

ON w -COPURE FLAT MODULES AND DIMENSION

EL MEHDI BOUBA, HWANKOO KIM, AND MOHAMMED TAMEKKANTE

Dedicated to Professor Fanggui Wang for the introduction and development of the w -operation theory

ABSTRACT. Let R be a commutative ring. An R -module M is said to be w -flat if $\text{Tor}_1^R(M, N)$ is GV -torsion for any R -module N . It is known that every flat module is w -flat, but the converse is not true in general. The w -flat dimension of a module is defined in terms of w -flat resolutions. In this paper, we study the w -flat dimension of an injective w -module. To do so, we introduce and study the so-called w -copure (resp., strongly w -copure) flat modules and the w -copure flat dimensions for modules and rings. The relations between the introduced dimensions and other (classical) homological dimensions are discussed. We also study change of rings theorems for the w -copure flat dimension in various contexts. Finally some illustrative examples regarding the introduced concepts are given.

1. Introduction

Throughout, all rings considered are commutative with unity and all modules are unital. Let R be a ring and M be an R -module. As usual, we use $\text{pd}_R(M)$, $\text{id}_R(M)$, and $\text{fd}_R(M)$ to denote, respectively, the classical projective dimension, injective dimension, and flat dimension of M , and $\text{wdim}(R)$ and $\text{gldim}(R)$ to denote, respectively, the weak and global homological dimensions of R .

Enochs and Jenda [8,9] introduced the notion of Gorenstein projective modules (G -projective modules for short), as an extension of the same notion to modules that are not necessarily finitely generated, and the Gorenstein injective modules (G -injective modules for short) as a dual notion of Gorenstein projective modules. To complete the analogy with the classical modules in the homological theory, Enochs, Jenda, and Torrecillas [10] introduced the Gorenstein flat modules (G -flat modules for short). Recall that an R -module M is called *Gorenstein flat*, if there exists an exact sequence of flat R -modules $\mathbf{F} : \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow F^0 \rightarrow F^1 \rightarrow \cdots$ such that $M \cong \text{Im}(F_0 \rightarrow F^0)$ and such

Received May 16, 2019; Revised August 10, 2019; Accepted September 5, 2019.

2010 *Mathematics Subject Classification.* 13D05, 13D07, 13H05.

Key words and phrases. w -copure flat module, strongly w -copure flat module, w -copure flat dimension, w -linked.

that the functor $- \otimes_R I$ leaves \mathbf{F} exact whenever I is an injective R -module. The complex \mathbf{F} is called a *complete flat resolution*. The *Gorenstein flat dimension* is defined in terms of Gorenstein flat resolutions, and denoted by $\text{Gfd}(-)$ [14].

In [2], the authors defined *the weak Gorenstein global dimension* of a ring R to be

$$\text{wGgldim}(R) = \sup\{\text{Gfd}_R(M) \mid M \text{ is an } R\text{-module}\}.$$

It is proved (in [17, Theorem 2.12]) that, for any ring R , we have

$$\text{wGgldim}(R) = \sup\{\text{fd}_R(E) \mid E \text{ is an injective } R\text{-module}\}.$$

The weak Gorenstein global dimension of a ring R is well studied, by N. Ding and J. Chen (1993), in [5] under the name $\text{IFD}(R)$. Another approach of the $\text{IFD}(R)$ was also done by considering the (strongly) copure flat modules. Recall that an R -module M is called *copure flat* if $\text{Tor}_1^R(E, M) = 0$ for any injective R -module E , and M is said to be *strongly copure flat* if $\text{Tor}_i^R(E, M) = 0$ for any injective R -modules E and any $i \geq 1$ [7]. Copure flat modules and strongly copure flat modules were discovered when studying flat preenvelopes. Let M be an R -module. The *copure flat dimension* of M [7], denoted by $\text{cfd}_R(M)$, is the smallest integer $n > 0$ such that $\text{Tor}_{n+i}^R(E, M) = 0$ for any injective R -module E and any integer $i \geq 1$. If no such n exists, then $\text{cfd}_R(M) = \infty$. The *copure flat dimension of a ring R* [11], denoted by $\text{cfd}(R)$, is defined to be the supremum of copure flat dimensions of R -modules. It is proved that $\text{cfd}(R) = \text{IFD}(R)$ ([11, Theorem 3.8]).

Since the concept of semi-divisorial modules, which generalizes both divisorial modules and injective modules, was introduced by Glaz and Vasconcelos ([13]) and was modified to allow the semi-divisorial closure (or w -closure) by the second author, the so-called w -operation has proved to be useful in the study of multiplicative ideal theory and module theory. The introduction of the w -operation in the class of flat modules has been successful, see for instance [1, 16, 19, 21, 23, 25]. The notion of w -flat modules appeared first in [19] when R is a domain and was extended to arbitrary commutative rings in [16].

Let J be an ideal of R . Following [30], J is called a *Glaz-Vasconcelos ideal* (a GV -ideal for short) if J is finitely generated and the natural homomorphism $\varphi : R \rightarrow J^* = \text{Hom}_R(J, R)$ is an isomorphism. Let M be an R -module and define

$$\text{tor}_{GV}(M) = \{x \in M \mid Jx = 0 \text{ for some } J \in GV(R)\},$$

where $GV(R)$ is the set of GV -ideals of R . It is clear that $\text{tor}_{GV}(M)$ is a submodule of M . Now M is said to be *GV -torsion* (resp., *GV -torsion-free*) if $\text{tor}_{GV}(M) = M$ (resp., $\text{tor}_{GV}(M) = 0$). A GV -torsion-free module M is called a *w -module* if $\text{Ext}_R^1(R/J, M) = 0$ for any $J \in GV(R)$. Projective modules and reflexive modules are w -modules. In the recent paper [31], it was shown that flat modules are w -modules. The notion of w -modules was introduced firstly over a domain [24] in the study of Strong Mori domains and was extended to

commutative rings with zero divisors in [30]. Let $w\text{-Max}(R)$ denote the set of maximal w -ideals of R , i.e., w -ideals of R maximal among proper integral w -ideals of R . Following [30, Proposition 3.8], every maximal w -ideal is prime.

