Bull. Korean Math. Soc. **0** (0), No. 0, pp. 1–0
https://doi.org/10.4134/BKMS.b190535
pISSN: 1015-8634 / eISSN: 2234-3016

SOME RESULTS ON MEROMORPHIC SOLUTIONS OF CERTAIN NONLINEAR DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we investigate the transcendental meromorphic solutions for the nonlinear differential equations $f^n f^{(k)} + Q_{d_*}(z,f) = R(z)e^{\alpha(z)}$ and $f^n f^{(k)} + Q_d(z,f) = p_1(z)e^{\alpha_1(z)} + p_2(z)e^{\alpha_2(z)}$, where $Q_{d_*}(z,f)$ and $Q_d(z,f)$ are differential polynomials in f with small functions as coefficients, of degree $d_* (\leq n-1)$ and $d\,(\leq n-2)$ respectively, R, p_1, p_2 are non-vanishing small functions of f, and α, α_1, α_2 are non-constant entire functions. In particular, we give out the conditions for ensuring the existence of these kinds of meromorphic solutions and their possible forms of the above equations.

1. Introduction

Let $f(z)$ be a transcendental meromorphic function in the complex plane $\mathbb{C}$. We assume that the reader is familiar with the standard notations and main results in Nevanlinna theory (see [1, 2, 6]). Throughout this paper, the term $S(r,f)$ always has the property that $S(r,f) = o(T(r,f))$ as $r \to \infty$, possibly outside a set E (which is not necessarily the same at each occurrence) of finite linear measure. A meromorphic function $a(z)$ is said to be a small function with respect to $f(z)$ if and only if $T(r,a) = S(r,f)$. A differential polynomial $Q_d(z,f)$ in f of degree d is a polynomial in f and its derivatives of a total degree at most d with small functions of f as the coefficients.

Recently, many scholars focus on the meromorphic solutions of the nonlinear differential equations of the form

$$f^n f' + Q_d(z,f) = h, \tag{1}$$

Received May 29, 2019; Revised September 16, 2019; Accepted October 16, 2019.
2010 *Mathematics Subject Classification.* Primary 34M05, 30D30, 30D35.
Key words and phrases. meromorphic functions, nonlinear differential equations, small functions, differential polynomials.
This work was supported by NNSF of China (No. 11801215 & No. 11626112 & No. 11371225), the NSF of Shandong Province, P. R. China (No. ZR2016AQ20 & No. ZR2018MA021), and the Fund of Doctoral Program Research of University of Jinan (No. XBS1630).

where $Q_d(z, f)$ denotes a polynomial in f and its derivatives with a total degree $d \le n-1$ with small functions of f as the coefficients, and h is a given meromorphic function.

In 2014, Liao and Ye [3] investigated the forms of meromorphic solutions of equation (1) for specific $Q_d(z, f)$ and h, and obtained the following result.

Theorem 1 ([3]). *Let $Q_d(z, f)$ be a differential polynomial in f of degree d with rational function coefficients. Suppose that u is a nonzero rational function and v is a nonconstant polynomial. If $n \ge d+1$ and the differential equation*

$$f^n f' + Q_d(z, f) = u(z)e^{v(z)} \tag{2}$$

admits a meromorphic solution f with finitely many poles, then f has the following form:

$$f(z) = s(z)e^{v(z)/(n+1)} \quad \text{and} \quad Q_d(z, f) \equiv 0,$$

where $s(z)$ is a rational function with $s^n((n+1)s' + v's) = (n+1)u$. In particular, if u is a polynomial, then s is a polynomial, too.

Later Lü etc., [5] changed the condition on the coefficients of the differential polynomial from rational functions to small functions, and extended Theorem 1 to the following result.

Theorem 2 ([5]). *Let $P_{n-1}(f)$ be a differential polynomial in f with coefficients being small functions, and let* $\deg P_{n-1}(f) \le n-1$. *Then for any positive integer n, any entire function α and any small function R, the equation*

$$f^n f' + P_{n-1}(f) = Re^{\alpha} \tag{3}$$

does not posses any transcendental meromorphic solution $f(z)$ with $N(r, f) = S(r, f)$ unless $P_{n-1}(f) \equiv 0$. Moreover, if the equation (3) possesses a meromorphic solution f with $N(r, f) = S(r, f)$, then (3) will become $f^n f' = Re^{\alpha}$ and $f(z)$ has the form $f(z) = u\exp(\alpha/(n+1))$ as the only possible admissible solution of (3), where u is a small function of f.

Then it is natural to ask what will happen if the dominant term is replaced by $f^n f^{(k)}$ when $k \ge 2$? Unfortunately, the method used in the proof of [5, Theorem 1.1] is not valid when $k \ge 2$ by a carefully observation, so in this paper we consider the above problem from a new angle by using deficiency, and obtain the following Theorem 1.1.

We need the following notations in order to state our results. Let p be a positive integer and $a \in \mathbb{C} \cup \{\infty\}$. We use $N_{p)}\left(r, \frac{1}{f-a}\right)$ ($N_{(p+1}\left(r, \frac{1}{f-a}\right)$) to denotes the counting function of the zeros of $f - a$, whose multiplicities are not greater than p (less than $p+1$). Define

$$\delta(a, f) = 1 - \limsup_{r\to\infty} \frac{N\left(r, \frac{1}{f-a}\right)}{T(r, f)}, \text{ and } \delta_{p)}(a, f) = 1 - \limsup_{r\to\infty} \frac{N_{p)}\left(r, \frac{1}{f-a}\right)}{T(r, f)}.$$

Theorem 1.1. *Let $n \geq 2$, $k \geq 2$ be integers, $Q_{d_*}(z, f)$ be a differential polynomial in f with coefficients being small functions, and $d_* \leq n-1$. Then for any entire function α and any small function R, the equation*

$$f^n f^{(k)} + Q_{d_*}(z, f) = Re^{\alpha} \tag{4}$$

has a transcendental meromorphic solution f with $\delta(\infty, f) = 1$ and $\delta_{1)}(0, f) > 0$ if and only if $Q_{d_*}(z, f) \equiv 0$. Moreover, if the equation (4) possesses a transcendental meromorphic solution f with $\delta(\infty, f) = 1$ and $\delta_{1)}(0, f) > 0$, then (4) will become $f^n f^{(k)} = Re^{\alpha}$ and $f(z)$ has the form $f(z) = u \exp(\alpha/(n+1))$ as the only possible admissible solution of (4), where u is a small function of f.

Remark 1. Actually, in Theorem 1.1 "if" part, i.e., if $Q_{d_*}(z, f) \equiv 0$, then we have not only $\delta_{1)}(0, f) > 0$ but also $\delta_{1)}(0, f) = \delta(0, f) = 1$ by using the following Lemma 2.3 directly.

Being enlightened by Theorem 2, we pose the following question.

