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A NOTE ON MULTILINEAR PSEUDO-DIFFERENTIAL OPERATORS AND ITERATED COMMUTATORS

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Abstract. This paper gives a sparse domination for the iterated commutators of multilinear pseudo-differential operators with the symbol σ belonging to the Hörmander class, and establishes the quantitative bounds of the Bloom type estimates for such commutators. Moreover, the C_p estimates for the corresponding multilinear pseudo-differential operators are also obtained.

1. Introduction

It is well known that the pseudo-differential operators play important roles in harmonic analysis and other fields, such as PDE and mathematical physics. Particularly, pseudo-differential operators are used extensively in partial differential equation, quantum field and index theory. It is worthy to pointing out that pseudo-differential operators played an influential role in the first proof of the Atiyah-Singer index theorem [2]. The investigation of pseudo-differential operator was initiated by Kohn and Nirenberg [26], and Hörmander [22]. Since then, great achievements have been made in the study of the properties of this operator. For examples, the local and global L^2 bounds of pseudo-differential operator were shown by Hörmander [23], Kumano-Go [28], Calderón and Vaillancourt [10]; the L^p-boundedness of this operator is given by Hörmander [22], Fefferman [18] and Stein [41]. The weighted theory for pseudo-differential operators has also been established. In order to state some known results, we need to introduce two definitions and briefly recount part of the history of this subject.

Definition. Let $r \in \mathbb{R}$ and $0 \leq \rho, \delta \leq 1$. We say a smooth function σ belongs to the Hörmander class $S^r_{\rho,\delta}(n,m)$ if for every tripe of multi-indices α and

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$\beta_1,\dots,\beta_m$, there is a constant $C_{\alpha,\beta}>0$ such that

$$|\partial_x^\alpha \partial_{\xi_1}^{\beta_1}\cdots\partial_{\xi_m}^{\beta_m}\sigma(x,\vec{\xi})|\le C_{\alpha,\beta}(1+|\xi_1|+\cdots+|\xi_m|)^{r-\rho\sum_{j=1}^m|\beta_j|+\delta|\alpha|}.$$

Definition. Given a symbol σ and for each locally integrable function b_{i_l}, the m-linear pseudo-differential operators T_σ and their commutators are given by

$$T_\sigma(\vec{f})(x):=\int_{(\mathbb{R}^n)^m}\sigma(x,\vec{\xi})e^{2\pi i x\cdot\vec{\xi}}\hat{f}_1(\xi_1)\cdots\hat{f}_m(\xi_1)d\vec{\xi},$$

$$[b_{i_1},\dots,[b_{i_l},T_\sigma]_{i_l}\cdots]_{i_1}(f_1,\dots,f_m)(x),$$

where $\hat{f}$ is the Fourier transform of f, $\{i_1,\dots,i_l\}\subset\{1,\dots,m\}$, and

$$[b_{i_l},T_\sigma]_{i_l}(f_1,\dots,f_m):=b_{i_l}T_\sigma(f_1,\dots,f_m)-T_\sigma(f_1,\dots,b_{i_l}f_{i_l},\dots,f_m).$$

In the linear case $m=1$, Chanillo and Torchinksy [16] proved that T_σ is bounded on $L^p(\omega)$ for $\omega\in A_{p/2}$ $(2\le p<\infty)$ provided that $\sigma\in S_{\rho,\delta}^{-n(1-\rho)/2}(n,1)$ with $0<\delta<\rho<1$. Subsequently, Michalowski, Rule and Staubach [32] extended the condition of [16] to $\delta=\rho$, they also gave the $L^p(\omega)$ bounds for $\omega\in A_p$ $(1<p<\infty)$ with σ belongs to smaller classes $S_{\rho,\delta}^{-n(1-\rho)}(n,1)$ $(0<\rho\le 1,\ 0\le\delta<1)$, see [33]. Recently, Beltran [3] proved that if $\sigma\in S_{\rho,\delta}^{r}(n,1)$ with $-n(1-\rho)/2<r<0,\ 0\le\delta\le\rho<1$, then T_σ is bounded on $L^p(\omega)$ for $\omega\in A_{p/2}\cap RH_{2t'/p'}$ and $2\le\rho<2t'$, where $t=(\rho-1)n/(2r)$. Still more recently, Beltran and Cladek [4] established the sparse bounds for pseudo-differential operators and thus they obtained some weighted estimates, which may recover, except for some critical cases, almost all the weighted results mentioned above. The commutators of the pseudo-differential operators have also been studied, we refer readers to [24, 25, 42, 45] and the references therein.

The earliest work of bilinear pseudo-differential operator origined from Coifman and Meyer [15], in which they used bilinear pseudo-differential operator as a model of Calderón commutator. Although bilinear pseudo-differential operators are the natural extension of linear cases, they do not always mimic the mapping behaviour of the linear pseudo-differential operators, this fact was first observed by Bényi and Torres [8], they showed that T_σ with $\sigma\in S_{0,0}^0(n,2)$ does not satisfy $L^2\times L^2\to L^1$. Moreover, it was shown in [5] that if $\sigma\in S_{\rho,\delta}^r(n,2)$ with $r<-n(1-\rho)$, then T_σ is bounded from $L^\infty(\mathbb{R}^n)\times L^\infty(\mathbb{R}^n)$ into $L^\infty(\mathbb{R}^n)$. However, Miyachi and Tomita [34] proved that T_σ fails to be bounded from $L^\infty(\mathbb{R}^n)\times L^\infty(\mathbb{R}^n)$ into $BMO(\mathbb{R}^n)$ with $r>-n(1-\rho)$. Therefore $r=-n(1-\rho)$ becomes a critical point.