For any GV -torsion-free module M ,

$$M_w := \{x \in E(M) \mid Jx \subseteq M \text{ for some } J \in GV(R)\}$$

is a w -submodule of $E(M)$ containing M and is called the w -envelope of M , where $E(M)$ denotes the injective hull of M . It is clear that a GV -torsion-free module M is a w -module if and only if $M_w = M$.

Let M and N be R -modules and $f : M \rightarrow N$ be a homomorphism. Following [20], f is called a w -monomorphism (resp., w -epimorphism, w -isomorphism) if $f_{\mathfrak{m}} : M_{\mathfrak{m}} \rightarrow N_{\mathfrak{m}}$ is a monomorphism (resp., an epimorphism, an isomorphism) for all $\mathfrak{m} \in w\text{-Max}(R)$. An R -module M is called a w -flat module if the induced map $1 \otimes f : M \otimes A \rightarrow M \otimes B$ is a w -monomorphism for any w -monomorphism $f : A \rightarrow B$. Certainly flat modules are w -flat, but the converse implication is not true in general. Recently, modules of this type have received attention in several papers in the literature (see for example [16, 21, 25]). Characterizations of w -flat modules are given in [16, Theorem 3.3].

Lemma 1.1. *Let R be a ring and M be an R -module. Then the following are equivalent.*

- (1) M is w -flat.
- (2) $M_{\mathfrak{m}}$ is a flat $R_{\mathfrak{m}}$ -module for all $\mathfrak{m} \in w\text{-Max}(R)$.
- (3) $\text{Tor}_1^R(M, N)$ is GV -torsion for all R -modules N .
- (4) $\text{Tor}_n^R(M, N)$ is GV -torsion for all R -modules N and all $n \geq 1$.

In [25], the authors introduced and investigated the w -flat dimensions of modules and rings. Let R be a ring and n be a nonnegative integer. We say that an R -module M has w -flat dimension less or equal to n , denoted by $w\text{-fd}_R(M) \leq n$, if $\text{Tor}_{n+1}^R(M, N)$ is a GV -torsion R -module for all R -modules N . Hence, the w -weak global dimension of R is defined to be

$$w\text{-wdim}(R) = \sup\{w\text{-fd}_R(M) \mid M \text{ is an } R\text{-module}\}.$$

In Section 2, we introduce and characterize the (strongly) w -copure flat modules and the w -copure flat dimension of modules. We show then that the w -copure flat dimension is a refinement of the w -flat dimension. Section 3 deals with the w -copure flat dimension of rings. In Proposition 3.2, it is proved that the w -copure flat dimension of a ring R is equal to the supremum of w -flat dimensions of injective w -modules. The relations between the w -copure flat dimension and the weak (Gorenstein) global dimension of a ring are discussed. In Section 4, we also study change of rings theorems for the w -copure flat dimension in various contexts. Some illustrative examples are given.

Any undefined terminology or notation is standard, as in [3, 13, 23].

2. The w -copure flat dimension of modules

We begin this section by introducing key concepts which will be used throughout the paper.

Definition 2.1. Let R be a ring and M be an R -module.

- (1) M is called *w-copure flat* if $\text{Tor}_1^R(E, M)$ is a GV -torsion R -module for any injective w -module E , and M is said to be *strongly w-copure flat* if $\text{Tor}_n^R(E, M)$ is a GV -torsion R -module for any injective w -module E and any $n \geq 1$.
- (2) The *w-copure flat dimension* of M , denoted by $w\text{-cfd}_R(M)$, is defined to be the smallest integer $n \geq 0$ such that $\text{Tor}_{n+i}^R(E, M)$ is GV -torsion for any injective w -module E and any $i \geq 1$. If there is no such n , set $w\text{-cfd}_R(M) = \infty$.

Obviously by Lemma 1.1, every w -flat module is strongly w -copure flat (and so w -copure flat) and every copure flat (resp., strongly copure flat) is w -copure flat (resp., strongly w -copure flat). Recall that a ring R is called a *DW-ring* if every ideal of R is a w -ideal, or equivalently $GV(R) = \{R\}$ [18]. In this case, the only GV -torsion module is (0) , and so over such a ring, copure flat (resp., strongly copure flat) module and w -copure flat (resp., strongly w -copure flat) coincide. Examples of DW -rings are Prüfer domains, domains with Krull dimension one, and rings with Krull dimension zero.

Next we give an example of a strongly w -copure flat module which is not w -flat.

Example 2.2. Let R be a QF-ring, but not semisimple. For example, $R := k[X]/(X^2)$, where k is a field. Then every flat module is projective and every injective module is flat, and so every R -module is a strongly copure flat module. Since R is not a semisimple ring, there exists an R -module which is not flat. Thus every strongly copure flat R -module is not necessarily flat. Since $\dim(R) = 0$, R is a DW -ring. Thus every strongly w -copure flat R -module is not necessarily w -flat.