Question 1. *Can the condition $\delta_{1)}(0, f) > 0$ in Theorem 1.1 "only if" part be omitted or not? That means, if the equation (4) possesses a transcendental meromorphic solution f with $\delta(\infty, f) = 1$, can we get $Q_{d_*}(z, f) \equiv 0$ and the related results?*

It is also interesting and difficult to consider what is the form of meromorphic solutions of the following differential equations:

$$f^n f' + Q_d(z, f) = p_1(z)e^{\alpha_1(z)} + p_2(z)e^{\alpha_2(z)}, \tag{5}$$

where $Q_d(z, f)$ is a differential polynomial in f with small functions of f as the coefficients, p_1, p_2 are small functions of f, $\alpha_1(z), \alpha_2(z)$ are nonconstant entire functions.

Recently, Zhang [7] gave the forms of transcendental meromorphic solutions of equation (5) for a particular case, when $Q_d(z, f)$ is a rational function of z (i.e. $d = 0$), p_1, p_2 are nonzero rational functions and α_1, α_2 are nonconstant polynomials. And they showed that the conditions concerning α_1'/α_2' could ensure the existence of the possible meromorphic solutions of the above equation. Later, in [9] Zhang etc., further their results to the case when $Q_d(z, f)$ is a differential polynomial in f of degree $d \leq n-2$ with rational functions as its coefficients.

In 2017, Lu [4] replaced the dominant term $f^n f'$ in equation (5) by $f^n f^{(k)}$, the rational coefficients of the differential polynomial by small functions, changed nonconstant polynomials α_1 and α_2 to entire functions satisfying one of the following three conditions (a) $T(r, e^{\alpha_1}) = S(r, e^{\alpha_2})$ (b) $T(r, e^{\alpha_2}) = S(r, e^{\alpha_1})$ (c) $T(r, e^{\alpha_1}) = O(T(r, e^{\alpha_2}))$ & $T(r, e^{\alpha_2}) = O(T(r, e^{\alpha_1}))$, and obtained the following result.

Theorem 3 ([4])**.** *Let $n \geq 3$, $k \geq 2$ be integers, and $P_{n-3}(z, f)$ be a differential polynomial in f of degree at most $n-3$ with small functions as its coefficients,*

α_1, α_2 be nonconstant entire functions, p_1, p_2 be nonzero small functions of both $e^{\alpha_1(z)}$ and $e^{\alpha_2(z)}$. If $f(z)$ is a transcendental meromorphic solution of the following nonlinear differential equation

$$f^n f^{(k)} + P_{n-3}(z, f) = p_1(z)e^{\alpha_1(z)} + p_2(z)e^{\alpha_2(z)} \tag{6}$$

satisfying $N(r, f) = S(r, f)$, then there exist two cases:

(I) *$T(r, e^{\alpha_1}) = S(r, e^{\alpha_2})$, and $f^n f^{(k)} = p_2e^{\alpha_2}$, $P_{n-3}(f) = p_1e^{\alpha_1}$; Or $T(r, e^{\alpha_2}) = S(r, e^{\alpha_1})$, and $f^n f^{(k)} = p_1e^{\alpha_1}$, $P_{n-3}(f) = p_2e^{\alpha_2}$;*

(II) *$T(r, e^{\alpha_1}) = O(T(r, e^{\alpha_2}))$. In this case, we have that*

$$T(r, f) = O(T(r, e^{\alpha_1})) = O(T(r, e^{\alpha_2}))$$

and therefore $S(r, f) = S(r, e^{\alpha_1}) = S(r, e^{\alpha_2})$. We use $T(r)$, resp. $S(r)$ to denote these two quantities. Then one of the following holds:
(1) $T(r, e^{\alpha_2-\alpha_1}) = S(r)$. In this case, $P_{n-3}(f) \equiv 0$ and $f^n f^{(k)} = (p_1 + \varphi p_2)e^{\alpha_1}$, where $\varphi = e^{\alpha_2-\alpha_1}$;
(2) $T(r, e^{k\alpha_1-(n+1)\alpha_2}) = S(r)$, where k is an integer satisfying $1 \le k \le n-3$. In this case, $f^n f^{(k)} = p_1e^{\alpha_1}$ and $P_{n-3}(f) = p_2e^{\alpha_2}$, which actually means $f = s_1(z)e^{\frac{\alpha_1}{n+1}}$ with $T(r, s_1) = S(r)$;
(3) $T(r, e^{k\alpha_2-(n+1)\alpha_1}) = S(r)$, where k is an integer satisfying $1 \le k \le n-3$. In this case, $f^n f^{(k)} = p_2e^{\alpha_2}$ and $P_{n-3}(f) = p_1e^{\alpha_1}$, which actually means $f = s_2(z)e^{\frac{\alpha_2}{n+1}}$ with $T(r, s_2) = S(r)$.

In Theorem 3, the degree of the differential polynomial $P_{n-3}(z, f)$ is under the condition "at most $n-3$", then it is natural to ask what will happen if the degree of the differential polynomial is bigger than $n-3$? In this paper, we study the above problem, consider the form of solutions of the following equation (7) when $d \le n-2$ and entire functions α_1 and α_2 satisfying one of the above three conditions (a), (b), (c) by using deficiency, and obtain the following Theorem 1.2.

Theorem 1.2. *Let f be a transcendental meromorphic function in the plane with $\delta(\infty, f) = 1$ and $\delta_{1)}(0, f) > 0$, $n \ge 2$, $k \ge 1$ be integers, and $Q_d(z, f)$ be a differential polynomial in f of degree $d \le n-2$ with small functions as its coefficients, α_1, α_2 be nonconstant entire functions satisfying one of the above three conditions* (a), (b), (c), *and p_1, p_2 be nonzero small functions of f. Suppose the following nonlinear differential equation*

$$f^n(z)f^{(k)}(z) + Q_d(z, f) = p_1(z)e^{\alpha_1(z)} + p_2(z)e^{\alpha_2(z)}, \tag{7}$$

holds, then

(I) *if $T(r, e^{\alpha_1}) = S(r, e^{\alpha_2})$, then $f^n f^{(k)} = p_2e^{\alpha_2}$, $Q_d(z, f) = p_1e^{\alpha_1}$, and $f(z) = u_1(z)e^{\frac{\alpha_2}{n+1}}$, where $u_1(z)$ is a small function of f;*

(II) *if $T(r, e^{\alpha_2}) = S(r, e^{\alpha_1})$, then $f^n f^{(k)} = p_1e^{\alpha_1}$, $Q_d(z, f) = p_2e^{\alpha_2}$, and $f(z) = u_2(z)e^{\frac{\alpha_1}{n+1}}$, where $u_2(z)$ is a small function of f;*