For the critical point $r=-n(1-\rho)$, if $\rho\in[0,1/2)$ and $\sigma\in S_{\rho,0}^{-n(1-\rho)}(n,2)$, Bényi et al. [5] first demonstrated that T_σ is bounded from $L^\infty(\mathbb{R}^n)\times L^\infty(\mathbb{R}^n)$ into $BMO(\mathbb{R}^n)$. Later on, Naibo [37] extended the condition in [5] to $\sigma\in S_{\rho,\delta}^{-n(1-\rho)}(n,2)$ with $0\le\delta\le\rho<1/2$. In the recent work of [35], they showed that if $\rho\in[0,1)$, $\sigma\in S_{\rho,\rho}^{-n(1-\rho)}(n,2)$, then T_σ is bounded from $L^\infty(\mathbb{R}^n)\times L^\infty(\mathbb{R}^n)$ into $BMO(\mathbb{R}^n)$.

One may ask what is the relationship between the bilinear pseudo-differential operators and the bilinear Calderón-Zygmund operators. If $\sigma \in S^0_{1,\delta}(n,2)$ ($\delta \in [0,1)$) or $\sigma \in S^r_{\rho,\delta}(n,2)$ with $r < -2n(1-\rho)$, $0 \le \delta \le \rho \le 1$, $\delta < 1$ and $\rho > 0$, in [5], Bényi et al. pointed out that T_σ is a bilinear Calderón-Zygmund operator, hence, T_σ shares the same boundedness properties with bilinear Calderón-Zygmund operator. However, the Hörmander classes $S^r_{\rho,\delta}(n,2)$ do not necessarily give rise to bilinear Calderón-Zygmund operators except for some ranges of indices mentioned above. We refer readers to [6, 7, 12, 38] for the boundedness properties for the corresponding pseudo-differential operators and the compact properties [43] for their commutators. Among these, we would like to mention the recent work of Cao, Xue and Yabuta [12], in which they established the weak and strong estimates including local exponential estimates, weighted mixed weak type inequality and sharp weighted estimates for T_σ and $T_{\sigma,\Sigma\vec{b}}$ with $\vec{b} \in BMO^m$, where $T_{\sigma,\Sigma\vec{b}}$ is the commutators of T_σ with single entry.

In this paper, we will study the C_p estimates and Bloom type estimates for multilinear pseudo-differential operators and their iterated commutators. The investigation of the weighted inequalities for weights in the class of C_p originated from the classical Coifman-Fefferman inequality [14], that is, for every $p \in (0,\infty)$ and any weight $\omega \in A_\infty$, there exists a constant $C > 0$ such that

$$\|T^*(f)\|_{L^p(\omega)} \le C\|M(f)\|_{L^p(\omega)}, \tag{1}$$

where T^* is the maximal truncate Calderón-Zygmund operator. Muckenhoupt [36] pointed out that C_p condition (see the definition in Section 2) instead of A_∞ condition is the appropriate necessary condition for (1) holding to be true. While the sufficiency was demonstrated by Sawyer [39], which states that: let $p \in (1,\infty)$ and $\omega \in C_q$ ($q > p$), then (1) holds. However, it's still unknown whether $\omega \in C_p$ is the sufficient condition.

In 2017, Cejas, Li, Pérez and Rivera-Ríos [13] used a technique of [44] to extend (1) to a wider class of operators, including linear pseudo-differential operators T_σ with σ belongs to Hörmander classes. Recently, Canto [11] gave a quantitative weighted norm inequality for (1), which improved Sawyer's result.

On the other hand, the two weight estimates for commutators of a variety of operators have attracted a great amount of interest. The first work of these is the characterized theorem given by Bloom [9]:

$$\|[b,H]\|_{L^p(\alpha)\to L^p(\beta)} \simeq \|b\|_{BMO_v},$$

where H is the Hilbert transform, $1 < p < \infty$, $\alpha, \beta \in A_p$, $v = (\alpha/\beta)^{1/p}$ and $b \in BMO_v$ (see its definition in Section 2). Two weight inequalities for commutators are usually referred as Bloom type estimates. For other works, we refer to [1, 17, 19–21, 31, 40] and references therein. Among these, we would like to mention the remarkable work of Lerner et al. [31], in which they introduced a new method called sparse dominations to obtain the two weight quantitative estimates for Calderón-Zygmund operators. Inspired by this work, Kunwar and

Ou [27] first solved the Bloom type estimates for commutators of multilinear Calderón-Zygmund operators.

A natural question would ask, can we extend the C_p estimate and Bloom type estimates to multilinear pseudo-differential operators and their commutators, respectively? We give a firm answer in this article. Our main results are as follows:

Theorem 1.1. *Let $I = \{i_1, \dots, i_l\} \subset \{1, \dots, m\}$ and T_σ be a m-linear pseudo-differential operator. Suppose that $\sigma \in S^r_{\rho,\delta}(n,m)$ with $0 \le \rho, \delta \le 1$ and $r < 2n(\rho - 1)$. For locally integrable functions $\vec{b} = (b_{i_1}, \dots, b_{i_l})$ defined on $\mathbb{R}^n$ and for any bounded functions $\vec{f} = (f_1, \dots, f_m)$ with compact support, there exist 3^n sparse collections $\{\mathcal{S}_j\}_{j=1}^{3^n}$ such that*

$$|[b_{i_1}, \dots, [b_{i_l}, T_\sigma]_{i_l} \cdots]_{i_1}(\vec{f})| \le C\Big(\sum_{j=1}^{3^n} \sum_{\vec{\gamma} \in \{1,2\}^l} \mathcal{T}^{\vec{\gamma}}_{\mathcal{S}_j, b}(\vec{f})\Big), \quad a.e. \ \ x \in \mathbb{R}^n$$

where

$$\mathcal{T}^{\vec{\gamma}}_{\mathcal{S}_j, b}(\vec{f}) := \sum_{Q \in \mathcal{S}_j} \Big(\prod_{s=1}^{l} T(b_{i_s}, f_{i_s}, Q, \gamma_{i_s})\Big)\Big(\prod_{j \notin I} \langle |f_j| \rangle_Q\Big)\chi_Q, \quad \textit{with}$$

$$T(b, f, Q, \gamma) = \begin{cases} |b - \langle b \rangle_Q| \langle |f| \rangle_Q & \textit{if } \gamma = 1, \\ \langle |(b - \langle b \rangle_Q) f| \rangle_Q & \textit{if } \gamma = 2. \end{cases}$$

