infty})$ and

$$G(t) := C(e+t) + C\int_0^t ||w(\tau)||_{L^\infty}d\tau \quad \text{with} \quad G(0) := Ce. \tag{49}$$

It follows from (48) that $F(t) \leq G(t)$ and $F(t)\ln F(t) \leq G(t)\ln G(t)$, then

$$\frac{d}{dt}\ln G(t) = \frac{C + C||w(t)||_{L^\infty}}{G(t)} \leq C + C\frac{||w(t)||_{L^\infty}}{F(t)\ln F(t)}\ln G(t), \tag{50}$$

therefore the Gronwall Lemma gives

$$\ln G(t) \lesssim (e+t)\exp\Big\{C\int_0^t \frac{||w||_{L^\infty}}{\ln\ln(e^3 + ||w||_{L^\infty})\ln\ln\ln(e^3 + ||w||_{L^\infty})}d\tau\Big\},$$

which follows from (8), (47)-(49) that

$$(u,v,\theta) \in (L^\infty([0,T];H^s))^3$$

and

$$(v,\theta) \in L^2([0,T];H^{s+1}) \times L^2([0,T];H^{s+\gamma}).$$

This is enough to ensure that the local smooth solutions can be extended to $T = \infty$. Thus, we complete the proof of Theorem 1.5. □

Acknowledgments. The author would like to thank Dr.Yanghai Yu for his valuable suggestions. The author also want to thank the referees for their careful review and constructive comments which greatly improved the presentation of this paper.

References

[1] L. Agélas, *Beyond the BKM criterion for the 2D resistive magnetohydrodynamic equations*, Analysis & PDE **11** (2018), no. 4, 899–918.

[2] H. Brézis and S. Wainger, *A note on limiting cases of Sobolev embeddings and convolution inequalities*, Comm. Partial Differential Equations **5** (1980), no. 7, 773–789. https://doi.org/10.1080/03605308008820154

[3] A. Córdoba and D. Córdoba, *A maximum principle applied to quasi-geostrophic equations*, Comm. Math. Phys. **249** (2004), no. 3, 511–528. https://doi.org/10.1007/s00220-004-1055-1

[4] M. Dabkowski, A. Kiselev, L. Silvestre, and V. Vicol, *Global well-posedness of slightly supercritical active scalar equations*, Analysis & PDE **7** (2014), no. 1, 43–72.

[5] B. Dong, W. Wang, J. Wu, Z. Ye and H. Zhang, *Global regularity for a class of 2D generalized tropical climate models*, J. Differential Equations **266** (2019), no. 10, 6346–6382. https://doi.org/10.1016/j.jde.2018.11.007

[6] B. Dong, W. Wang, J. Wu, and H. Zhang, *Global regularity results for the climate model with fractional dissipation*, Discrete Contin. Dyn. Syst. Ser. B **24** (2019), no. 1, 211–229.

[7] B.-Q. Dong, J. Wu, and Z. Ye, *Global regularity for a 2D tropical climate model with fractional dissipation*, J. Nonlinear Sci. **29** (2019), no. 2, 511–550. https://doi.org/10.1007/s00332-018-9495-5

[8] J. Fan, S. Jiang, G. Nakamura, and Y. Zhou, *Logarithmically improved regularity criteria for the Navier-Stokes and MHD equations*, J. Math. Fluid Mech. **13** (2011), no. 4, 557–571. https://doi.org/10.1007/s00021-010-0039-5

[9] J. Fan, H. Malaikah, S. Monaquel, G. Nakamura, and Y. Zhou, *Global Cauchy problem of 2D generalized MHD equations*, Monatsh. Math. **175** (2014), no. 1, 127–131. https://doi.org/10.1007/s00605-014-0652-0

[10] D. M. W. Frierson, A. J. Majda, and O. M. Pauluis, *Large scale dynamics of precipitation fronts in the tropical atmosphere: a novel relaxation limit*, Commun. Math. Sci. **2** (2004), no. 4, 591–626. http://projecteuclid.org/euclid.cms/1109885499

[11] A. Gill, *Some simple solutions for heat-induced tropical circulation*, Q. J. R. Meteorol. Soc. **106** (1980), 447–462.