(III) *if* $T(r, e^{\alpha_1}) = O(T(r, e^{\alpha_2}))$ *and* $T(r, e^{\alpha_2}) = O(T(r, e^{\alpha_1}))$, *then one of the following holds*:
(1) $T(r, e^{\alpha_2-\alpha_1}) = S(r, f)$. *In this case,* $Q_d(z, f) \equiv 0$ *and* $f^n f^{(k)} = (p_1 + \varphi p_2)e^{\alpha_1} = (p_2 + 1/\varphi \cdot p_1)e^{\alpha_2}$, *where* $\varphi = e^{\alpha_2-\alpha_1}$, *and* $f(z) = u_3(z)e^{\frac{\alpha_1}{n+1}} = u_4(z)e^{\frac{\alpha_2}{n+1}}$, *where* $u_3(z)$, $u_4(z)$ *are small functions of* f;
(2) $T(r, e^{l\alpha_1-(n+1)\alpha_2}) = S(r, f)$, *where* l *is an integer satisfying* $1 \leq l \leq n-2$. *In this case,* $f^n f^{(k)} = p_1 e^{\alpha_1}$, $Q_d(z, f) = p_2 e^{\alpha_2}$, *and* $f(z) = u_5(z)e^{\frac{\alpha_1}{n+1}}$, *where* $u_5(z)$ *is a small function of* f;
(3) $T(r, e^{l\alpha_2-(n+1)\alpha_1}) = S(r, f)$, *where* l *is an integer satisfying* $1 \leq l \leq n-2$. *In this case,* $f^n f^{(k)} = p_2 e^{\alpha_2}$, $Q_d(z, f) = p_1 e^{\alpha_1}$, *and* $f(z) = u_6(z)e^{\frac{\alpha_2}{n+1}}$, *where* $u_6(z)$ *is a small function of* f.

Specifically, when α_1 and α_2 be polynomials, we get the following Corollary.

Corollary 1.3. *Let* $n \geq 2$, $k \geq 1$ *be integers, and* $Q_d(z, f)$ *be a differential polynomial in* f *of degree* $d \leq n-2$ *with small functions as its coefficients,* α_1, α_2 *be nonconstant polynomials, and* p_1, p_2 *be nonzero small functions of* f. *Suppose the nonlinear differential equation* (7) *has a transcendental meromorphic solution* f *with* $\delta(\infty, f) = 1$ *and* $\delta_{1)}(0, f) > 0$, *then*
(I) *if* $\deg \alpha_1 < \deg \alpha_2$, *then Theorem* 1.2(I) *holds*;
(II) *if* $\deg \alpha_2 < \deg \alpha_1$, *then Theorem* 1.2(II) *holds*;
(III) *if* $\deg \alpha_1 = \deg \alpha_2$, *then Theorem* 1.2(III) *holds.*

Remark 2. Actually, by using the similar method as in the proof of Theorem 1.2, the condition on p_1, p_2 in Theorem 3 can be changed from "small functions of both $e^{\alpha_1(z)}$ and $e^{\alpha_2(z)}$" to "small functions of f", and the same conclusion still holds. Moreover, the forms of its transcendental solutions can also be given.

Example 1. $f_0(z) = e^{e^z}$ is a solution of the following equation
$$f^3 f' + f'' = e^z e^{4e^z} + (e^{2z} + e^z)e^{e^z},$$
where $n = 3$, $d = k = 1$, $\alpha_1(z) = 4e^z$, $\alpha_2(z) = e^z$, $p_1 = e^z$, $p_2 = e^{2z} + e^z$, $\delta(\infty, f_0) = \delta_{1)}(0, f_0) = 1$.

The above Example 1 shows that the solution in Theorem 1.2(III)(2) can exist. However, we raise the following question.

Question 2. *Can the condition* $\delta_{1)}(0, f) > 0$ *in Theorem 1.2 be omitted or not?*

The following corollary deals with a particular case that the degree of the non-dominant term is at most n.

Corollary 1.4. *Let* f *be a transcendental meromorphic function in the plane with* $\delta(\infty, f) = 1$ *and* $\delta_{1)}(0, f) > 0$, $n \geq 2$ *be an integer,* q *be a constant, and* $Q_d(z, f)$ *be a differential polynomial in* f *of degree* d *with small functions as* *coefficients. Suppose p_1, p_2 are nonzero small functions and α_1, α_2 are nonconstant entire functions. If $n \geq d+2$ and the differential equation*

$$f^n f' - qf^{n-1}f' + \frac{n-1}{2n}q^2 f^{n-2}f' + Q_d(z,f) = p_1(z)e^{\alpha_1(z)} + p_2(z)e^{\alpha_2(z)}, \tag{8}$$

holds, then the conclusion in Theorem 1.2 holds.

2. Preliminary lemmas

The following lemma plays an important role in uniqueness problems of meromorphic functions.

Lemma 2.1 ([6])**.** *Let $f_j(z)\,(j = 1,\ldots,n)\,(n \geq 2)$ be meromorphic functions, and let $g_j(z)\,(j = 1,\ldots,n)$ be entire functions satisfying*

(i) $\sum_{j=1}^{n} f_j(z)e^{g_j(z)} \equiv 0$;
(ii) *when $1 \leq j < k \leq n$, then $g_i(z) - g_k(z)$ is not a constant*;
(iii) *when $1 \leq j \leq n, 1 \leq h < k \leq n$, then*

$$T(r, f_j) = o\{T(r, e^{g_h - g_k})\} \quad (r \to \infty, r \notin E),$$

where $E \subset (1,\infty)$ is of finite linear measure or logarithmic measure.

Then, $f_j(z) \equiv 0\,(j = 1,\ldots,n)$.

Lemma 2.2 (Clunie Lemma [2])**.** *Let f be a transcendental meromorphic solution of the equation*:

$$f^n P(z,f) = Q(z,f),$$

where $P(z,f)$ and $Q(z,f)$ are polynomials in f and its derivatives with meromorphic coefficients $\{a_\lambda \,|\, \lambda \in I\}$ such that $m(r,a_\lambda) = S(r,f)$ for all $r \in I$. If the total degree of $Q(z,f)$ as a polynomial in f and its derivatives is at most n, then $m(r,P(z,f)) = S(r,f)$.

The following two lemmas are crucial to the proofs of Theorems 1.1 and 1.2.

Lemma 2.3. *Let f be a transcendental meromorphic function in the plane satisfying*

$$f^n f^{(k)} = Re^{\alpha}, \tag{9}$$

where $n \geq 1$, $k \geq 1$ are integers, α is a nonconstant entire function, and R is a nonzero small function of f. Then $f(z) = u\exp(\alpha/(n+1))$, where u is a small function of f.

Proof. From equation (9), we have

$$nN\left(r,\frac{1}{f}\right) = N\left(r,\frac{1}{f^n}\right) \leq N\left(r,\frac{1}{R}\right) = S(r,f),$$

and

$$N(r,f^{(k)}) \leq N(r,R) = S(r,f).$$

Therefore,

$$T\left(r, \frac{Re^{\alpha}}{f^{n+1}}\right) = T\left(r, \frac{f^n f^{(k)}}{f^{n+1}}\right) = m\left(r, \frac{f^{(k)}}{f}\right) + N\left(r, \frac{f^{(k)}}{f}\right) = S(r,f).$$

Set $\beta(z) = \frac{Re^{\alpha}}{f^{n+1}}$, then we have

$$f(z) = \left(\frac{R}{\beta}\right)^{\frac{1}{n+1}} e^{\frac{\alpha}{n+1}} = u(z)e^{\frac{\alpha}{n+1}},$$

where $T(r,u) = S(r,f)$. □

By the proof of [8, Theorem 1.3] (or [2, Lemma 2.4.2, Clunie Lemma]), we have the following lemma. Here for convenience of the readers we also give the sketch of its proof.