Proposition 2.3. Let R be a ring and $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ be a w -exact sequence of R -modules, where C is strongly w -copure flat. Then A is strongly w -copure flat if and only if B is strongly w -copure flat.

Proof. Let E be an injective w -module and let $i \geq 1$. By [23, Theorem 6.6.2], the induced homomorphism $\text{Tor}_i^R(E, A) \rightarrow \text{Tor}_i^R(E, B)$ is a w -isomorphism. Thus the assertion follows. \square

Corollary 2.4. Let R be a ring and M be a GV -torsion-free R -module. Then M is strongly w -copure flat if and only if M_w is strongly w -copure flat.

Proof. By [23, Proposition 6.2.5], M_w/M is GV -torsion, and so strongly w -copure flat. Hence applying Proposition 2.3 to the short exact sequence $0 \rightarrow M \rightarrow M_w \rightarrow M_w/M \rightarrow 0$ gives the result. \square

Proposition 2.5. *The following statements are equivalent for an R -module M .*

- (1) M is strongly w -copure flat.
- (2) $\text{Ext}_R^i(L, \text{Hom}_R(M, E)) = 0$ for any injective w -modules E and L , and any integer $i \geq 1$.
- (3) $M \otimes F$ is strongly w -copure flat for any w -flat R -module F .
- (4) $M \otimes F$ is strongly w -copure flat for any flat R -module F .
- (5) If $\xi : \cdots \rightarrow F_n \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0$ is w -exact with each F_i w -flat, then $E \otimes \xi$ remains w -exact for any injective w -module E .
- (6) There exists a w -exact sequence $\xi : \cdots \rightarrow F_n \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0$ with each F_i w -flat such that $E \otimes \xi$ remains w -exact for any injective w -module E .
- (7) If $\xi : \cdots \rightarrow F_n \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0$ is w -exact with each F_i flat, then $E \otimes \xi$ remains w -exact for any injective w -module E .
- (8) There exists a w -exact sequence $\xi : \cdots \rightarrow F_n \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0$ with each F_i flat such that $E \otimes \xi$ remains w -exact for any injective w -module E .

Proof. Let E be an injective w -module and A and B be any two R -modules. By [12, Theorem 1.1.8], we have the following isomorphism

$$(2.1) \quad \text{Ext}_R^i(A, \text{Hom}_R(B, E)) \cong \text{Hom}_R(\text{Tor}_i^R(A, B), E)$$

for any $i \geq 1$.

(1) \Rightarrow (2) Let L be an arbitrary injective w -module. Since $\text{Tor}_n^R(L, M)$ is GV -torsion, we have $\text{Hom}_R(\text{Tor}_i^R(L, M), E) = 0$. By the isomorphism (2.1), we have $\text{Ext}_R^i(L, \text{Hom}_R(B, E)) = 0$.

(2) \Rightarrow (1) Again by the isomorphism (2.1), we have $\text{Hom}_R(\text{Tor}_i^R(L, M), E) = 0$. By [23, Exercise 6.22], $\text{Tor}_i^R(L, M)$ is GV -torsion. Thus M is strongly w -copure flat.

(1) \Rightarrow (4) Let E be an injective w -module. By [23, Theorem 3.4.10], we have the natural isomorphism $\text{Tor}_i^R(E, M \otimes_R F) \cong \text{Tor}_i^R(E, M) \otimes_R F$ for any $i \geq 1$. Since $\text{Tor}_i^R(E, M)$ is a GV -torsion module for any $i \geq 1$, so is $\text{Tor}_i^R(E, M \otimes_R F)$ for any $i \geq 1$. Thus $M \otimes F$ is strongly w -copure flat.

(4) \Rightarrow (3) Let E be an injective w -module. By Lemma 1.1, $F_{\mathfrak{m}}$ is a flat $R_{\mathfrak{m}}$ -module, and so a flat R -module, for all $\mathfrak{m} \in w\text{-Max}(R)$. Thus $\text{Tor}_i^R(E, M \otimes_R F_{\mathfrak{m}})$ is a GV -torsion R -module. By [23, Proposition 6.2.18], $\text{Tor}_i^R(E, M \otimes_R F_{\mathfrak{m}})$ is also a w -module. Thus $\text{Tor}_i^R(E, M \otimes_R F_{\mathfrak{m}}) = 0$. By the natural isomorphism $\text{Tor}_i^R(E, M \otimes_R F)_{\mathfrak{m}} \cong \text{Tor}_i^R(E, M \otimes_R F_{\mathfrak{m}}) = 0$, we get that $\text{Tor}_i^R(E, M \otimes_R F)$ is GV -torsion. Thus $M \otimes F$ is strongly w -copure flat.

(3) \Rightarrow (1) This follows immediately by taking $F := R$.

(1) \Rightarrow (5) Let $\xi : \cdots \rightarrow F_n \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0$ be a w -exact sequence with each F_i flat. Set $K_0 := \ker(F_0 \rightarrow M)$ and $K_i := \ker(F_i \rightarrow F_{i-1})$ for any $i \geq 1$. Then $0 \rightarrow K_0 \rightarrow F_0 \rightarrow M \rightarrow 0$ and $0 \rightarrow K_i \rightarrow F_i \rightarrow K_{i-1} \rightarrow 0$ for any $i \geq 1$ are w -exact. Let E be an injective w -module over R . Since

$\text{Tor}_1^R(E, M)$ is GV -torsion, $E \otimes_R -$ leaves the sequence $0 \rightarrow K_0 \rightarrow F_0 \rightarrow M \rightarrow 0$ w -exact. It follows from Proposition 2.3 that K_0 is strongly w -copure flat since F_0 is w -flat. Then we can show that $E \otimes \xi$ leaves the sequence ξ w -exact by repeating the same procedure.