Theorem 1.2. *Let T_σ be a m-linear pseudo-differential operator, $\sigma \in S^r_{\rho,\delta}(n,m)$ with $0 \le \rho, \delta \le 1$ and $r < 2n(\rho - 1)$. Let $\vec{p} = (p_1, \dots, p_m)$, $1 < p_j < \infty$ and $1/p = 1/p_1 + \cdots + 1/p_m$. Let $\vec{\alpha} = (\alpha_1, \dots, \alpha_m)$ and $\vec{\beta} = (\beta_1, \dots, \beta_m)$ be vector weights satisfy $\alpha_j, \beta_j \in A_{p_j}$ $(j = 1, \dots, m)$. Denote $v_j = (\alpha_j/\beta_j)^{1/p_j}$ $(j = 1, \dots, m)$ and $v_{\vec{\beta}} = \prod_{j=1}^m \beta_j^{p/p_j}$. Then for $b_j \in BMO_{v_j}$ $(j = 1, \dots, m)$, it holds that*

$$\begin{aligned} &\|[b_m, \dots, [b_1, T_\sigma]_1 \cdots]_m(f_1, \dots, f_m)\|_{L^p(v_{\vec{\beta}})} \\ \lesssim &\Big(\prod_{j=1}^m [\alpha_j]_{A_{p_j}}^{\max\{1, \frac{1}{p_j - 1}\}} [\beta_j]_{A_{p_j}}^{\max\{p_j, p_1', \dots, p_m'\}/p_j}\Big) \prod_{j=1}^m \|b_j\|_{BMO_{v_j}} \prod_{j=1}^m \|f_j\|_{L^{p_j}(\alpha_j)}. \end{aligned}$$

Theorem 1.3. *Let T_σ be a m-linear pseudo-differential operator, $\sigma \in S^r_{\rho,\delta}(n,m)$ with $0 \le \rho, \delta \le 1$ and $r < 2n(\rho - 1)$. Let $\vec{p} = (p_1, \dots, p_m)$, $1 < p_j < \infty$ and $1/p = 1/p_1 + \cdots + 1/p_m$. Let $I = \{i_1, \dots, i_l\} \subset \{1, \dots, m\}$ and $b_{i_s} \in BMO_{v_{i_s}}$, $s = 1, \dots, l$. Suppose that $\vec{\alpha} = (\alpha_1, \dots, \alpha_m)$ and $\vec{\beta} = (\beta_1, \dots, \beta_m)$ are vector weights satisfying $\alpha_{i_s}, \beta_{i_s} \in A_{p_{i_s}}$ $(s = 1, \dots, l)$ and $\alpha_j = \beta_j$ $(j \notin I)$. Denote $\vec{q} = (p_j)_{j \notin I}$, $\vec{\omega} = (\alpha_j)_{j \notin I}$ with $\vec{\omega} \in A_{\vec{q}}$. Let $\vec{v} = (v_{i_1}, \dots, v_{i_l})$ with $v_{i_s} = (\alpha_{i_s}/\beta_{i_s})^{1/p_{i_s}}$ and $v_{\vec{\beta}} = \prod_{j=1}^m \beta_j^{p/p_j}$. Then*

$$\|[b_{i_1}, \dots, [b_{i_l}, T_\sigma]_{i_l} \cdots]_{i_1}(f_1, \dots, f_m)\|_{L^p(v_{\vec{\beta}})}$$

$$\lesssim C(\vec{\alpha},\vec{\beta},\vec{p})\prod_{s=1}^{l}\|b_{i_s}\|_{BMO_{v_{i_s}}}\prod_{j=1}^{m}\|f_j\|_{L^{p_j}(\alpha_j)},$$

where

$$C(\vec{\alpha},\vec{\beta},\vec{p}):=\Big(\prod_{s=1}^{l}[\alpha_{i_s}]_{A_{p_j}}^{\max\{1,\frac{1}{p_{i_s}-1}\}}[\beta_{i_s}]_{A_{p_{i_s}}}^{\max\{p_{i_s},q,p_1',\dots,p_m'\}/p_{i_s}}\Big)[\vec{\omega}]_{A_{\vec{q}}}^{\max\{q,p_1',\dots,p_m'\}/q},$$

$$1/q:=\sum_{j\notin I}1/q_j.$$

Theorem 1.4. *Let T_σ be a m-linear pseudo-differential operator, $\sigma\in S^r_{\rho,\delta}(n,m)$ with $0\le\rho,\delta\le 1$ and $r<2n(\rho-1)$. Let $0<p<\infty$ and ω belongs to the class of $C_{m\max\{1,p\}+\epsilon}$ weights for some $\epsilon>0$. Then*

$$\|T_\sigma(\vec{f})\|_{L^p(\omega)}\lesssim\|\mathcal{M}(\vec{f})\|_{L^p(\omega)},$$

where $\mathcal{M}$ denotes the multilinear maximal operator (see Subsection 2.3 for its definition).