Lemma 2.4. *Let $Q(z,f)$ be a differential polynomial in f of degree d with small functions of f as coefficients. Then we have $m(r,Q) \le dm(r,f) + S(r,f)$.*

Proof. Defining $E_1 := \{\theta \in [0,2\pi) \,|\, |f(re^{i\theta})| < 1\}$, $E_2 := [0,2\pi) \setminus E_1$, we may consider the proximity function $m(r,Q_d)$ in two parts:

$$m(r,Q) = \frac{1}{2\pi}\int_{E_1} \log^+ |Q| d\theta + \frac{1}{2\pi}\int_{E_2} \log^+ |Q| d\theta. \tag{10}$$

Writing, with $\lambda = (l_0, \dots, l_\nu)$,

$$Q(z,f) = \sum_{\lambda \in I} Q_\lambda(z,f) = \sum_{\lambda\in I} a_\lambda(z) f^{l_0}(f')^{l_1}\cdots(f^{(\nu)})^{l_\nu}.$$

For $z \in E_1$, we have

$$\begin{aligned}|Q_\lambda(z,f)| &= |a_\lambda(z) f^{l_0}(f')^{l_1}\cdots(f^{(\nu)})^{l_\nu}|\\ &\le |a_\lambda| \left|\frac{f'}{f}\right|^{l_1} \cdots \left|\frac{f^{(\nu)}}{f}\right|^{l_\nu}.\end{aligned}$$

Therefore, by the logarithmic derivative lemma, we obtain

$$\frac{1}{2\pi}\int_{E_1} \log^+ |Q_\lambda| d\theta \le m(r,a_\lambda) + \sum_{j=1}^{\nu} l_j m\left(r, \frac{f^{(j)}}{f}\right) = S(r,f).$$

Hence

$$\frac{1}{2\pi}\int_{E_1} \log^+ |Q| d\theta \le \sum_{\lambda\in I}\int_{E_1} \log^+ |Q_\lambda| d\theta + O(1) = S(r,f). \tag{11}$$

For $z \in E_2$, as $l_0 + l_1 + \cdots + l_\nu \le d$ for all $\lambda \in I$, we have

$$\begin{aligned}|Q(z,f)| &\le \sum_{\lambda\in I} \left|a_\lambda(z) f^{l_0}(f')^{l_1}\cdots(f^{(\nu)})^{l_\nu}\right|\\ &\le |f|^d \left(\sum_{\lambda\in I} |a_\lambda| \left|\frac{f'}{f}\right|^{l_1} \cdots \left|\frac{f^{(\nu)}}{f}\right|^{l_\nu}\right).\end{aligned}$$

Thus we have

$$\frac{1}{2\pi}\int_{E_2}\log^+|Q|d\theta \leq dm(r,f)+\sum_{\lambda\in I}m(r,a_\lambda) + \sum_{\lambda\in I}\left(\sum_{j=1}^{\nu} l_j m\left(r,\frac{f^{(j)}}{f}\right)\right)+O(1) = dm(r,f)+S(r,f). \tag{12}$$

By combining (10), (11) with (12), we obtain the conclusion. □

Lemma 2.5. *Let $n\geq 2$, $k\geq 1$ be integers and $Q_d(z,f)$ denote an algebraic differential polynomial in $f(z)$ of degree $d\leq n-1$ with small functions of f as its coefficients. If $p_1(z)$, $p_2(z)$ are small functions of f, $\alpha_1(z),\alpha_2(z)$ are nonconstant entire functions and if f is a transcendental meromorphic solution of the equation (7) with $N(r,f)=S(r,f)$, then we have $T(r,f)=O\left(T\left(r,e^{\alpha_1}\right)+T\left(r,e^{\alpha_2}\right)\right)$, $T(r,p_1e^{\alpha_1}+p_2e^{\alpha_2})=O(T(r,f))$, and $T(r,f^nf^{(k)}+Q_d(z,f))\neq S(r,f)$.*

Proof. By Lemma 2.4, we get that

$$m\left(r,Q_d(z,f)\right)\leq dm(r,f)+S(r,f). \tag{13}$$

By combining (13) with $N(r,f)=S(r,f)$, we get that

$$\begin{aligned}(n+1)T(r,f)=T(r,f^{n+1})&=T\left(r,\frac{1}{f^{n+1}}\right)+S(r,f)\\ &\leq m\left(r,\frac{1}{f^nf^{(k)}}\right)+m\left(r,\frac{f^{(k)}}{f}\right)+N\left(r,\frac{1}{f^nf^{(k)}}\right)\\ &\quad -N\left(r,\frac{1}{f^{(k)}}\right)+N\left(r,\frac{1}{f}\right)+S(r,f)\\ &\leq T\left(r,f^nf^{(k)}\right)+N\left(r,\frac{1}{f}\right)+S(r,f)\\ &= m\left(r,f^nf^{(k)}\right)+N\left(r,\frac{1}{f}\right)+S(r,f)\\ &\leq m\left(r,p_1e^{\alpha_1}\right)+m\left(r,p_2e^{\alpha_2}\right)+m\left(r,Q_d(z,f)\right)\\ &\quad +N\left(r,\frac{1}{f}\right)+S(r,f)\\ &\leq T\left(r,e^{\alpha_1}\right)+T\left(r,e^{\alpha_2}\right)+(d+1)T(r,f)+S(r,f).\end{aligned}$$

This gives that

$$(n-d)T(r,f)\leq T\left(r,e^{\alpha_1}\right)+T\left(r,e^{\alpha_2}\right)+S(r,f),$$

i.e., $T(r,f)=O\left(T\left(r,e^{\alpha_1}\right)+T\left(r,e^{\alpha_2}\right)\right)$.

From (13), $N(r,f)=S(r,f)$ and equation (7), we can also get $T(r,p_1e^{\alpha_1}+p_2e^{\alpha_2})=O(T(r,f))$.

Next, we prove that $T(r, f^n f^{(k)} + Q_d(z,f))$ can not be a small function of f. Otherwise, we will have $f^n f^{(k)} + Q_d(z,f) = \beta$ with $T(r,\beta) = S(r,f)$. Thus $f^n f^{(k)} = \beta - Q_d(z,f)$. Since $d \le n-1$, from Lemma 2.2, we get $m(r, f^{(k)}) = S(r,f)$ and $m(r, ff^{(k)}) = S(r,f)$. Then $T(r, f^{(k)}) = S(r,f)$ and $T(r, ff^{(k)}) = S(r,f)$ since $N(r,f) = S(r,f)$. By $f^{(k)} \not\equiv 0$ from the assumption that f is transcendental, we have $T(r,f) \le T(r, ff^{(k)}) + T(r, 1/f^{(k)}) = S(r,f)$, which yields a contradiction. □

3. Proof of Theorem 1.1

The sufficiency can be deduced by using Lemma 2.3 directly, so next we prove the necessity.

Let f be a transcendental meromorphic solution of the equation (4) with $\delta(\infty, f) = 1$ and $\delta_{1)}(0,f) > 0$. Then obviously we have $N(r,f) = S(r,f)$.

