

## DEPENDENT SUBSETS OF EMBEDDED PROJECTIVE VARIETIES

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ABSTRACT. Let  $X \subset \mathbb{P}^r$  be an integral and non-degenerate variety. Set  $n := \dim(X)$ . Let  $\rho(X)''$  be the maximal integer such that every zero-dimensional scheme  $Z \subset X$  smoothable in  $X$  is linearly independent. We prove that  $X$  is linearly normal if  $\rho(X)'' \geq \lceil (r+2)/2 \rceil$  and that  $\rho(X)'' < 2\lceil (r+1)/(n+1) \rceil$ , unless either  $n = r$  or  $X$  is a rational normal curve.

### 1. Introduction

Let  $X \subset \mathbb{P}^r$  be an integral and non-degenerate variety defined over an algebraically closed field with characteristic zero. Set  $n := \dim X$ . We recall that a zero-dimensional scheme  $Z \subset X$  is said to be *smoothable in  $X$*  if it is a flat limit of a family of finite subsets of  $X$  with cardinality  $\deg(Z)$  (see [16] for a discussion of it). If  $X$  is smooth (or if  $Z$  is contained in the smooth locus of  $X$ )  $Z$  is smoothable in  $X$  if and only if it is smoothable in  $\mathbb{P}^r$  and the notion of smoothability in  $\mathbb{P}^r$  does not depend on the choice of the embedding of  $Z$  in a projective space ([16, Proposition 2.1]). Let  $\rho(X)$  (resp.  $\rho(X)'$ , resp.  $\rho(X)''$ ) denote the maximal integer  $t > 0$  such that each zero-dimensional scheme (resp. each finite set, resp. each zero-dimensional scheme smoothable in  $X$ )  $Z \subset X$  with  $\deg(Z) = t$  is linearly independent. Obviously  $\rho(X) \leq \rho(X)'' \leq \rho(X)'$ . Since  $X$  is embedded in  $\mathbb{P}^r$ , we have  $\rho(X) \geq 2$ .

The integer  $\rho(X)'$  appears in very classical projective geometry papers and books (see [22, Ch. 8,9,10,12], [23, Ch. 27] and references therein for the case of a finite field). When  $X$  is a finite set of a finite dimensional vector space over a finite field, this integer is related to the minimum distance of the dual of the code obtained evaluating linear forms at the points of  $X$  ([18]). In the set-up of the  $X$ -rank described below the integer  $\rho(X)'$  gives a uniqueness result (see Remarks 2.3 for details and references).

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For algebraic varieties the integer  $\rho(X)$  is very natural. Zero-dimensional schemes appeared in several papers concerning the additive decomposition of polynomials and the equations of embedded projective varieties ([7, 14, 16]) and they allowed the introduction of a name, the cactus rank (of a point, of a homogeneous polynomial or a tensor) ([12, 13, 25]); in [24] it was called the scheme-rank. Over a finite field combining zero-dimensional schemes (all of them smoothable) with [18] is a standard feature to get bounds on the minimum distance and number of minimal weight codewords for an evaluation code coming from certain projective curves ([3, 8–10]).

To justify the integer  $\rho(X)''$  one should justify the use of smoothable zero-dimensional schemes, not just of zero-dimensional schemes. Smoothable zero-dimensional subschemes of  $X$  and the integer  $\rho(X)''$  arise in the study of secant varieties and border ranks described below (see [16, Proposition 2.5] for the case of additive decompositions of homogeneous polynomials). Computing  $\rho(X)''$  gives a lower bound for  $\rho(X)'$  and an upper bound for  $\rho(X)$ . Thus one can try to compute  $\rho(X)''$ , when computing  $\rho(X)$  fails. At least after [16] each time a lower bound for  $\rho(X)'$  is computed, it seems useful to ask oneself if the same proof works (with minimal modifications) for  $\rho(X)$  or at least for  $\rho(X)''$ . Using only smoothable zero-dimensional schemes instead of arbitrary ones allows the check of a shorter list of schemes in several proofs ([4–7, 11, 15]).

For any  $q \in \mathbb{P}^r$  the  $X$ -rank  $r_X(q)$  of  $q$  is the minimal positive integer  $t$  such that  $q \in \langle S \rangle$  for some finite subset  $S \subset X$  with  $\sharp(S) = t$ , where  $\langle \cdot \rangle$  denotes the linear span. For any positive integer  $t$  the  $t$ -secant variety  $\sigma_t(X)$  of  $X$  is the closure in  $\mathbb{P}^r$  of the union of all  $\langle S \rangle$  with  $S$  a finite subset of  $X$  with cardinality  $t$ .