(5) \Rightarrow (7) This is trivial.

(7) \Rightarrow (8) This follows immediately by taking a flat resolution of M .

(8) \Rightarrow (6) This is trivial.

(6) \Rightarrow (1) Let E be an injective w -module over R and consider the w -exact sequence $\xi : \cdots \rightarrow F_n \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0$ of w -flat modules F_i such that $E \otimes \xi$ remains w -exact. Set $K_0 := \ker(F_0 \rightarrow M)$. Then there is a w -exact sequence $0 \rightarrow K_0 \rightarrow F_0 \rightarrow M \rightarrow 0$. Clearly $E \otimes_R -$ preserves the w -exactness of this sequence. Hence $\text{Tor}_1^R(E, M)$ is GV -torsion. Now by induction, $\text{Tor}_i^R(E, M)$ is GV -torsion for any $i \geq 1$. So M is strongly w -copure flat. \square

For the w -copure flat module, we can have the corresponding result. But we omit its proof since it is easy and similar.

Proposition 2.6. *The following statements are equivalent for an R -module M .*

- (1) M is w -copure flat.
- (2) $\text{Ext}_R^1(L, \text{Hom}_R(M, E)) = 0$ for any injective w -modules E and L .
- (3) $M \otimes F$ is w -copure flat for any w -flat R -module F .
- (4) $M \otimes F$ is w -copure flat for any flat R -module F .
- (5) If $\xi : 0 \rightarrow A \rightarrow F \rightarrow M \rightarrow 0$ is w -exact with F w -flat, then $E \otimes \xi$ remains w -exact for any injective w -module E .
- (6) There exists a w -exact sequence $\xi : 0 \rightarrow A \rightarrow F \rightarrow M \rightarrow 0$ with F w -flat such that $E \otimes \xi$ remains w -exact for any injective w -module E .
- (7) If $\xi : 0 \rightarrow A \rightarrow F \rightarrow M \rightarrow 0$ is w -exact with F flat, then $E \otimes \xi$ remains w -exact for any injective w -module E .
- (8) There exists a w -exact sequence $\xi : 0 \rightarrow A \rightarrow F \rightarrow M \rightarrow 0$ with F flat such that $E \otimes \xi$ remains w -exact for any injective w -module E .

Recall that an R -module M is called a *strong w -module* if $\text{Ext}_R^i(C, M) = 0$ for any GV -torsion module C and any $i \geq 1$. See [26] for more results of strong w -modules.

Proposition 2.7. *The following statements are equivalent for an R -module M .*

- (1) M is strongly w -copure flat.
- (2) If $0 \rightarrow N \rightarrow E_0 \rightarrow E_1 \rightarrow \cdots \rightarrow E_{n-1} \rightarrow E_n \rightarrow 0$ is w -exact with each E_i injective, then $\text{Tor}_i^R(N, M)$ is GV -torsion for any injective w -module M and any $i \geq 1$.
- (3) If N is a strong w -module with $\text{id}_R(N) < \infty$, then $\text{Tor}_i^R(N, M)$ is GV -torsion for any $i \geq 1$.

Proof. (1) \Rightarrow (2) Let $A := \ker(E_{n-1} \rightarrow E_n)$. Then $0 \rightarrow A \rightarrow E_{n-1} \rightarrow E_n$ is a w -exact sequence. By [23, Theorem 6.7.2], we have a w -exact sequence

$$\mathrm{Tor}_{i+1}^R(E_n, M) \rightarrow \mathrm{Tor}_i^R(A, M) \rightarrow \mathrm{Tor}_i^R(E_{n-1}, M).$$

Thus we get that $\mathrm{Tor}_i^R(A, M)$ is GV -torsion. Then the assertion follows by applying [23, Theorem 6.7.2] repeatedly.

(2) \Rightarrow (3) Since N is a strong w -module with $\mathrm{id}_R(N) < \infty$, by [26, Proposition 2.2] there exists an exact sequence

$$0 \rightarrow N \rightarrow E_0 \rightarrow E_1 \rightarrow \cdots \rightarrow E_{n-1} \rightarrow E_n \rightarrow 0$$

such that each E_i is an injective w -module. Now the assertion follows by hypothesis.

(3) \Rightarrow (1) This is trivial since every injective w -module is a strong w -module. \square

Proposition 2.8. *Let R be a ring. Then the class of all w -copure flat (resp., strongly w -copure flat) R -modules is closed under direct sums and direct summands.*

Proof. As seen in [22, Lemma 0.1], it is clear that the set of GV -torsion modules is closed under direct sums and direct summands. Hence our result follows immediately from the definition of w -copure flat (resp., strongly w -copure flat) modules. \square

Before giving a functorial description of the w -copure flat dimension of modules, we need the following lemma, which is similar to [25, Lemma 2.2].