We assert that $R \not\equiv 0$. Otherwise, from (4), we get that

$$f^n f^{(k)} = -Q_{d_*}(z,f).$$

Since $d_* \le n-1$, then by Lemma 2.2 we have

$$m(r, f^{(k)}) = S(r,f),\ m(r, ff^{(k)}) = S(r,f).$$

Combining with $N(r,f) = S(r,f)$, we get that

$$T(r, f^{(k)}) = S(r,f),\ T(r, ff^{(k)}) = S(r,f).$$

Since f is transcendental, we have that $f^{(k)} \not\equiv 0$. Therefore,

$$T(r,f) \le T(r, ff^{(k)}) + T(r, 1/f^{(k)}) = S(r,f),$$

which yields a contradiction. So we have $R \not\equiv 0$. Thus from (4) we get

$$e^{\alpha} = \frac{f^n f^{(k)} + Q_{d_*}(z,f)}{R}.$$

Therefore, by using Lemma 2.4 to the differential polynomial $f^n f^{(k)} + Q_{d_*}(z,f)$ with degree $n+1$, we get that

$$\begin{aligned} T(r, e^{\alpha}) &\le T(r, f^n f^{(k)} + Q_{d_*}(z,f)) + T(r,R) \\ &= m(r, f^n f^{(k)} + Q_{d_*}(z,f)) + S(r,f) \\ &\le (n+1) m(r,f) + S(r,f) \\ &= (n+1) T(r,f) + S(r,f), \end{aligned}$$

which means a small function of e^{α} is also a small function of f. So we have $T(r, \alpha') = S(r,f)$.

By differentiating both sides of (4) we have

$$nf^{n-1} f' f^{(k)} + f^n f^{(k+1)} + Q'_{d_*} = (R' + R\alpha') e^{\alpha}. \tag{14}$$

Multiplying (4) by $(R'+R\alpha')$ and (14) by R, and then subtracting the resulting equations, we get

$$f^{n-1}\phi = RQ'_{d_*} - (R'+R\alpha')Q_{d_*}, \tag{15}$$

where

$$\phi = (R'+R\alpha')ff^{(k)} - Rnf'f^{(k)} - Rff^{(k+1)}. \tag{16}$$

It follows from Lemma 2.2 that $m(r,\phi) = S(r,f)$. Combining with $N(r,f) = S(r,f)$, we have $T(r,\phi) = S(r,f)$.

Next we prove that $\phi \equiv 0$. Otherwise, from formula (16), we get that

$$\frac{\phi}{f^2} = (R'+R\alpha')\frac{f^{(k)}}{f} - nR\frac{f'}{f}\frac{f^{(k)}}{f} - R\frac{f^{(k+1)}}{f}.$$

Thus,

$$2m\left(r,\frac{1}{f}\right) \le m\left(r,\frac{\phi}{f^2}\right) + m\left(r,\frac{1}{\phi}\right) = S(r,f). \tag{17}$$

It follows from (16) that

$$\begin{aligned}\frac{1}{2}N_{(2}\left(r,\frac{1}{f}\right) &\le N\left(r,\frac{1}{\phi}\right) + N(r,R'+R\alpha') + N(r,R)\\ &\le T(r,\phi) + S(r,f) = S(r,f),\end{aligned}$$

implying that the zeros of f are mainly simple zeros. Thus, by combining with (17), we obtain

$$T(r,f) = N\left(r,\frac{1}{f}\right) + S(r,f) = N_{1)}\left(r,\frac{1}{f}\right) + S(r,f),$$

which contradicts with the assumption that $\delta_{1)}(0,f) > 0$. Therefore,

$$(R'+R\alpha')ff^{(k)} - Rnf'f^{(k)} - Rff^{(k+1)} \equiv 0, \tag{18}$$

and

$$RQ'_{d_*} - (R'+R\alpha')Q_{d_*} \equiv 0. \tag{19}$$

If $Q_{d_*} \equiv 0$, then equation (4) reduces to

$$f^n f^{(k)} = Re^{\alpha}.$$

Thus by Lemma 2.3, we get the conclusion.

If $Q_{d_*} \not\equiv 0$, then from (19), we have

$$\frac{Q'_{d_*}}{Q_{d_*}} = \frac{R'}{R} + \alpha'.$$

Therefore

$$Q_{d_*} = cRe^{\alpha}, \tag{20}$$

where c is a nonzero constant.

By substituting (20) into equation (4), we get

$$f^n f^{(k)} = \left(\frac{1}{c} - 1\right) Q_{d_*}. \tag{21}$$

If $c = 1$, then we have $f^n f^{(k)} \equiv 0$. Thus we have $f \equiv 0$, or f is a polynomial, a contradiction. Therefore, we have $c \neq 1$. Since $d_* \leq n-1$, by using Lemma 2.2 to (21), we have $m(r, f^{(k)}) = S(r, f)$ and $m(r, ff^{(k)}) = S(r, f)$. By combining with $N(r, f) = S(r, f)$, we get $T(r, f^{(k)}) = S(r, f)$ and $T(r, ff^{(k)}) = S(r, f)$. Thus by $f^{(k)} \not\equiv 0$, we have $T(r, f) \leq T(r, ff^{(k)}) + T(r, 1/f^{(k)}) = S(r, f)$, which yields a contradiction.

4. Proof of Theorem 1.2

Let f be a transcendental meromorphic solution of the equation (7) with $\delta(\infty, f) = 1$ and $\delta_{1)}(0, f) > 0$. Then obviously we have $N(r, f) = S(r, f)$. It follows from Lemma 2.5 and the assumption $d \leq n - 2 < n - 1$ that

$$T(r, f) \leq K_0 \left(T\left(r, e^{\alpha_1}\right) + T\left(r, e^{\alpha_2}\right)\right), \tag{22}$$

$$T(r, p_1 e^{\alpha_1} + p_2 e^{\alpha_2}) \leq K_0 T(r, f), \tag{23}$$

and

$$T(r, f^n f^{(k)} + Q_d(z, f)) \neq S(r, f) \tag{24}$$

as $r \to \infty$, where $K_0\,(> 0)$ is a constant.

From (7), we have

$$n f^{n-1} f' f^{(k)} + f^n f^{(k+1)} + Q_d' = (p_1' + p_1 \alpha_1') e^{\alpha_1} + (p_2' + p_2 \alpha_2') e^{\alpha_2}. \tag{25}$$

By eliminating $e^{\alpha_2(z)}$ from equations (7) and (25), we have

$$\begin{aligned} &(p_2' + p_2\alpha_2') f^n f^{(k)} - n p_2 f^{n-1} f' f^{(k)} - p_2 f^n f^{(k+1)} + (p_2' + p_2 \alpha_2') Q_d - p_2 Q_d' \\ &\quad = A_1 e^{\alpha_1}, \text{ where } A_1 = \left(p_2' + p_2 \alpha_2'\right) p_1 - p_2 \left(p_1' + p_1 \alpha_1'\right). \end{aligned} \tag{26}$$

Next we discuss the following three cases.