The *border  $X$ -rank*  $b_X(q)$  of  $q \in \mathbb{P}^r$  is the minimal integer  $k$  such that  $q \in \sigma_k(X)$ . The *generic rank*  $r_{X,\text{gen}}$  is the minimal integer  $k > 0$  such that  $\sigma_k(X) = \mathbb{P}^r$ . There is a non-empty open subset  $U \subset \mathbb{P}^r$  such that  $r_X(q) = r_{X,\text{gen}}$  for all  $q \in U$ .

In this paper we prove that if  $\rho(X)''$  is large, then  $X$  is linearly normal and that  $\rho(X)''$  cannot be very large for  $n > 1$  (Theorem 1.1).

We prove the following results.

**Proposition 1.1.** *Assume that  $X$  is a curve and  $\rho(X)'' \geq \lceil (r+2)/2 \rceil$ . Then  $X$  is linearly normal.*

**Proposition 1.2.** *Assume  $n := \dim X \geq 2$ ,  $r_{X,\text{gen}} = \lceil (r+1)/(n+1) \rceil$  and  $\rho(X)'' > \lceil (r+1)/(n+1) \rceil$ . Then  $X$  is linearly normal.*

**Theorem 1.3.** *Let  $X \subset \mathbb{P}^r$  be an integral and non-degenerate variety. Set  $n := \dim X$ . We have  $\rho(X)'' \geq 2\lceil (r+1)/(n+1) \rceil$  if and only if either  $r = n$  (i.e.  $X = \mathbb{P}^r$ ) or  $n = 1$ ,  $r$  is odd and  $X$  is a rational normal curve.*

If  $n = r$  we have  $\rho(X)' = \rho(X) = 2$ . If  $X$  is a rational normal curve we have  $\rho(X) = \rho(X)' = r + 1$ . This is the only case with  $\rho(X)' = r + 1$  (Lemma 2.4). Theorem 1.3 implies that  $\rho(X)'' < 2\lceil (r+1)/(n+1) \rceil$  if  $(r+1)/(n+1) \notin \mathbb{Z}$ .

The example of a general linear projection in  $\mathbb{P}^4$  of the Veronese surface shows that in Proposition 1.2 it is not sufficient to assume that  $\rho(X)'' \geq \lceil (r+2)/(n+1) \rceil$ .

We point out that to get our results we only use a small family of zero-dimensional schemes, each of them with connected components of degree 1 or 2, but that this family contains a complete family covering  $X$ : each  $p \in X$  is contained in some scheme  $Z$  of the family.

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## 2. Preliminaries

Let  $X \subset \mathbb{P}^r$  be an integral and non-degenerate variety. Set  $n := \dim X$ . For any  $q \in \mathbb{P}^r$  let  $\mathcal{S}(X, q)$  be the set of all  $S \subset X$  such that  $|S| = r_X(q)$  and  $q \in \langle S \rangle$ .

*Remark 2.1.* Fix  $q \in \sigma_k(X)$ . If  $k \leq \rho(X)''$  there is a zero-dimensional scheme  $Z \subset X$  smoothable in  $X$  and such that  $\deg(Z) \leq k$  and  $q \in \langle Z \rangle$  ([14, Lemma 2.6, Theorem 1.18] and [16, Proposition 2.5]).

*Remark 2.2.* Let  $X \subset \mathbb{P}^r$  be a smooth variety with  $\dim X \leq 2$ . Every zero-dimensional subscheme of  $X$  is smoothable ([19]) and hence  $\rho(X)'' = \rho(X)$ . Easy examples show that we may have  $\rho(X) < \rho(X)'$  for a smooth curve (Example 4.2).

*Remark 2.3* ([14, Theorem 1.17]). Fix  $q \in \mathbb{P}^r$  and  $A, B \in \mathcal{S}(X, q)$ . Set  $x := r_X(q)$  and assume  $\rho(X)' \geq 2x$ . Since  $|A \cup B| \leq 2x$ ,  $A \cup B$  is linearly independent. Thus  $A = B$  ([4, Lemma 1]). Hence  $|\mathcal{S}(X, q)| = 1$ .

In following extremal case we are able to use only  $\rho(X)'$  instead of  $\rho(X)''$ .

**Lemma 2.4.** *The following conditions are equivalent:*

- (1)  $X$  is a rational normal curve;
- (2)  $\rho(X)' = r + 2 - n$ ;
- (3)  $\rho(X)' \geq r + 2 - n$ .