Lemma 2.9. *Let E be an injective w -module over R and $0 \rightarrow A \rightarrow F \rightarrow C \rightarrow 0$ be a w -exact sequence of R -modules with F a strongly w -copure flat module. Then the induced map $\mathrm{Tor}_{i+1}^R(E, C) \rightarrow \mathrm{Tor}_i^R(E, A)$ is a w -isomorphism for any $i \geq 1$. Hence for any $i \geq 1$, $\mathrm{Tor}_{i+1}^R(E, C)$ is GV -torsion if and only if so is $\mathrm{Tor}_i^R(E, A)$.*

Proposition 2.10. *Let R be a ring, M be an R -module, and n be a nonnegative integer. Then the following statements are equivalent.*

- (1) $w\text{-cfd}_R(M) \leq n$.
- (2) $\mathrm{Tor}_{n+i}^R(E, M)$ is GV -torsion for any injective w -module E and any $i \geq 1$.
- (3) $\mathrm{Tor}_{n+i}^R(N, M)$ is GV -torsion for any strong w -module N with $\mathrm{id}_R(N) < \infty$ and any $i \geq 1$.
- (4) If a sequence $0 \rightarrow F_n \rightarrow F_{n-1} \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0$ is exact with F_0, \dots, F_{n-1} strongly w -copure flat, then F_n is also strongly w -copure flat.
- (5) If a sequence $0 \rightarrow F_n \rightarrow F_{n-1} \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0$ is exact with F_0, \dots, F_{n-1} flat, then F_n is strongly w -copure flat.

- (6) If a sequence $0 \rightarrow F_n \rightarrow F_{n-1} \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0$ is w -exact with F_0, \dots, F_{n-1} strongly w -copure flat, then F_n is also strongly w -copure flat.
- (7) If a sequence $0 \rightarrow F_n \rightarrow F_{n-1} \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0$ is w -exact with F_0, \dots, F_{n-1} flat, then F_n is strongly w -copure flat.

Proof. (1) \Leftrightarrow (2) This is just the definition of the w -copure flat dimension of modules.

(1) \Leftrightarrow (3) This follows from Proposition 2.7.

(2) \Leftrightarrow (5) Let $0 \rightarrow F_n \rightarrow F_{n-1} \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0$ be any exact sequence with F_0, \dots, F_{n-1} flat. Then F_n is strongly w -copure flat if and only if $\text{Tor}_{n+i}^R(E, M) \cong \text{Tor}_i^R(E, F_n)$ is GV -torsion for any injective module E and any $i \geq 1$.

(6) \Rightarrow (7) \Rightarrow (5) and (6) \Rightarrow (4) \Rightarrow (5) These are trivial.

(2) \Rightarrow (6) Let E be an injective w -module over R . Set $L_n := F_n$ and $L_i := \text{Im}(F_i \rightarrow F_{i-1})$ for $i = 1, \dots, n-1$. Then both $0 \rightarrow L_{i+1} \rightarrow F_i \rightarrow L_i \rightarrow 0$ and $0 \rightarrow L_1 \rightarrow F_0 \rightarrow M \rightarrow 0$ are w -exact. By using Lemma 2.9 repeatedly, we see that for any $k \geq 1$, $\text{Tor}_k^R(E, F_n)$ is GV -torsion if and only if $\text{Tor}_{n+k}^R(E, M)$ is GV -torsion. By hypothesis, $\text{Tor}_k^R(E, F_n)$ is GV -torsion for any $k \geq 1$. Thus F_n is strongly w -copure flat. \square

Proposition 2.11. *Let R be a ring and M be an R -module. Then:*

- (1) $w\text{-cfd}_R(M) \leq w\text{-fd}_R(M)$ with equality if $w\text{-fd}_R(M) < \infty$.
- (2) If M is a w -copure flat R -module and $w\text{-fd}_R(M) \leq 1$, then M is w -flat.

Proof. (1) It suffices to prove that if $n := w\text{-fd}_R(M) < \infty$, then every strongly w -copure flat module is w -flat, i.e., $n = 0$. Let M be a strongly w -copure flat module. Assume on the contrary that $n > 0$. Then there exists an R -module N such that $\text{Tor}_n^R(N, M)$ is not GV -torsion. Without loss of generality, we assume that N is GV -torsion free. Thus $E := E(N)$ is an injective w -module. Hence the exact sequence

$$0 \rightarrow N \rightarrow E \rightarrow E/N \rightarrow 0$$

gives rise to the exactness of the sequence

$$(2.2) \quad \text{Tor}_{n+1}^R(E/N, M) \rightarrow \text{Tor}_n^R(N, M) \rightarrow \text{Tor}_n^R(E, M).$$

The left term of (2.2) is a GV -torsion module since $w\text{-fd}_R(M) = n$, and the right term of (2.2) is a GV -torsion module since M is strongly w -copure flat. Hence by [22, Lemma 0.1], $\text{Tor}_n^R(N, M)$ is a GV -torsion module, a contradiction. Thus $n = 0$.

(2) Since $w\text{-fd}_R(M) \leq 1$, M is a strongly w -copure flat module. Now the assertion follows by (1). \square

The next example shows that every strongly w -copure flat module is not necessarily strongly copure flat.

Example 2.12. Let (R, \mathfrak{m}) be a regular local ring with $\text{gldim}(R) = 2$. Then $\text{fd}_R(R/\mathfrak{m}) = 2$, and hence $\text{fd}_R(\mathfrak{m}) = 1$. By [25], every strongly copure flat module is a flat module. Thus $\text{cfd}_R(\mathfrak{m}) = 1$. Since R is a UFD, and hence is a PvMD. By [25], $w\text{-dim}(R) \geq 1$. By Proposition 2.11, every strongly w -copure flat module is w -flat, and so $w\text{-cfd}_R(\mathfrak{m}) = 0$. Therefore every strongly w -copure flat module is not necessarily strongly copure flat.

The proof of the next result is standard homological algebra. Thus we omit its proof.