Case 1. $T(r, e^{\alpha_1}) = S(r, e^{\alpha_2})$. Then from (22) we have

$$T(r, f) \leq 2K_0 \cdot T\left(r, e^{\alpha_2}\right) \text{ as } r \to \infty,$$

which means that a small function of f is also a small function of e^{α_2}. So from (23), we get

$$(1 + o(1)) T(r, e^{\alpha_2}) = T(r, p_1 e^{\alpha_1} + p_2 e^{\alpha_2}) \leq K_0 T(r, f) \text{ as } r \to \infty,$$

which means that a small function of e^{α_2} is also a small function of f. So we have $T(r, e^{\alpha_1}) = S(r, f)$. We rewritten (7) as follows:

$$f^n(z) f^{(k)}(z) + Q_d(z, f) - p_1 e^{\alpha_1} = p_2 e^{\alpha_2}.$$

Therefore, by using Theorem 1.1 and Theorem 2, we get that $Q_d(z, f) = p_1 e^{\alpha_1}$, $f^n f^{(k)} = p_2 e^{\alpha_2}$, and $f = u_1 \exp(\alpha_2/(n+1))$, where u_1 is a small function of f.

Case 2. $T(r, e^{\alpha_2}) = S(r, e^{\alpha_1})$. The argument is similar as in Case 1.

Case 3. $T(r, e^{\alpha_1}) = O(T(r, e^{\alpha_2}))$ and $T(r, e^{\alpha_2}) = O(T(r, e^{\alpha_1}))$. Then there exist constants K_1 and $L_1\,(>0)$ such that

$$T(r, e^{\alpha_2}) \le K_1 \cdot T(r, e^{\alpha_1}), \quad T(r, e^{\alpha_1}) \le L_1 \cdot T(r, e^{\alpha_2}) \tag{27}$$

as $r \to \infty$, which means that a small function of e^{α_2} is also a small function of e^{α_1}, while a small function of e^{α_1} is also a small function of e^{α_2}.

Subcase 3.1. $A_1(z) \equiv 0$. Then we have

$$(p_2' + p_2\alpha_2')\,p_1 = p_2\,(p_1' + p_1\alpha_1')\,.$$

Therefore

$$p_2e^{\alpha_2} = c_0p_1e^{\alpha_1}, \tag{28}$$

where c_0 is a nonzero constant. So we have $T(r, e^{\alpha_1-\alpha_2}) = S(r, f)$.

Substituting (28) into equation (7), we get

$$f^n(z)f^{(k)}(z) + Q_d(z, f) = (1 + c_0)p_1e^{\alpha_1} = \left(1 + \frac{1}{c_0}\right)p_2e^{\alpha_2}.$$

Obviously, from (24) we have that $1 + c_0 \ne 0$ and $1 + 1/c_0 \ne 0$. Therefore, by using Theorem 1.1 and Theorem 2, we get that $Q_d(z, f) \equiv 0$, $f^nf^{(k)} = (p_1 + p_2e^{\alpha_2-\alpha_1})e^{\alpha_1} = (p_2 + p_1e^{\alpha_1-\alpha_2})e^{\alpha_2}$, and $f = s_1\exp(\alpha_1/(n + 1)) = s_2\exp(\alpha_2/(n + 1))$, where s_1, s_2 are small functions of f. This belongs to Case III (1) in Theorem 1.2.

Subcase 3.2. $A_1(z) \not\equiv 0$. By combining (22) with (27), we get that

$$T(r, f) \le K_0(1 + K_1) \cdot T(r, e^{\alpha_1}), \quad T(r, f) \le K_0(1 + L_1) \cdot T(r, e^{\alpha_2}) \tag{29}$$

as $r \to \infty$, which means that a small function of f is also a small function of e^{α_1} and e^{α_2}.

By combining (26) with (27), we get that there exists $K_2\,(>0)$ such that

$$T(r, e^{\alpha_1}) \le K_2T(r, f), \text{ and } T(r, e^{\alpha_2}) \le K_1K_2T(r, f)$$

as $r \to \infty$, which means that any small function of e^{α_1} (or e^{α_2}) is also a small function of f. Therefore, we have $S(r, f) = S(r, e^{\alpha_1}) = S(r, e^{\alpha_2})$.

For convenience of calculation, we denote $A_2 = p_2' + p_2\alpha_2'$ and $g = (p_2' + p_2\alpha_2')Q_d - p_2Q_d'$. Obviously, $A_2 \not\equiv 0$. Otherwise we will get that

$$p_2 = c_1e^{-\alpha_2},$$

where c_1 is a nonzero constant, which yields a contradiction by the fact that $T(r, p_2) = S(r, e^{\alpha_2})$.

Thus equation (26) becomes

$$A_2f^nf^{(k)} - np_2f^{n-1}f'f^{(k)} - p_2f^nf^{(k+1)} + g = A_1e^{\alpha_1}. \tag{30}$$

Differentiating both sides of (30), we have

$$A_2'f^nf^{(k)} + n(A_2 - p_2')f^{n-1}f'f^{(k)} + (A_2 - p_2')f^nf^{(k+1)}$$
$$- np_2f^{n-1}f''f^{(k)} - n(n - 1)p_2f^{n-2}(f')^2f^{(k)} - 2np_2f^{n-1}f'f^{(k+1)}$$

$$-p_2 f^n f^{(k+2)} + g' = (A_1' + A_1\alpha_1')e^{\alpha_1}. \tag{31}$$

By eliminating e^{α_1} from equations (30) and (31), we obtain

$$\begin{aligned}
&[A_1A_2' - (A_1' + A_1\alpha_1')A_2]\, f^n f^{(k)}\\
&+ [nA_1p_2\alpha_2' + np_2(A_1' + A_1\alpha_1')]\, f^{n-1}f'f^{(k)}\\
&+ [A_1p_2\alpha_2' + (A_1' + A_1\alpha_1')p_2]\, f^n f^{(k+1)} - n(n-1)A_1p_2 f^{n-2}(f')^2 f^{(k)}\\
&- nA_1p_2 f^{n-1}f''f^{(k)} - 2np_2A_1 f^{n-1}f'f^{(k+1)} - p_2A_1 f^n f^{(k+2)}\\
&= (A_1' + A_1\alpha_1')g - A_1g'.
\end{aligned} \tag{32}$$

Set $B_1 = A_1A_2' - (A_1' + A_1\alpha_1')A_2$, $B_2 = nA_1p_2\alpha_2' + np_2(A_1' + A_1\alpha_1')$, $B_3 = A_1p_2\alpha_2' + (A_1' + A_1\alpha_1')p_2$, $Q_1 = (A_1' + A_1\alpha_1')g - A_1g'$, then we have

$$f^{n-2}Q = Q_1, \tag{33}$$

where

$$\begin{aligned}
Q = {} & B_1f^2f^{(k)} + B_2ff'f^{(k)} + B_3f^2f^{(k+1)} - n(n-1)A_1p_2(f')^2f^{(k)}\\
& - nA_1p_2ff''f^{(k)} - 2np_2A_1ff'f^{(k+1)} - p_2A_1f^2f^{(k+2)}.
\end{aligned} \tag{34}$$

It follows from Lemma 2.2 that $m(r,Q) = S(r,f)$. By combining with the fact that $N(r,f) = S(r,f)$, we have $T(r,Q) = S(r,f)$.