*Proof.* It is sufficient to prove that (3) implies (1).

First assume  $n = 1$ . Let  $H \subset \mathbb{P}^r$  be a general linear hyperplane. Since  $X \cap H$  is formed by  $\deg(X)$  points, if  $\rho(X)' > r$  we have  $\deg(X) = r$  and hence  $X$  is a rational normal curve.

Now assume  $n > 1$ . Take a general linear subspace  $V \subset \mathbb{P}^r$  with codimension  $n - 1$ . The scheme  $X \cap V$  is an integral curve spanning  $V$ . If  $X \cap V$  is not a rational normal curve we have  $\rho(X)' \leq \rho(X \cap V)' \leq r - n + 1$  by the case  $n = 1$  just proved. Now assume that  $X \cap V$  is a rational normal curve, i.e. assume  $\deg(X) = r + 1 - n$ . The classification of minimal degree subvarieties of projective spaces ([21, Proposition 3.10]) show that  $X$  contains lines (and hence  $\rho(X)' = 3$ ) unless  $r = 5$ ,  $n = 2$  and  $X$  is the Veronese surface. If  $X$  is the Veronese surface of  $\mathbb{P}^5$  we have  $\rho(X)' = 3$ , because no 3 points of  $X$  are coplanar, but  $X$  contains plane conics.  $\square$

### 3. The proofs

*Proof of Proposition 1.1.* Assume that  $X$  is not linearly normal. Thus there is a non-degenerate curve  $Y \subset \mathbb{P}^{r+1}$  such that  $X$  is an isomorphic linear projection of  $Y$  from some  $o \in \mathbb{P}^{r+1} \setminus Y$ . Set  $b := b_Y(o)$ . Each secant variety of a curve has the expected dimension ([1, Remark 1.6]). Thus  $r_{Y,\text{gen}} = \lceil (r+2)/2 \rceil$ . Hence  $b \leq \lceil (r+2)/2 \rceil$ . Let  $\ell : \mathbb{P}^{r+1} \setminus \{o\} \rightarrow \mathbb{P}^r$  denote the linear projection from  $o$ . By assumption  $o \notin Y$  and  $\ell|_Y$  is an embedding with  $\ell(Y) = X$ . Let  $W \subset Y$  be any zero-dimensional scheme. Since  $\ell|_Y : Y \rightarrow X$  is an isomorphism,  $W$  is smoothable in  $Y$  if and only if  $\ell(W)$  is smoothable in  $X$  and any degree  $b$  smoothable zero-dimensional subscheme of  $X$  is the image of a unique degree  $b$  zero-dimensional subscheme of  $Y$ . Thus  $\rho(Y)'' \geq \rho(X)''$ . The image in  $\mathbb{P}^r$  of a linear subspace  $V \subset \mathbb{P}^{r+1}$  has either dimension  $\dim V$  (case  $o \notin V$ ) or dimension  $\dim V - 1$  (case  $o \in V$ ). Since  $\rho(Y)'' \geq \rho(X)'' \geq \lceil (r+2)/2 \rceil = r_{Y,\text{gen}}$  and  $b \leq r_{Y,\text{gen}}$ , there is a smoothable zero-dimensional scheme  $W \subset Y$  such that  $o \in \langle W \rangle$  and  $\deg(W) = b$  (Remark 2.1). Since  $\ell(W)$  is not linearly independent, we have  $\rho(X)'' \leq b - 1$ , a contradiction.  $\square$

*Proof of Proposition 1.2.* Assume that  $X$  is not linearly normal. Thus there is a non-degenerate variety  $Y \subset \mathbb{P}^{r+1}$  such that  $X$  is an isomorphic linear projection of  $Y$  from some  $o \in \mathbb{P}^{r+1} \setminus Y$ . Set  $b := b_Y(o)$  and  $a := r_{X,\text{gen}} = \lceil (r+1)/(n+1) \rceil$ . Let  $\ell : \mathbb{P}^{r+1} \setminus \{o\} \rightarrow \mathbb{P}^r$  denote the linear projection from  $o$ . By assumption  $o \notin Y$  and  $\ell|_Y$  is an embedding with  $\ell(Y) = X$ . As in the proof of Proposition 1.1 we have  $\rho(Y)'' \geq \rho(X)''$  and to get a contradiction it is sufficient to prove that  $b \leq \rho(X)''$ . Assume  $b > \rho(X)''$ , i.e., assume  $b \geq a + 2$ . Since  $b > a$ , we have  $o \notin \sigma_a(Y)$ . Hence  $\ell|_{\sigma_a(Y)} : \sigma_a(Y) \rightarrow \mathbb{P}^r$  is a finite map. Since  $\ell(\sigma_a(Y)) = \sigma_a(X)$ , we get  $\dim \sigma_a(Y) = r$ . Since  $\dim \sigma_{a+1}(Y) > \dim \sigma_a(Y)$  ([1, Proposition 1.3]), we get  $\sigma_{a+1}(Y) = \mathbb{P}^{r+1}$ . Thus  $b \leq a + 1$ , a contradiction.  $\square$