Proposition 2.13. *Let R be a ring and $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ be a w -exact sequence of R -modules. If two of $w\text{-cfd}_R(A)$, $w\text{-cfd}_R(B)$, and $w\text{-cfd}_R(C)$ are finite, so is the third. Moreover,*

- (1) $w\text{-cfd}_R(B) \leq \sup\{w\text{-cfd}_R(A), w\text{-cfd}_R(C)\}$.
- (2) $w\text{-cfd}_R(A) \leq \sup\{w\text{-cfd}_R(B), w\text{-cfd}_R(C) - 1\}$.
- (3) $w\text{-cfd}_R(C) \leq \sup\{w\text{-cfd}_R(B), w\text{-cfd}_R(A) + 1\}$.

Corollary 2.14. *Let R be a ring and $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ be a w -exact sequence of R -modules, where B is strongly w -copure flat. Then either the three modules are strongly w -copure flat or $w\text{-cfd}_R(C) = w\text{-cfd}_R(A) + 1$.*

The next result characterizes coherent rings with finite weak Gorenstein global dimension over which every strongly w -copure flat module is strongly copure flat.

Proposition 2.15. *Let R be a coherent ring with finite weak Gorenstein global dimension. Then the following statements are equivalent.*

- (1) *Every strongly w -copure flat R -module is strongly copure flat.*
- (2) *R is a DW-ring.*

Proof. (1) \Rightarrow (2) Let J be a GV -ideal. Then by [23, Theorem 6.2.6] R/J is a GV -torsion R -module, and so by [23, Corollary 6.7.4(1)] R/J is a w -flat R -module. Hence by hypothesis, R/J is strongly copure flat. Then by [11, Theorem 2.12], R/J is Gorenstein flat. Thus R/J is a submodule of a flat R -module, and so it is a GV -torsion free R -module (as a submodule of a w -module). Then $R/J = (0)$, and so $R = J$. Consequently R is a DW -ring.

(2) \Rightarrow (1) This is clear. \square

3. The w -copure flat dimension of rings

We start with the following definition.

Definition 3.1. Let R be a ring. The w -copure flat dimension of R , denoted by $w\text{-cfd}(R)$, is defined as the supremum of the w -copure flat dimensions of R -modules.

Proposition 3.2. *Let R be a ring and n be a nonnegative integer. Then the following statements are equivalent.*

- (1) $w\text{-cfD}(R) \leq n$.
- (2) For any R -module M and any exact sequence $0 \rightarrow P_n \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$ with P_0, P_1, \dots, P_{n-1} projective, we have P_n is a w -copure flat module.
- (3) $w\text{-fd}_R(E) \leq n$ for any injective w -module E .
- (4) Every GV -torsion-free R -module can be embedded in an R -module with w -flat dimension at most n .
- (5) Every strong w -module can be embedded in an R -module with w -flat dimension at most n .
- (6) $w\text{-fd}_R(\text{Hom}_R(F, E)) \leq n$ for any w -flat R -module F and any injective w -module E .
- (7) $w\text{-fd}_R(\text{Hom}_R(F, E)) \leq n$ for any flat R -module F and any injective w -module E .
- (8) $\text{id}_R(\text{Hom}_R(L, E)) \leq n$ for any injective w -modules E and L .

Proof. (1) \Rightarrow (2) This follows from Proposition 2.10.

(2) \Rightarrow (3) Let E be an injective w -module. Consider an arbitrary R -module M and pick an exact sequence $0 \rightarrow P_n \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$ such that P_0, \dots, P_{n-1} are projective. By hypothesis, P_n is w -copure flat, and then $\text{Tor}_{n+1}^R(E, M) \cong \text{Tor}_1^R(E, P_n)$ is a GV -torsion R -module. Hence by [25, Proposition 2.3], we have $w\text{-fd}_R(E) \leq n$.

(3) \Rightarrow (1) Let M be an R -module and E be any injective w -module. Since $w\text{-fd}_R(E) \leq n$, we obtain that $\text{Tor}_{n+1}^R(E, M)$ is a GV -torsion R -module. By Proposition 2.10, $w\text{-cfD}_R(M) \leq n$. Thus $w\text{-cfD}(R) \leq n$.

(3) \Rightarrow (4) Let N be a GV -torsion-free module. Then $E(N)$ is an injective w -module. Now the assertion follows immediately.

(4) \Rightarrow (5) This is trivial.

(5) \Rightarrow (3) Let E be an injective w -module. By hypothesis, E can be embedded in an R -module L with $w\text{-fd}_R(L) \leq n$. Thus we have $L = E \oplus B$, where B is a submodule of L . By using [25, Proposition 2.3] and the fact that the set of GV -torsion R -modules is closed under direct sums and direct summands, we can prove easily that for any two R -modules A and B , we have $w\text{-fd}_R(A \oplus B) = \sup\{w\text{-fd}_R(A), w\text{-fd}_R(B)\}$. Thus we have $w\text{-fd}_R(E) \leq n$.

(3) \Rightarrow (6) Let A be an R -module. Since F is a w -flat module, $\text{Tor}_1^R(A, F)$ is GV -torsion. By the isomorphism (2.1), we have that $\text{Ext}_R^1(A, \text{Hom}_R(F, E)) = 0$. Hence $\text{Hom}_R(F, E)$ is an injective module. Therefore it follows from the hypothesis that $w\text{-fd}_R(\text{Hom}_R(F, E)) \leq n$.

(6) \Rightarrow (7) This is trivial.

(7) \Rightarrow (3) This follows by letting $F = R$.