Remark 1.5. We would like to give two comments:

(i) In Theorem 1.3 when $I=\emptyset$, the conclusion degenerates to

$$\|T_\sigma(f_1,\dots,f_m)\|_{L^p(v_{\vec{\omega}})}\lesssim[\vec{\omega}]_{A_{\vec{p}}}^{\max\{p,p_1',\dots,p_m'\}/p}\prod_{j=1}^{m}\|f_j\|_{L^{p_j}(\omega_j)},\quad\forall\vec{\omega}\in A_{\vec{p}},$$

hence, we extends the conclusion in [12].

(ii) Theorem 1.4 extends the linear case of [13] to the multilinear case.

This paper is organized as follows. In Section 2, we will present some related definitions and lemmas. The proofs of main theorems will be given in Section 3.

Through out the rest of our paper, we will denote positive constants by C, which may change at each occurrence. If $f\le Cg$ and $f\lesssim g\lesssim f$, we denote $f\lesssim g$, $f\sim g$, respectively. Given a cube Q, χ_Q means the characteristic function of Q and $\langle f\rangle_Q$ means the mean value of f over Q.

2. Preliminaries

In this section, we recall some necessary definitions and lemmas..

2.1. Weights and weighted BMO spaces

Definition (Multiple weights, [30])**.** Let $1<p_1,\dots,p_m<\infty$, $\vec{\omega}=(\omega_1,\dots,\omega_m)$, where each ω_i is a nonnegative and locally integrable function on $\mathbb{R}^n$, $\vec{\omega}$ is said to be a multiple $A_{\vec{p}}$ weight if

$$[\vec{\omega}]_{A_{\vec{p}}}:=\sup_Q\Big(\frac{1}{|Q|}\int_Q v_{\vec{\omega}}(x)dx\Big)\prod_{j=1}^{m}\Big(\frac{1}{|Q|}\int_Q\omega_i^{1-p_i'}(x)dx\Big)^{p/p_i'},$$

where $v_{\vec{\omega}} = \prod_{j=1}^{m} \omega_j^{p/p_j}$ and the supremum is taken over all cubes $Q \subset \mathbb{R}^n$.

When $m = 1$, the multilinear $A_{\vec{p}}$ weights coincide with the classical A_p weights. We also define the A_∞ constant by

$$[\omega]_{A_\infty} := \sup_Q \frac{1}{\omega(Q)} \int_Q M(\omega\chi_Q)(x)dx.$$

The weighted BMO space associated to weight $\omega \in A_\infty$ is the space of functions whose norm satisfies

$$\|b\|_{BMO_v} := \sup_Q \frac{1}{v(Q)} \int_Q |b - \langle b\rangle_Q| < \infty,$$

where $v(Q) := \int_Q v(x)dx$. Clearly, the weighted BMO space is the classical BMO space when $v = 1$.

A more larger class of weights is the following class of C_p weights.

Definition (C_p weights class, [39])**.** For $1 < p < \infty$, a nonnegative and locally integrable function ω belongs to the class of C_p weights, if there exist C, $\epsilon > 0$ such that for each cube Q and each measurable $E \subset Q$, we have

$$\omega(E) \le C\Big(\frac{|E|}{|Q|}\Big)^\epsilon \int_{\mathbb{R}^n} M(\chi_Q)(x)^p \omega(x)dx.$$

2.2. Sparse family

In this subsection, we will introduce a quite useful tool, which was introduced in [29].

In the following, we call $\mathcal{D}(Q)$ the dyadic grid obtained by repeatedly subdividing Q and its descendants in 2^n cubes with the same side length.

Definition (Dyadic lattice, [29])**.** A family of cubes is said to be a dyadic lattice $\mathcal{D}$ if it satisfies the following properties:

(1) if $Q \in \mathcal{D}$, then every descendant of Q is also in $\mathcal{D}$;
(2) for every two cubes $Q_1, Q_2 \in \mathcal{D}$, we can find a common ancestor $Q \in \mathcal{D}$ such that $Q_1, Q_2 \in \mathcal{D}(Q)$;
(3) for every compact set $K \subseteq \mathbb{R}^n$, we can find a cube $Q \in \mathcal{D}$ such that $K \subseteq Q$.

The following lemma is called the Three Lattice Theorem, which provides a foundation for our analysis.

Lemma 2.1 ([29])**.** *Given a dyadic lattice $\mathcal{D}$, there exist 3^n dyadic lattices $\mathcal{D}_1, \dots, \mathcal{D}_{3^n}$ such that*

$$\{3Q : Q \in \mathcal{D}\} = \bigcup_{j=1}^{3^n} \mathcal{D}_j$$

and for each cube $Q \in \mathcal{D}$ we can find a cube R_Q in each $\mathcal{D}_j$ such that $Q \subseteq R_Q$ and $3l_Q = l_{R_Q}$.

Remark 2.2. Fix a dyadic lattice $\mathcal{D}$. For any cube $Q \subset \mathbb{R}^n$, we can always find a cube $Q' \in \mathcal{D}$ such that $l_Q/2 < l_{Q'} \leq l_Q$ and $Q \subset 3Q'$. By Lemma 2.1, for some $j \in \{1, \dots, 3^n\}$, one can see that $3Q' =: P \in \mathcal{D}_j$. Hence, for any cube $Q \subset \mathbb{R}^n$, there is a cube $P \in \mathcal{D}_j$ satisfying $Q \subset P$ and $l_P \leq 3l_Q$.

Now we give the definition of sparse family.

Definition (Sparse family)**.** Let $\mathcal{D}$ be a dyadic lattice. $\mathcal{S} \subset \mathcal{D}$ is a η-sparse family with $\eta \in (0,1)$ if for every cube $Q \in \mathcal{S}$, we can find a measurable subset $E_Q \subset Q$ such that $\eta|Q| \leq |E_Q|$, where all the E_Q are pairwise disjoint.

We also need the following lemmas.