**Lemma 3.1.** *Assume  $\rho(X)' \geq 2\lceil (r+1)/(n+1) \rceil$ . Then  $X$  is not defective,  $r+1 \equiv 0 \pmod{n+1}$  and for a general  $q \in \mathbb{P}^r$  we have  $|\mathcal{S}(X, q)| = 1$ .*

*Proof.* Set  $a := \lceil (r+1)/(n+1) \rceil$ . Fix any  $q \in \mathbb{P}^r$  such that  $r_X(q) \leq a$ . Remark 2.3 gives  $|\mathcal{S}(X, q)| = 1$  if  $\rho(X)' \geq 2r_X(q)$ . In particular  $|\mathcal{S}(X, q)| = 1$  for a general  $q \in \sigma_a(X)$ . Thus a dimensional count shows that  $\dim \sigma_a(X) = a(n+1) - 1$ . Since  $\dim \sigma_a(X) \leq r$ , we get  $a \in \mathbb{Z}$ ,  $\mathbb{P}^r = \sigma_a(X)$  and that  $X$  is not defective. Since  $\mathbb{P}^r = \sigma_a(X)$ , we have  $r_X(q) = a$  for a general  $q \in \mathbb{P}^r$ . Hence  $|\mathcal{S}(X, q)| = 1$  for a general  $q \in \mathbb{P}^r$ .  $\square$

The (smooth)  $n$ -dimensional varieties  $X \subset \mathbb{P}^{2n+1}$  such that  $\sigma_2(X) = \mathbb{P}^{2n+1}$  and  $|\mathcal{S}(X, q)| = 1$  are classically called OADP (or varieties with only one apparent double point), because projecting them from a general point of  $X$  one gets a variety with a unique singular point ([17]). They are always linearly normal ([17, Remark 1.2]). In [17] there are also older references and the classification of the smooth ones with dimension up to 3 ([26], [17, Theorem 7.1]). Thus the thesis of Lemma 3.1 is a generalization of this concept to

the case in which  $(r+1)/(n+1)$  is an integer  $> 2$ . But the assumption “ $\rho(X)' \geq 2\lceil(r+1)/(n+1)\rceil$ ” of the lemma is too strong to be interesting for the classification of extremal varieties. Just assuming  $\rho(X)' > 2$  excludes all  $X$  containing lines and hence all smooth OADP's of dimension 2 and 3.

**Corollary 3.2.** *Assume  $n := \dim X \geq 2$ ,  $\rho(X)'' > \lceil(r+1)/(n+1)\rceil$  and  $\rho(X)' \geq 2\lceil(r+1)/(n+1)\rceil$ . Then  $X$  is linearly normal, non-defective,  $r+1 \equiv 0 \pmod{n+1}$  and  $|\mathcal{S}(X, q)| = 1$  for a general  $q \in \mathbb{P}^r$ .*

*Proof of Corollary 3.2.* Apply Lemma 3.1 and Proposition 1.2.  $\square$

*Proof of Theorem 1.3.* Assume the existence of  $X$  with  $\rho(X)'' \geq 2\lceil(r+1)/(n+1)\rceil$ . We may assume  $n < r$ , i.e.,  $X \neq \mathbb{P}^r$ .

First assume  $n = 1$ . Lemma 2.4 gives that  $X$  is a rational normal curve, that  $r$  is odd and that  $\rho(X) = \rho(X)' = r + 1$ .

Now assume  $n \geq 2$ . By Lemma 3.1,  $a := (r+1)/(n+1)$  is an integer. Since  $r > n$ , we have  $a \geq 2$ . Fix a general  $S \subset X$  such that  $|S| = a - 1$ . Since  $S$  is general, each  $p \in S$  is a smooth point of  $X$ . We saw in the proof of Proposition 1.2 that  $V := \langle \cup_{p \in S} T_p X \rangle$  has dimension  $(a-1)(n+1) - 1$ . Fix  $o \in X \setminus S$ .