(1) \Rightarrow (8) Let M be any R -module and E, L be injective w -modules. Thus by using the isomorphism (2.1), Proposition 2.10, and [22, Lemma 0.1], we have

$$\text{Ext}_R^{n+i}(M, \text{Hom}_R(L, E)) \cong \text{Hom}_R(\text{Tor}_{n+i}^R(M, L), E) = 0$$

for any $i \geq 1$, and so $\text{id}_R(\text{Hom}_R(E, L)) \leq n$.

(8) \Rightarrow (1) Let M be an R -module and E, L be injective w -modules. Since $\text{id}_R(\text{Hom}_R(L, E)) \leq n$, it follows from the isomorphism (2.1) that

$$\text{Hom}_R(\text{Tor}_{n+i}^R(L, M), E) \cong \text{Ext}_R^{n+i}(M, \text{Hom}_R(L, E)) = 0$$

for any $i \geq 1$. By [23, Exercise 6.22], $\text{Tor}_{n+i}^R(L, M)$ is a GV -torsion module for any $i \geq 1$. Thus $w\text{-cfd}_R(M) \leq n$. Hence $w\text{-cfd}(R) \leq n$. \square

Corollary 3.3. *Let R be a ring. Then*

$$w\text{-cfd}(R) = \sup\{w\text{-fd}_R(E) \mid E \text{ is an injective } w\text{-module}\}.$$

Proof. This follows immediately from Proposition 3.2. \square

Corollary 3.4. *Let R be a ring. Then*

$$\sup\{w\text{-fd}_R(M) \mid w\text{-fd}_R(M) < \infty\} \leq w\text{-cfd}(R) \leq w\text{-wdim}(R).$$

Moreover,

- (1) *If $w\text{-wdim}(R) < \infty$, then the two above inequalities become equalities.*
- (2) *If $w\text{-cfd}(R) < \infty$, then*

$$\sup\{w\text{-fd}_R(M) \mid w\text{-fd}_R(M) < \infty\} = w\text{-cfd}(R).$$

Proof. (1) This follows from Proposition 2.11.

(2) By Corollary 3.3, there is an injective w -module E such that $w\text{-fd}_R(E) = w\text{-cfd}(R) < \infty$. Then $w\text{-cfd}(R) \leq \sup\{w\text{-fd}_R(M) \mid w\text{-fd}_R(M) < \infty\}$, and so we have the desired equality. \square

Following [4], we say that a ring R is an *IF ring* if every injective R -module is flat. In order to characterize the rings of w -copure flat dimension 0, we introduce a new class of rings, which is a w -version of IF rings.

Definition 3.5. A ring R is called a *w -IF-ring* if every injective w -module is w -flat.

Corollary 3.6. *Let R be a ring. Then R is a w -IF-ring if and only if $w\text{-cfd}(R) = 0$.*

Proposition 3.7. *Let R be a ring. Then $w\text{Ggldim}(R) \leq 1$ if and only if R is DW and $w\text{-cfd}(R) \leq 1$.*

Proof. As seen in [17, Theorem 2.12] and Proposition 3.2, it is clear that $w\text{-cfd}(R) \leq w\text{Ggldim}(R)$ with equality if R is a DW -ring. Hence it suffices to prove that if $w\text{Ggldim}(R) \leq 1$, then R is DW . Let $J \in GV(R)$. Since $w\text{Ggldim}(R) \leq 1$, J is Gorenstein flat. Then there exists a short exact sequence $0 \rightarrow J \rightarrow F \rightarrow G \rightarrow 0$ where F is flat and G is Gorenstein flat. Since F is a w -module and G is a GV -torsion-free module (as a submodule of a flat module), it follows from [15, Lemma 3.2] that J is a w -module. Hence $J_w = J$. On the other hand, by [30, Proposition 3.5] $J_w = R$. Thus $GV(R) = \{R\}$, which means that R is a DW -ring. \square

Corollary 3.8. *A ring R is IF if and only if it is a DW-ring and every module is (strongly) w -copure flat.*

Proof. Recall that R is IF if and only if $w\text{Gldim}(R) = 0$. Hence the assertion follows from Proposition 3.7. \square

Proposition 3.9. *A ring R is von Neumann regular if and only if every R -module is (strongly) w -copure flat and $w\text{-wdim}(R) < \infty$.*

Proof. If R is von Neumann regular, then $w\text{-wdim}(R) = 0$ and every R -module is flat (and so strongly w -copure flat). Conversely, if every R -module is w -copure flat, then by the equivalence of (1) and (2) in Proposition 3.2 we have $w\text{-cfD}(R) = 0$. Now, since $w\text{-wdim}(R) < \infty$, we get $w\text{-wdim}(R) = 0$ (by Corollary 3.4). Then by [21, Theorem 4.4], R is von Neumann regular. \square

Proposition 3.10. *The following statements are equivalent for a domain R .*

- (1) R is a field.
- (2) Every R -module is strongly w -copure flat.
- (3) Every R -module is w -copure flat.
- (4) Every cyclic R -module is strongly w -copure flat.
- (5) Every cyclic R -module is w -copure flat.
- (6) $w\text{-cfD}(R) = 0$.

Proof. (1) \Rightarrow (2) \Rightarrow (4) These are clear.

(4) \Rightarrow (1) Let a be a nonzero element of R . Clearly $w\text{-fd}_R(R/aR) \leq \text{fd}_R(R/aR) \leq \text{pd}_R(R/aR) \leq 1$ since R is a domain. Thus R/aR is w -flat (by Proposition 2.11). By applying [21, Proposition 2.2] to the exact sequence $0 \rightarrow aR \rightarrow R \rightarrow R/aR \rightarrow 0$, we obtain that $(a)_w = (a^2)_w$. Hence $a \in (a^2)_w$. Thus by [21, Theorem 4.4], R is von Neumann regular. Consequently R is a field since it is a domain.