Lemma 2.3 ([12])**.** *Let $0 \leq \rho, \delta < 1$ and $r < 2n(\rho - 1)$. If $\sigma \in S^r_{\rho,\delta}(n,m)$, then the following pointwise estimates hold*:

(i) *for a.e. $x \in Q_0$*

$$|T_\sigma(\vec{f} \cdot \chi_{3Q_0})| \leq c_n \mathcal{N}_{weak} \prod_{i=1}^m |f_i(x)| + \mathcal{M}_{T_\sigma, Q_0}(\vec{f})(x),$$

where $\mathcal{N}_{weak} := \|T_\sigma\|_{L^1 \times \cdots \times L^1 \to L^{1/m,\infty}}$;

(ii) *for any $x \in \mathbb{R}^n$ and $0 < \epsilon < 1/m$,*

$$\mathcal{M}_{T_\sigma}(\vec{f})(x) \leq c_{n,r,\epsilon}(\mathcal{N}_{weak}\mathcal{M}(\vec{f})(x) + M_\epsilon(T_\sigma(\vec{f}))(x)).$$

In particular, we have

$$\mathcal{M}_{T_\sigma} : L^1(\mathbb{R}^n) \times \cdots \times L^1(\mathbb{R}^n) \to L^{1/m,\infty}(\mathbb{R}^n).$$

Lemma 2.4 ([27])**.** *Let $\vec{p} = (p_1, \dots, p_m)$, $1 < p_1, \dots, p_m < \infty$ and $1/p = 1/p_1 + \cdots + 1/p_m$. For any $I = \{i_1, \dots, i_l\} \subset \{1, \dots, m\}$, let $\vec{b} = (b_{i_1}, \dots, b_{i_l})$ with $b_{i_s} \in BMO_{v_{i_s}}$, $s = 1, \dots, l$. Suppose that $\vec{\alpha} = (\alpha_1, \dots, \alpha_m)$ and $\vec{\beta} = (\beta_1, \dots, \beta_m)$ are vector weights satisfying $\alpha_{i_s}, \beta_{i_s} \in A_{p_{i_s}}$ $(s = 1, \dots, l)$ and $\alpha_j = \beta_j$ $(j \notin I)$. Denote $\vec{q} = (p_j)_{j \notin I}$, $\vec{\omega} = (\alpha_j)_{j \notin I}$ with $\vec{\omega} \in A_{\vec{q}}$. Let $\vec{v} = (v_{i_1}, \dots, v_{i_l})$ with $v_{i_s} = (\alpha_{i_s}/\beta_{i_s})^{1/p_{i_s}}$ and $v_{\vec{\beta}} = \prod_{j=1}^m \beta_j^{p/p_j}$. Then*

$$\|\mathcal{T}^{\vec{\gamma}}_{\mathcal{S},\vec{b}}(\vec{f})\|_{L^p(v_{\vec{\beta}})} \lesssim C(\vec{\alpha}, \vec{\beta}, \vec{p}) \prod_{s=1}^l \|b_{i_s}\|_{BMO_{v_{i_s}}} \prod_{j=1}^m \|f_j\|_{L^{p_j}(\alpha_j)},$$

where

$$C(\vec{\alpha}, \vec{\beta}, \vec{p}) := \Big(\prod_{s=1}^l [\alpha_{i_s}]_{A_{p_j}}^{\max\{1, \frac{1}{p_{i_s}-1}\}} [\beta_{i_s}]_{A_{p_{i_s}}}^{\max\{p_{i_s}, q, p_1', \dots, p_m'\}/p_{i_s}} \Big) [\vec{\omega}]_{A_{\vec{q}}}^{\max\{q, p_1', \dots, p_m'\}/q},$$

$$1/q := \sum_{j \notin I} 1/q_j.$$

2.3. Maximal function, sharp maximal function and grand maximal function

The multilinear maximal function is defined by

$$\mathcal{M}(\vec{f})(x) := \sup_{Q \ni x} \prod_{i=1}^{m} \frac{1}{|Q|} \int_Q |f_i(y)| dy.$$

Let $Q \subset \mathbb{R}^n, \eta \in (0,1)$, the sharp maximal function $M^\sharp$, $M^\sharp_\eta$ and the grand maximal function $\mathcal{M}_{T_\sigma}$ are defined by

$$M^\sharp(f)(x) := \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - \langle f \rangle_Q| dy, \ M^\sharp_\eta(f) := M^\sharp(|f|^\eta)^{1/\eta},$$

$$\mathcal{M}_{T_\sigma}(\vec{f})(x) := \sup_{Q \ni x} \operatorname*{ess\,sup}_{\xi \in Q} |T_\sigma(\vec{f})(\xi) - T_\sigma(\vec{f}\chi_{3Q})(\xi)|.$$

Given a cube $Q_0 \subset \mathbb{R}^n$, we also define the local grand maximal operator $\mathcal{M}_{T_\sigma, Q_0}$ by

$$\mathcal{M}_{T_\sigma, Q_0}(\vec{f})(x) := \sup_{Q \ni x, Q \subset Q_0} \operatorname*{ess\,sup}_{\xi \in Q} |T_\sigma(\vec{f}\chi_{3Q_0})(\xi) - T_\sigma(\vec{f}\chi_{3Q})(\xi)|.$$

The following theorem is used to prove the C_p estimate.

Lemma 2.5 ([44]). *Let $1 < p < \infty$ and ω satisfy the $C_{p+\epsilon}$ condition for some $\epsilon > 0$. Then*

$$\|M(f)\|_{L^p(\omega)} \leq C \|M^\sharp(f)\|_{L^p(\omega)}.$$

3. Proofs of Theorems

This section is devoted to the proofs of our main results.