Next we assert that $Q \equiv 0$. Otherwise, from formula (34), we get that

$$\begin{aligned}
3m\left(r,\frac{1}{f}\right) &\le m\left(r,\frac{Q}{f^3}\right) + m\left(r,\frac{1}{Q}\right)\\
&\le 4m\left(r,\frac{f^{(k)}}{f}\right) + 4m\left(r,\frac{f'}{f}\right) + 2m\left(r,\frac{f^{(k+1)}}{f}\right) + m\left(r,\frac{f''}{f}\right)\\
&\quad + m\left(r,\frac{f^{(k+2)}}{f}\right) + S(r,f)\\
&= S(r,f).
\end{aligned} \tag{35}$$

It follows from (34) that

$$\begin{aligned}
N_{(2}\left(r,\frac{1}{f}\right) &\le N(r,B_1) + N(r,B_2) + N(r,B_3) + N(r,A_1p_2) + N\left(r,\frac{1}{Q}\right)\\
&\le T(r,Q) + S(r,f) = S(r,f),
\end{aligned}$$

implying that the zeros of f are mainly simple zeros. Thus, combining with (35), we obtain

$$T(r,f) = N\left(r,\frac{1}{f}\right) + S(r,f) = N_{1)}\left(r,\frac{1}{f}\right) + S(r,f),$$

which contradicts with the assumption that $\delta_{1)}(0,f) > 0$. Therefore,

$$\begin{aligned}
& B_1f^2f^{(k)} + B_2ff'f^{(k)} + B_3f^2f^{(k+1)} - n(n-1)A_1p_2(f')^2f^{(k)}\\
& - nA_1p_2ff''f^{(k)} - 2np_2A_1ff'f^{(k+1)} - p_2A_1f^2f^{(k+2)} \equiv 0,
\end{aligned} \tag{36}$$

and

$$Q_1 = (A_1' + A_1\alpha_1')g - A_1 g' \equiv 0. \tag{37}$$

Subcase 3.2.1. $g(z) \equiv 0$, i.e.,

$$(p_2' + p_2\alpha_2')Q_d - p_2 Q_d' \equiv 0. \tag{38}$$

If $Q_d \equiv 0$, then equation (7) becomes

$$f^n f^{(k)} = p_1 e^{\alpha_1} + p_2 e^{\alpha_2}. \tag{39}$$

Next, we prove that $N(r, 1/f) = S(r, f)$. Otherwise, consider equation (36), let z_0 be a zero of f with multiplicity p, which is not a zero or pole of B_1, B_2, B_3 and $p_2 A_1$, then $(f'(z_0))^2 f^{(k)}(z_0) = 0$. Suppose $f(z) = a_p(z - z_0)^p + a_{p+1}(z - z_0)^{p+1} + \cdots$, $a_p \neq 0$.

For the case $k = 1$, we have $p \geq 2$ from the fact that $f'(z_0) = 0$.

If $p = 2$, by calculating the coefficient of the lowest power of $z - z_0$ in the left of equation (36), we have

$$n(n-1)(2a_2)^3 + 3n2^2a_2^3 = 0.$$

Thus

$$2n^2 + n = 0,$$

which is impossible since $n \geq 2$.

If $p \geq 3$, then by calculating the coefficient of the lowest power of $z - z_0$ in the left of equation (36), we have

$$n(n-1)(pa_p)^3 + 3np^2(p-1)a_p^3 + p(p-1)(p-2)a_p^3 = 0,$$

thus

$$[(n+1)p - 1]\,[(n+1)p - 2] = 0.$$

This gives that

$$p = \frac{1}{n+1}, \text{ or } p = \frac{2}{n+1},$$

which is impossible since $n \geq 2$ and $p \geq 3$.

For the case $k \geq 2$, suppose $f^{(k)}(z) = b_m(z - z_0)^m + b_{m+1}(z - z_0)^{m+1} + \cdots$, $b_m \neq 0$. Then there exist the following three subcases.

I. $f'(z_0) = 0$ and $f^{(k)}(z_0) = 0$. Then $p \geq 2$ and $m \geq 1$.

If $p \geq 2$ and $m = 1$. By calculating the coefficient of the lowest power of $z - z_0$ in the left of equation (36), we have

$$n(n-1)(pa_p)^2 b_1 + np(p-1)a_p^2 b_1 + 2npa_p^2 b_1 = 0,$$

thus,

$$(n-1)p + p - 1 + 2 = 0,$$

which yields a contradiction.

If $p \geq 2$ and $m \geq 2$. By calculating the coefficient of the lowest power of $z - z_0$ in the left of equation (36), we have

$$n(n-1)(pa_p)^2 b_m + np(p-1)a_p^2 b_m + 2npa_p^2 m b_m + a_p^2 m(m-1)b_m = 0,$$

thus

$$(np+m)(np+m-1) = 0,$$

which also yields a contradiction.

II. $f'(z_0) = 0$ and $f^{(k)}(z_0) \neq 0$. Then $2 \leq p \leq k$ and $m = 0$. By calculating the coefficient of the lowest power of $z - z_0$ in the left of equation (36), we have

$$n(n-1)(pa_p)^2 b_0 + na_p p(p-1)a_p b_0 = 0,$$

thus

$$(n-1)p + (p-1) = 0,$$

which yields a contradiction.

III. $f'(z_0) \neq 0$ and $f^{(k)}(z_0) = 0$. Then $p = 1$ and $m \geq 1$.

If $p = 1$ and $m = 1$, then by calculating the coefficient of the lowest power of $z - z_0$ in the left of equation (36), we have

$$n(n-1)a_1^2 b_1 + 2na_1^2 b_1 = 0,$$

thus

$$n(n+1) = 0,$$

which yields a contradiction.

If $p = 1$ and $m \geq 2$, then by calculating the coefficient of the lowest power of $z - z_0$ in the left of equation (36), we have

$$n(n-1)a_1^2 b_m + 2na_1^2 m b_m + a_1^2 m(m-1)b_m = 0,$$

thus

$$(m+n)(m+n-1) = 0,$$

which also yields a contradiction.

Hence, for $k \geq 1$ we have

$$N\left(r, \frac{1}{f}\right) = S(r, f). \tag{40}$$

Rewrite (30) as

$$\frac{A_2}{A_1}\frac{f^{(k)}}{f} - \frac{np_2}{A_1}\frac{f'}{f}\frac{f^{(k)}}{f} - \frac{p_2}{A_1}\frac{f^{(k+1)}}{f} = \frac{e^{\alpha_1}}{f^{n+1}}. \tag{41}$$

Then by Logarithmic Derivative Lemma, from (41) we get

$$m\left(r, \frac{e^{\alpha_1}}{f^{n+1}}\right) = S(r, f).$$

Therefore, by combining with (40),

$$T\left(r, \frac{e^{\alpha_1}}{f^{n+1}}\right) = m\left(r, \frac{e^{\alpha_1}}{f^{n+1}}\right) + N\left(r, \frac{e^{\alpha_1}}{f^{n+1}}\right) = S(r, f).$$

We set

$$\beta(z) = \frac{e^{\alpha_1}}{f^{n+1}},$$

then $T(r, \beta) = S(r, f)$, and

$$f = \left(\frac{1}{\beta}\right)^{\frac{1}{n+1}} e^{\frac{\alpha_1}{n+1}} = t_1(z) e^{\frac{\alpha_1}{n+1}}, \tag{42}$$

where $t_1(z)$ is a small function of f.