**Claim 1.**  $o \notin V$ .

*Proof of Claim 1.* Assume  $o \in V$ . We saw in the proof of Proposition 1.2 that there are connected degree 2 zero-dimensional schemes  $v_p \subset X$  such that  $(v_p)_{\text{red}} = \{p\}$  and  $o \in \langle Z \rangle$ , where  $Z := \cup_{p \in S} v_p$ . Since  $o \notin S$ , we have  $o \notin Z$ . Thus the scheme  $Z \cup \{o\}$  is linearly dependent. Since  $\deg(Z \cup \{o\}) = 2a - 1 < \rho(X)''$  and  $Z \cup \{o\}$  is smoothable, we got a contradiction.  $\square$

Let  $\ell : \mathbb{P}^r \setminus V \rightarrow \mathbb{P}^n$  denote the linear projection from  $V$ . By Claim 1 we have  $S = X \cap V$  and hence  $\mu = \ell|_{X \setminus S} : X \setminus S \rightarrow \mathbb{P}^n$  is a morphism. Fix  $o \in X \setminus S$  and assume the existence of  $o' \in X \setminus S$  such that  $o \neq o'$  and  $\mu(o) = \mu(o')$ . Thus  $o' \in \langle \{o\} \cup Z \rangle$ . Hence  $\{o, o'\} \cup Z$  is linearly dependent. The zero-dimensional scheme  $\{o, o'\} \cup Z$  is smoothable and it has degree  $2a \leq \rho(X)''$ , a contradiction.

Thus  $\mu : X \setminus S \rightarrow \mathbb{P}^n$  is an injective morphism between two quasi-projective varieties. Since  $\mathbb{P}^n$  is smooth (it would be sufficient to assume that the target,  $X$ , is normal or even less (weakly normal)) and we are in characteristic zero,  $\mu$  is an open map which is an isomorphism onto its image ([20]). Since  $X$  is smooth at each point of  $S$ ,  $X$  is smooth. Since  $S$  is finite, the étale cohomology of  $X \setminus S$  in dimension  $n - 1$  shows that  $\mathbb{P}^n \setminus \mu(X \setminus S)$  is finite with cardinality  $\sharp(S)$ . Thus  $\mu$  extends to an isomorphism  $u : X \rightarrow \mathbb{P}^n$ . However, by the definition of linear projection from  $V = \langle \cup_{p \in S} T_p X \rangle$ , the linear independence of the linear spaces  $T_p$ ,  $x \in S$ , and the smoothness of  $X$ ,  $\mu$  extends to some non-empty open subset of the exceptional divisor of the blowing-up of  $X$  at the points of  $S$ . Since  $S \neq \emptyset$ , this is absurd.  $\square$

#### 4. Elementary examples

By Proposition 1.1 to complete the picture for curves we need to describe the linearly normal curves with very high  $\rho(X)'$ ,  $\rho(X)$  and  $\rho(X)''$ . We also gives examples of smooth curves  $X$  with prescribed  $\rho(X)$  or prescribed  $\rho(X)'$ .

*Remark 4.1.* Let  $X$  be an integral projective curve. To compute  $\rho(X)''$  we recall that every Cartier divisor of  $X$  is smoothable. Let  $\mathcal{F}$  be any torsion free sheaf of  $X$ . Duality gives  $h^1(\mathcal{F}) = \dim \text{Hom}(\mathcal{F}, \omega_X)$ , ([2, 1.1 at p. 5]). Thus for any zero-dimensional scheme  $Z \subset X$  we have  $h^1(\mathcal{I}_Z(1)) = \dim \text{Hom}(\mathcal{I}_Z, \omega_X(1))$ . If  $d \geq 3g - 2$  we have  $\rho(X) = d - 2g + 2$ . We have  $\rho(X)' = d - 2g + 2$  if and only if  $X$  is Gorenstein, i.e.,  $\omega_X$  is locally free. For lower  $d$  the integers  $\rho(X)$ ,  $\rho(X)'$  and  $\rho(X)''$  depends both from the Brill-Noether theory of the special line bundles on  $X$  and the choice of the very ample line bundle  $\mathcal{O}_X(1)$ , not just the integers  $d$  and  $g$ .