Proof of Theorem 1.1. For brevity, we only consider the case $m = 2$ and $I = \{1, 2\}$. Then, we need to prove the following domination

$$|[b_2, [b_1, T_\sigma]_1]_2(f_1, f_2)| \leq C \sum_{j=1}^{3^n} (\mathcal{T}^{(1,1)}_{\mathcal{S}_j, \vec{b}}(\vec{f}) + \mathcal{T}^{(1,2)}_{\mathcal{S}_j, \vec{b}}(\vec{f}) + \mathcal{T}^{(2,1)}_{\mathcal{S}_j, \vec{b}}(\vec{f}) + \mathcal{T}^{(2,2)}_{\mathcal{S}_j, \vec{b}}(\vec{f})).$$

By Remark 2.2, there are 3^n dyadic lattices $\mathcal{D}_j$ such that for any cube $Q \subset \mathbb{R}^n$, there is a cube $R_Q \in \mathcal{D}_j$ for some j such that $3Q \subset R_Q$ and $|R_Q| \leq 9^n |Q|$.

Fix a cube $Q_0 \subset \mathbb{R}^n$, we first claim that there exists a 1/2-sparse family $\mathcal{F} \subset \mathcal{D}(Q_0)$ such that for a.e. $x \in Q_0$,

$$\begin{aligned} &|[b_2, [b_1, T_\sigma]_1]_2(f_1\chi_{3Q_0}, f_2\chi_{3Q_0})| \\ &\leq C \sum_{Q \in \mathcal{F}} \Big(\sum_{\vec{\gamma} \in \{1,2\}^2} T'(b_1, f_1, Q, \gamma_1) T'(b_2, f_2, Q, \gamma_2) \Big) \chi_Q, \end{aligned} \tag{2}$$

where

$$T'(b_i, f_i, Q, \gamma_i) := \begin{cases} |b_i - \langle b_i \rangle_{R_Q}| \langle |f_i| \rangle_{3Q} & \text{if } \gamma_i = 1, \\ \langle |(b_i - \langle b_i \rangle_{R_Q}) f_i| \rangle_{3Q} & \text{if } \gamma_i = 2. \end{cases}$$

In order to prove (2), it suffices to prove the following claim: there exist mutually disjoint cubes $P_j \in \mathcal{D}(Q_0)$ such that $\sum_j |P_j| \le \frac{1}{2}|Q_0|$ and

$$\begin{aligned} &|[b_2,[b_1,T_\sigma]_1]_2(f_1\chi_{3Q_0}, f_2\chi_{3Q_0})(x)|\chi_{Q_0}(x) \\ \le\, & C\Big(\sum_{\vec{\gamma}\in\{1,2\}^2} T'(b_1,f_1,Q_0,\gamma_1)T'(b_2,f_2,Q_0,\gamma_2)\Big) \\ &+\sum_j |[b_2,[b_1,T_\sigma]_1]_2(f_1\chi_{3P_j}, f_2\chi_{3P_j})(x)|\chi_{P_j}(x). \end{aligned} \tag{3}$$

In fact, by integrating the above estimate, we can obtain (2) with $\mathcal{F} = \{P_j^k\}$, $k \in \mathbb{Z}^+$, where $\{P_j^0\} = \{Q_0\}$, $\{P_j^1\} = \{P_j\}$ and $\{P_j^k\}$ are the cubes achieved at the k-th stage of the iterative process.

For any mutually disjoint cubes $P_j \in \mathcal{D}(Q_0)$, it is easy to see that (3) follows from the following estimate.

$$\begin{aligned} &|[b_2,[b_1,T_\sigma]_1]_2(f_1\chi_{3Q_0}, f_2\chi_{3Q_0})|\chi_{Q_0\setminus\bigcup_j P_j} \\ &+\sum_j |[b_2,[b_1,T_\sigma]_1]_2(f_1\chi_{3Q_0}, f_2\chi_{3Q_0}) \\ &-[b_2,[b_1,T_\sigma]_1]_2(f_1\chi_{3P_j}, f_2\chi_{3P_j})|\chi_{P_j} \\ \le\, & C\Big(\sum_{\vec{\gamma}\in\{1,2\}^2} T'(b_1,f_1,Q_0,\gamma_1)T'(b_2,f_2,Q_0,\gamma_2)\Big). \end{aligned} \tag{4}$$

Now, we are in the position to prove (4). For any constant c, $[b_i, T_\sigma]_i = [b_i - c, T_\sigma]_i$, we get

$$\text{the left side of (4)} \le \sum_{i=1}^{8} I_i,$$

where

$$\begin{aligned} I_1 :=\, & |b_1 - \langle b_1\rangle_{R_{Q_0}}||b_2 - \langle b_2\rangle_{R_{Q_0}}| \\ &\times\sum_j |T_\sigma(f_1\chi_{3Q_0}, f_2\chi_{3Q_0}) - T_\sigma(f_1\chi_{3P_j}, f_2\chi_{3P_j})|\chi_{P_j}, \end{aligned}$$

$$I_2 := |b_1 - \langle b_1\rangle_{R_{Q_0}}||b_2 - \langle b_2\rangle_{R_{Q_0}}||T_\sigma(f_1\chi_{3Q_0}, f_2\chi_{3Q_0})|\chi_{Q_0\setminus\bigcup_j P_j},$$

$$\begin{aligned} I_3 :=\, & |b_2 - \langle b_2\rangle_{R_{Q_0}}|\sum_j |T_\sigma((b_1 - \langle b_1\rangle_{R_{Q_0}})f_1\chi_{3Q_0}, f_2\chi_{3Q_0}) \\ &- T_\sigma((b_1 - \langle b_1\rangle_{R_{Q_0}})f_1\chi_{3P_j}, f_2\chi_{3P_j})|\chi_{P_j}, \end{aligned}$$

$$I_4 := |b_2 - \langle b_2\rangle_{R_{Q_0}}||T_\sigma((b_1 - \langle b_1\rangle_{R_{Q_0}})f_1\chi_{3Q_0}, f_2\chi_{3Q_0})|\chi_{Q_0\setminus\bigcup_j P_j},$$