Substituting (42) into equation (39), we get that

$$q_{n+1}(z) e^{\alpha_1} = p_1 e^{\alpha_1} + p_2 e^{\alpha_2},$$

where $q_{n+1}(z)$ is a small function of f, which gives that $T(r, e^{\alpha_2 - \alpha_1}) = S(r, f)$. Then $f^n f^{(k)} = (p_1 + \varphi p_2) e^{\alpha_1}$, where $\varphi = e^{\alpha_2 - \alpha_1}$ such that $T(r, \varphi) = S(r, f)$. This belongs to Case III (1) in Theorem 1.2.

If $Q_d \not\equiv 0$, then equation (38) becomes

$$\frac{p_2'}{p_2} + \alpha_2' = \frac{Q_d'}{Q_d}. \tag{43}$$

Therefore

$$p_2 e^{\alpha_2} = Q_d c_2,$$

where c_2 is a nonzero constant. Substituting it into (7) we have

$$f^n f^{(k)} + (1 - c_2) Q_d = p_1 e^{\alpha_1}.$$

Then by Theorem 1.1 and Theorem 2, we have $c_2 = 1$, $f^n f^{(k)} = p_1 e^{\alpha_1}$, and $f = u_5 \exp(\alpha_1/(n+1))$, where u_5 is a small function of f. Therefore,

$$p_2 e^{\alpha_2} = Q_d. \tag{44}$$

By substituting $f = u_5 \exp(\alpha_1/(n+1))$ into (44), we get

$$\sum_{l=0}^{n-2} q_l(z) e^{\frac{l\alpha_1}{n+1}} = p_2 e^{\alpha_2},$$

where $q_l(z)$ are small functions of f. By Lemma 2.1, there must exists some $l\,(1 \le l \le n-2)$ such that $T(r, e^{\alpha_2 - \frac{l\alpha_1}{n+1}}) = S(r, f)$, i.e. $T(r, e^{(n+1)\alpha_2 - l\alpha_1}) = S(r, f)$. This belongs to Case III (2) in Theorem 1.2.

Subcase 3.2.2. $g(z) \not\equiv 0$. Then from (37) we have

$$\frac{A_1'}{A_1} + \alpha_1' = \frac{g'}{g}.$$

Therefore

$$A_1 e^{\alpha_1} = g c_3,$$

where c_3 is a nonzero constant.

Substituting it into (30) we have

$$A_2 f^n f^{(k)} - n p_2 f^{n-1} f' f^{(k)} - p_2 f^n f^{(k+1)} = (c_3 - 1) g. \tag{45}$$

Denote $\varphi = A_2 f f^{(k)} - np_2 f' f^{(k)} - p_2 f f^{(k+1)}$. If $c_3 \neq 1$, then $\varphi \not\equiv 0$. Thus by Lemma 2.2, we have $m(r,\varphi) = S(r,f)$ and $m(r, f\varphi) = S(r,f)$. Combining with $N(r,f) = S(r,f)$, we have $T(r,\varphi) = S(r,f)$ and $T(r, f\varphi) = S(r,f)$. Then, $T(r,f) \leq T(r,f\varphi) + T(r,\frac{1}{\varphi}) = S(r,f)$, which yields a contradiction. Therefore, $c_3 = 1$ and $\varphi \equiv 0$, i.e.,

$$A_2 f f^{(k)} - np_2 f' f^{(k)} - p_2 f f^{(k+1)} = 0.$$

This gives that

$$\frac{p_2'}{p_2} + \alpha_2' = n\frac{f'}{f} + \frac{f^{(k+1)}}{f^{(k)}}.$$

Thus

$$f^n f^{(k)} = c_4 p_2 e^{\alpha_2}, \tag{46}$$

where c_4 is a nonzero constant. Substituting (46) into (7), we have

$$\left(1 - \frac{1}{c_4}\right) f^n f^{(k)} + Q_d(z,f) = p_1 e^{\alpha_1}. \tag{47}$$

If $c_4 = 1$, then we have

$$f^n f^{(k)} = p_2 e^{\alpha_2}, \tag{48}$$

and

$$Q_d(z,f) = p_1 e^{\alpha_1}. \tag{49}$$

By using Lemma 2.3 to (48), we have

$$f = u_6(z) e^{\frac{\alpha_2}{n+1}}, \tag{50}$$

where $u_6(z)$ is a small function of f. Substituting (50) into (49), by using Lemma 2.1, there exists some $l\,(1 \leq l \leq n-2)$ such that $T(r, e^{\alpha_1 - \frac{l\alpha_2}{n+1}}) = S(r,f)$, i.e., $T(r, e^{(n+1)\alpha_1 - l\alpha_2}) = S(r,f)$. This belongs to Case III (3) in Theorem 1.2.

If $c_4 \neq 1$, then from (47) we have

$$f^n f^{(k)} + \frac{c_4}{c_4 - 1} Q_d(z,f) = \frac{c_4}{c_4 - 1} p_1 e^{\alpha_1}. \tag{51}$$

By using Theorem 1.1 and Theorem 2 to (51), we have

$$Q_d(z,f) \equiv 0.$$

Thus

$$g = (p_2' + p_2\alpha_2')Q_d - p_2 Q_d' \equiv 0,$$

a contradiction with $g \not\equiv 0$.

5. Proof of Corollary 1.3

Let $\alpha_1(z) = a_p z^p + a_{p-1} z^{p-1} + \cdots + a_1 z + a_0$, $\alpha_2(z) = b_q z^q + b_{q-1} z^{q-1} + \cdots + b_1 z + b_0$. It is well known [1, p. 7] that

$$T(r, e^{\alpha_1}) = \frac{|a_p|}{\pi} r^p + o(r^p) \text{ and } T(r, e^{\alpha_2}) = \frac{|b_q|}{\pi} r^q + o(r^q).$$

Therefore, by combining with Theorem 1.2 we can get the conclusion.

6. Proof of Corollary 1.4

Assume that f is a transcendental meromorphic solution with $\delta(\infty, f) = 1$ and $\delta_{1)}(\frac{q}{n}, f) > 0$ of equation (8). Let $g(z) = f(z) - \frac{q}{n}$, then g is a transcendental meromorphic solution with $\delta(\infty, g) = 1$ and $\delta_{1)}(0, g) > 0$ of the following differential equation

$$g^n g^{(k)} + Q^*(z, g) = p_1 e^{\alpha_1} + p_2 e^{\alpha_2},$$

where $Q^*(z, g)$ is a differential equation with degree $\leq n - 2$. The conclusion of the theorem follows immediately from Theorem 1.2.

Acknowledgements. The authors would like to thank the referee for his/her thorough review with constructive suggestions and comments on the paper.

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