**Example 4.2.** Fix integers  $r, a$  such that  $2 \leq a \leq r + 1$ . Here we prove the existence of a smooth and non-degenerate curve  $X \subset \mathbb{P}^r$  such that  $\rho(X)' = \rho(X) = a$  and, if  $a \leq r - 1$  and  $2a \geq r + 2$ , another example  $\tilde{X}$  with  $\rho(\tilde{X})' > \rho(\tilde{X}) = a$ . If  $a = r + 1$  we know that  $X$  is a rational normal curve. The case  $g = 1, d = r + 1$  covers the the case  $a = r$ . Now assume  $2 \leq a \leq r - 1$ .

(a) We first cover the case  $2a \leq r + 1$ . In this range we construct a smooth rational curve  $X$  with  $\rho(X) = \rho(X)' = a$ , but of course  $X$  is not linearly normal. Let  $Y \subset \mathbb{P}^{r+1}$  be a rational normal curve. Fix a set  $S \subset Y$  such that  $|S| = a + 1$  and take any  $o \in \langle S \rangle$  such that  $o \notin \langle S' \rangle$  for any  $S' \subsetneq S$ . Let  $\ell : \mathbb{P}^{r+1} \setminus \{o\} \rightarrow \mathbb{P}^r$  denote the linear projection from  $o$ . Since  $a \geq 2$  and  $\rho(Y) = r + 2$ , we have  $o \notin Y$ . Hence  $\ell|_Y$  is a morphism. Set  $X := \ell(Y)$ .

**Claim 1.**  $\ell|_Y$  is an embedding.

*Proof of Claim 1.* It is sufficient to prove that for any zero-dimensional scheme  $A \subset Y$  with  $\deg(A) \leq 2$  we have  $o \notin \langle A \rangle$ . Assume the existence of a zero-dimensional scheme  $A \subset Y$  with  $\deg(A) \leq 2$  and  $o \in \langle A \rangle$ . Since  $o \notin Y$ , we have  $\deg(A) = 2$ . Since  $o \notin \langle S' \rangle$  for any  $S' \subsetneq S$  and  $|S| = a + 1 > 2$ , we have  $A \not\subseteq S$ . Since  $o \in \langle A \rangle \cap \langle S \rangle$ ,  $A \cup S$  is linearly dependent. Since  $\deg(A \cup S) \leq a + 3$  and  $\rho(Y) = r + 1$ , we get a contradiction.  $\square$

By Claim 1  $X$  is a smooth rational curve and  $\deg(X) = r + 1$ .

**Claim 2.** We have  $\rho(X) = \rho(X)' = a$ .

*Proof of Claim 2.* Since  $\ell|_X$  is an embedding, we have  $|\ell(S)| = a + 1$ . Since  $o \in \langle S \rangle$ ,  $\ell(S)$  is linearly dependent and hence  $\rho(X)' \leq a$ . Assume  $\rho(X)' < a$  and take a zero-dimensional scheme  $Z \subset X$  such that  $\deg(Z) \leq a$  and  $Z$  is linearly dependent. Let  $W \subset Y$  be the only scheme such that  $\ell(W) = Z$ . Since  $Z$  is linearly dependent, we have  $o \in \langle W \rangle$ . Since  $\deg(W) \leq a$  and  $o \notin \langle S' \rangle$  for any  $S' \subsetneq S$ , we have  $W \not\subseteq S$ . Thus  $S \cup W$  is linearly dependent. Hence  $2a + 1 \geq r + 2$ , a contradiction.  $\square$

(b) Now assume  $2 \leq a \leq r - 1$  and  $2a \geq r + 2$ . Set  $g := r + 1 - a$ . Fix a smooth curve  $C$  of genus  $g$  and a zero-dimensional scheme  $A \subset X$  such that  $\deg(A) = a + 1$ . Since  $\deg(\omega_C(A)) = 2g + a + 1 - 2 \geq 2g + 1$ ,  $\omega_C(A)$  is very ample. By Riemann-Roch we have  $h^0(\omega_C(A)) = g + a = r + 1$ . Let  $f : C \rightarrow \mathbb{P}^r$ , be the embedding induced by  $|\omega_X(A)|$ . Set  $X := f(C)$  and  $Z := f(A)$ . By Riemann-Roch  $Z$  is linearly dependent and  $Z$  is the only linearly dependent zero-dimensional scheme  $W \subset X$  such that  $\deg(W) \leq a + 1$ . Thus  $\rho(X) = a$ . We have  $\rho(X)' = a$  if and only if  $A$  is a reduced set. If  $A$  is not reduced we get the promised curve  $\tilde{X}$ .

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