$$I_5 := |b_1 - \langle b_1\rangle_{R_{Q_0}}||T_\sigma(f_1\chi_{3Q_0}, (b_2 - \langle b_2\rangle_{R_{Q_0}})f_2\chi_{3Q_0})|\chi_{Q_0\setminus\bigcup_j P_j},$$

$$I_6 := |b_1 - \langle b_1\rangle_{R_{Q_0}}|\sum_j |T_\sigma(f_1\chi_{3Q_0}, (b_2 - \langle b_2\rangle_{R_{Q_0}})f_2\chi_{3Q_0})$$

$$- T_\sigma(f_1\chi_{3P_j}, (b_2 - \langle b_2\rangle_{R_{Q_0}})f_2\chi_{3P_j})|\chi_{P_j},$$

$$I_7 := |T_\sigma((b_1 - \langle b_1\rangle_{R_{Q_0}})f_1\chi_{3Q_0}, (b_2 - \langle b_2\rangle_{R_{Q_0}})f_2\chi_{3Q_0})|\chi_{Q_0\setminus\bigcup_j P_j},$$

$$I_8 := \sum_j |T_\sigma((b_1 - \langle b_1\rangle_{R_{Q_0}})f_1\chi_{3Q_0}, (b_2 - \langle b_2\rangle_{R_{Q_0}})f_2\chi_{3Q_0}) - T_\sigma((b_1 - \langle b_1\rangle_{R_{Q_0}})f_1\chi_{3P_j}, (b_2 - \langle b_2\rangle_{R_{Q_0}})f_2\chi_{3P_j})|\chi_{P_j}.$$

Now we temporarily stop the proof of inequality (4), turn to estimate the subsets of Q_0, and then return to the proof. Set $E = \bigcup_{j=1}^4 E_j$, where

$$E_1 := \{x \in Q_0 : \max\{|f_1(x)f_2(x)|, \mathcal{M}_{T_\sigma,Q_0}(f_1, f_2)(x)\} > C\langle|f_1|\rangle_{3Q_0}\langle|f_2|\rangle_{3Q_0}\},$$

$$E_2 := \{x \in Q_0 : \max\{|(b_1(x) - \langle b_1\rangle_{R_{Q_0}})f_1(x)f_2(x)|, \mathcal{M}_{T_\sigma,Q_0}((b_1 - \langle b_1\rangle_{R_{Q_0}})f_1, f_2)(x)\} > C\langle|(b_1 - \langle b_1\rangle_{R_{Q_0}})f_1|\rangle_{3Q_0}\langle|f_2|\rangle_{3Q_0}\},$$

$$E_3 := \{x \in Q_0 : \max\{|f_1(x)(b_2(x) - \langle b_2\rangle_{R_{Q_0}})f_2(x)|, \mathcal{M}_{T_\sigma,Q_0}(f_1, (b_2 - \langle b_2\rangle_{R_{Q_0}})f_2)(x)\} > C\langle|f_1|\rangle_{3Q_0}\langle|(b_2 - \langle b_2\rangle_{R_{Q_0}})f_2|\rangle_{3Q_0}\},$$

$$E_4 := \{x \in Q_0 : \max\{|(b_1(x) - \langle b_1\rangle_{R_{Q_0}})f_1(x)(b_2(x) - \langle b_2\rangle_{R_{Q_0}})f_2(x)|, \mathcal{M}_{T_\sigma,Q_0}((b_1 - \langle b_1\rangle_{R_{Q_0}})f_1, (b_2 - \langle b_2\rangle_{R_{Q_0}})f_2)(x)\} > C\langle|(b_1 - \langle b_1\rangle_{R_{Q_0}})f_1|\rangle_{3Q_0}\langle|(b_2 - \langle b_2\rangle_{R_{Q_0}})f_2|\rangle_{3Q_0}\}.$$

Notice that when C is large enough, it holds that

$$|\{x \in Q_0 : |f_1(x)f_2(x)| > C\langle|f_1|\rangle_{3Q_0}\langle|f_2|\rangle_{3Q_0}\}| \leq \frac{1}{8 \cdot 2^{n+2}}|Q_0|.$$

By Lemma 2.3, choose C big enough, one may obtain

$$|\{x \in Q_0 : \mathcal{M}_{T_\sigma,Q_0}(f_1, f_2)(x) > C\langle|f_1|\rangle_{3Q_0}\langle|f_2|\rangle_{3Q_0}\}| \leq \frac{1}{8 \cdot 2^{n+2}}|Q_0|.$$

Hence, we have $|E_1| \leq \frac{1}{4\cdot 2^{n+2}}|Q_0|$.

Similarly, we may obtain $|E_j| \leq \frac{1}{4\cdot 2^{n+2}}|Q_0|$ for $j = 2, 3, 4$.

Therefore, sum up in all, we have $|E| \leq \frac{1}{4\cdot 2^{n+2}}|Q_0|$ for C large enough.

Applying the Calderón-Zygmund decomposition to χ_E on Q_0 at height $h = 1/2^{n+1}$, we obtain mutually disjoint cubes $P_j \in \mathcal{D}(Q_0)$ such that for any j

$$2^{-n-1}|P_j| \leq |P_j \cap E| \leq 2^{-1}|P_j|, \quad |E\setminus \cup_j P_j| = 0.$$

Then, we have

$$\sum_j |P_j| \leq 2^{-1}|Q_0|, \quad P_j \cap E^c \neq \emptyset. \tag{5}$$

Now we return to the estimates of I_j ($j = 1, \ldots, 8$). We only consider the estimates of I_6, I_7, since the other terms can be dealt with by a similar method.