[12] N. Jacob, *Pseudo differential operators and Markov processes. Vol. III*, Imperial College Press, London, 2005. https://doi.org/10.1142/9781860947155

[13] T. Kato and G. Ponce, *Commutator estimates and the Euler and Navier-Stokes equations*, Comm. Pure Appl. Math. **41** (1988), no. 7, 891–907. https://doi.org/10.1002/cpa.3160410704

[14] C. E. Kenig, G. Ponce, and L. Vega, *Well-posedness of the initial value problem for the Korteweg-de Vries equation*, J. Amer. Math. Soc. **4** (1991), no. 2, 323–347. https://doi.org/10.2307/2939277

[15] H. Kozono and Y. Taniuchi, *Limiting case of the Sobolev inequality in BMO, with application to the Euler equations*, Comm. Math. Phys. **214** (2000), no. 1, 191–200. https://doi.org/10.1007/s002200000267

[16] J. Li and E. Titi, *Global well-posedness of strong solutions to a tropical climate model*, Discrete Contin. Dyn. Syst. **36** (2016), no. 8, 4495–4516. `https://doi.org/10.3934/dcds.2016.36.4495`

[17] Z. Lei and Y. Zhou, *BKM's criterion and global weak solutions for magnetohydrodynamics with zero viscosity*, Discrete Contin. Dyn. Syst. **25** (2009), no. 2, 575–583. `https://doi.org/10.3934/dcds.2009.25.575`

[18] P. G. Lemarié-Rieusset, *Recent developments in the Navier-Stokes problem*, Chapman & Hall/CRC Research Notes in Mathematics, **431**, Chapman & Hall/CRC, Boca Raton, FL, 2002. `https://doi.org/10.1201/9781420035674`

[19] C. Ma, Z. Jiang, and R. Wan, *Local well-posedness for the tropical climate model with fractional velocity diffusion*, Kinet. Relat. Models **9** (2016), no. 3, 551–570. `https://doi.org/10.3934/krm.2016006`

[20] C. Ma and R. Wan, *Spectral analysis and global well-posedness for a viscous tropical climate model with only a damp term*, Nonlinear Anal. Real World Appl. **39** (2018), 554–567. `https://doi.org/10.1016/j.nonrwa.2017.08.004`

[21] R. Wan, *Global small solutions to a tropical climate model without thermal diffusion*, J. Math. Phys. **57** (2016), no. 2, 021507, 13 pp. `https://doi.org/10.1063/1.4941039`

[22] A. J. Majda and J. A. Biello, *The nonlinear interaction of barotropic and equatorial baroclinic Rossby waves*, J. Atmospheric Sci. **60** (2003), no. 15, 1809–1821.

[23] T. Matsuno, *Quasi–geostrophic motions in the equatorial area*, J. Meteorol. Soc. Jpn. **44** (1966), 25–42.

[24] X. Wu, Y. Yu, and Y. Tang, *Global existence and asymptotic behavior for the 3D generalized Hall-MHD system*, Nonlinear Anal. **151** (2017), 41–50. `https://doi.org/10.1016/j.na.2016.11.010`

[25] Z. Ye, *Global regularity for a class of 2D tropical climate model*, J. Math. Anal. Appl. **446** (2017), no. 1, 307–321. `https://doi.org/10.1016/j.jmaa.2016.08.053`

[26] Y. Yu and Y. Tang, *A new blow–up criterion for the 2D generalized tropical climate model*, Bull. Malays. Math. Sci. Soc. **8** (2018), 1–16.

[27] Y. Yu, X. Wu, and Y. Tang, *Global regularity of the 2D liquid crystal equations with weak velocity dissipation*, Comput. Math. Appl. **74** (2017), no. 5, 920–933. `https://doi.org/10.1016/j.camwa.2016.11.008`

[28] ———, *Global well-posedness for the 2D Boussinesq system with variable viscosity and damping*, Math. Methods Appl. Sci. **41** (2018), no. 8, 3044–3061. `https://doi.org/10.1002/mma.4799`

[29] B. Yuan and J. Zhao, *Global regularity of 2D almost resistive MHD equations*, Nonlinear Anal. Real World Appl. **41** (2018), 53–65. `https://doi.org/10.1016/j.nonrwa.2017.10.006`

[30] Y. Zhou and J. Fan, *A regularity criterion for the 2D MHD system with zero magnetic diffusivity*, J. Math. Anal. Appl. **378** (2011), no. 1, 169–172. `https://doi.org/10.1016/j.jmaa.2011.01.014`

[31] M. Zhu, *Global regularity for the tropical climate model with fractional diffusion on barotropic mode*, Appl. Math. Lett. **81** (2018), 99–104. `https://doi.org/10.1016/j.aml.2018.02.003`

Mulan Zhou
School of Mathematics and Statistics
Anhui Normal University
Wuhu, Anhui 241002, P. R. China
Email address: `mlzhou916@ahnu.edu.cn`