For I_6, since P_j is mutually disjoint, by the definition of $\mathcal{M}_{T_\sigma,Q_0}$ and (5), we deduce that

$$\begin{aligned} I_6 &\le |b_1 - \langle b_1\rangle_{R_{Q_0}}| \operatorname*{ess\,sup}_{\xi\in P_j} |T_\sigma(f_1\chi_{3Q_0}, (b_2 - \langle b_2\rangle_{R_{Q_0}})f_2\chi_{3Q_0}) \\ &\quad - T_\sigma(f_1\chi_{3P_j}, (b_2 - \langle b_2\rangle_{R_{Q_0}})f_2\chi_{3P_j})| \\ &\le |b_1 - \langle b_1\rangle_{R_{Q_0}}| \mathcal{M}_{T_\sigma,Q_0}(f_1, (b_2 - \langle b_2\rangle_{R_{Q_0}})f_2) \\ &\le C|b_1 - \langle b_1\rangle_{R_{Q_0}}|\langle|f_1|\rangle_{3Q_0}\langle|(b_2 - \langle b_2\rangle_{R_{Q_0}})f_2|\rangle_{3Q_0} \\ &= CT'(b_1, f_1, Q_0, 1)T'(b_2, f_2, Q_0, 2). \end{aligned}$$

For I_7, since $x \in Q_0\backslash\bigcup_j P_j$ means that $x \notin E_4$, by the definition of E_4 and Lemma 2.3, it gives that

$$\begin{aligned} I_7 &\le C|(b_1(x) - \langle b_1\rangle_{R_{Q_0}})f_1(x)||(b_2(x) - \langle b_2\rangle_{R_{Q_0}})f_2| \\ &\quad + \mathcal{M}_{T_\sigma,Q_0}((b_1 - \langle b_1\rangle_{R_{Q_0}})f_1, ((b_2 - \langle b_2\rangle_{R_{Q_0}}))f_2)(x) \\ &\le C\langle|(b_1 - \langle b_1\rangle_{R_{Q_0}})f_1|\rangle_{3Q_0}\langle|(b_2 - \langle b_2\rangle_{R_{Q_0}})f_2|\rangle_{3Q_0} \\ &= CT'(b_1, f_1, Q_0, 2)T'(b_2, f_2, Q_0, 2). \end{aligned}$$

This proves our claim and therefore inequality (2) holds.

We continue to prove our Theorem. Decompose $\mathbb{R}^n$ by cubes Q_j such that $\operatorname{supp}(f) \subset 3Q_j$. Now for a.e. $x \in Q_j$, since $\operatorname{supp}(f) \subset 3Q_j$, we have

$$\begin{aligned} &|[b_2, [b_1, T_\sigma]_1]_2(f_1, f_2)(x)|\chi_{Q_j}(x) \\ &\le C \sum_{Q\in\mathcal{F}_j} \Big(\sum_{\vec{\gamma}\in\{1,2\}^2} T'(b_1, f_1, Q, \gamma_1)T'(b_2, f_2, Q, \gamma_2)\Big)\chi_Q(x), \end{aligned}$$

where $\mathcal{F}_j \subset \mathcal{D}(Q_j)$ is a 1/2-sparse family.

Denote $\mathcal{F} = \bigcup_j \mathcal{F}_j$, then $\mathcal{F}$ is also a 1/2-sparse family, this time for a.e. $x \in \mathbb{R}^n$,

$$\begin{aligned} &|[b_2, [b_1, T_\sigma]_1]_2(f_1, f_2)(x)| \\ &\le C \sum_{Q\in\mathcal{F}} \Big(\sum_{\vec{\gamma}\in\{1,2\}^2} T'(b_1, f_1, Q, \gamma_1)T'(b_2, f_2, Q, \gamma_2)\Big)\chi_Q(x). \end{aligned} \tag{6}$$

Keeping in mind that $\langle|f_i|\rangle_{3Q} \le c_n\langle|f_i|\rangle_{R_Q}$ $(i = 1, 2)$, further, denote $\mathcal{S}_j = \{R_Q \in \mathcal{D}_j : Q \in \mathcal{F}\}$, then the conclusion follows from (6). □

Proofs of Theorems 1.2 *and* 1.3. Theorem 1.2 follows by Theorem 1.1 and taking $I = \{1, \ldots, m\}$ in Lemma 2.4. Theorem 1.3 follows from Theorem 1.1 and Lemma 2.4 □

Proof of Theorem 1.4. Let $\eta \in (0, 1/m)$ be a parameter to be determined later. By Lebesgue differentiation theorem, we get

$$\|T_\sigma(\vec{f})\|_{L^p(\omega)} \le \|M(|T_\sigma(\vec{f})|^\eta)^{1/\eta}\|_{L^p(\omega)}.$$

It was proved in [12] that

$$M_\eta^\sharp(T_\sigma(\vec{f}))(x) \le C\mathcal{M}(\vec{f})(x). \tag{7}$$

Now we take η such that

$$m\max\{1,p\} < p/\eta < m\max\{1,p\} + \epsilon.$$

Setting $\epsilon_m = m\max\{1,p\} + \epsilon - p/\eta$, by our assumption, we have $\omega \in C_{p/\eta+\epsilon_m}$. Invoking Lemma 2.5 and (7), we deduce that

$$\begin{aligned}\|T_\sigma(\vec{f})\|_{L^p(\omega)} &\le C\|M^\sharp(|T_\sigma(\vec{f})|^\eta)\|_{L^{p/\eta}(\omega)}^{1/\eta}\\ &= C\|M_\eta^\sharp(T_\sigma(\vec{f}))\|_{L^p(\omega)} \le C\|\mathcal{M}(\vec{f})\|_{L^p(\omega)}.\end{aligned}$$

This finishes the proof of Theorem 1.4. □

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