(1) \Rightarrow (3) \Rightarrow (5) \Rightarrow (1) These are similar to the “strongly w -copure flat dimension” case.

(1) \Leftrightarrow (6) This follows immediately by the definition of the w -copure flat dimension of a ring. \square

Next we characterize the rings of w -copure flat dimension at most one.

Proposition 3.11. *The following conditions are equivalent for a ring R .*

- (1) $w\text{-cfD}(R) \leq 1$.
- (2) Every submodule of a w -copure flat R -module is strongly w -copure flat.
- (3) Every submodule of a w -copure flat R -module is w -copure flat.

Proof. (1) \Rightarrow (2) Let M be a w -copure flat module and let N be a submodule of M . Let E be an injective w -module and $n \geq 1$ be an integer. We have the following exact sequence of R -modules:

$$(3.1) \quad \text{Tor}_{i+1}^R(E, M/N) \rightarrow \text{Tor}_i^R(E, N) \rightarrow \text{Tor}_i^R(E, M).$$

The right term of (3.1) is always GV -torsion since M is w -flat and $w\text{-fd}(E) \leq 1$ by Proposition 2.11. Also the left term of (3.1) is GV -torsion since $w\text{-fd}(E) \leq 1$. Thus $\text{Tor}_i^R(E, N)$ is GV -torsion for any $i \geq 1$, which means that N is strongly w -copure flat.

(2) \Rightarrow (3) This is trivial.

(3) \Rightarrow (1) By hypothesis, every submodule of a projective R -module is w -copure flat. Hence the assertion follows from (2) \Rightarrow (1) in Proposition 3.2. \square

Proposition 3.12. *Let R be a coherent ring. Then $w\text{-cfD}(R) \leq 1$ if and only if every w -copure flat is strongly w -copure flat.*

Proof. (\Rightarrow) This follows from Proposition 3.11.

(\Leftarrow) Let M be an R -module. Pick a short exact sequence $0 \rightarrow K \rightarrow P \rightarrow M \rightarrow 0$ with P projective. Note that K has a flat preenvelope $f : K \rightarrow F$ since R is coherent [29, Theorem 2.5.1]. So f is a monomorphism, and we get an exact sequence $0 \rightarrow K \rightarrow F \rightarrow L \rightarrow 0$, where L is a copure flat module (by [6, Corollary 3.2]). Then L is strongly w -copure flat, and so is K . Then $w\text{-cfD}_R(M) \leq 1$, and so $w\text{-cfD}(R) \leq 1$. \square

Now we consider the w -copure flat dimension and w -weak global dimension of finite product rings.

Proposition 3.13. *Let R_1 and R_2 be two rings, M_1 and M_2 be an R_1 -module and an R_2 -module respectively. Set $R = R_1 \times R_2$ and $M = M_1 \times M_2$.*

- (1) *Every R_1 -module (resp., R_2 -module) which is GV -torsion as an R -module is a GV -torsion R_1 -module (resp., R_2 -module). If M_1 (resp., M_2) is a GV -torsion R_1 -module (resp., R_2 -module), then M is a GV -torsion R -module.*
- (2) *$\text{Tor}_{n+i}^{R_1}(M_1, E) \cong \text{Tor}_{n+i}^{R_1}(M \otimes_R R_1, E) \cong \text{Tor}_{n+i}^R(M, E)$ for any $i \geq 1$. In particular, M is a strongly w -copure flat R -module if and only if M_1 and M_2 are strongly w -copure flat R_1 -module and R_2 -module, respectively.*
- (3) *$w\text{-cfD}(R) = \sup\{w\text{-cfD}(R_1), w\text{-cfD}(R_2)\}$.*
- (4) *$w\text{-fd}_R(M) = \sup\{w\text{-fd}_{R_1}(M_1), w\text{-fd}_{R_2}(M_2)\}$ and $w\text{-wdim}(R) = \sup\{w\text{-wdim}(R_1), w\text{-wdim}(R_2)\}$.*

Proof. Note that any R -module N has a decomposition $N = N_1 \times N_2$, where N_1 (resp., N_2) is an R_1 -module (resp., R_2 -module). Now the assertion follows from the facts that

- (a) $\text{Hom}_R(M, N) \cong \text{Hom}_{R_1}(M_1, N_1) \times \text{Hom}_{R_2}(M_2, N_2)$,
- (b) $\text{Ext}_R^k(M, N) \cong \text{Ext}_{R_1}^k(M_1, N_1) \times \text{Ext}_{R_2}^k(M_2, N_2)$ for all $k \geq 1$,
- (c) $M \otimes_R N \cong (M_1 \otimes_{R_1} N_1) \times (M_2 \otimes_{R_2} N_2)$, and
- (d) $\text{Tor}_k^R(M, N) \cong \text{Tor}_k^{R_1}(M_1, N_1) \times \text{Tor}_k^{R_2}(M_2, N_2)$ for all $k \geq 1$. \square

Example 3.14. Let (R, \mathfrak{m}) be a regular local ring with $\text{gldim}(R) = 2$ and R' be an IF ring which is not von Neumann regular (take for example the ring $\mathbb{Z}/4\mathbb{Z}$). Then

- (1) $1 = w\text{-wdim}(R) = w\text{-cfD}(R) < \text{wdim}(R) = \text{wGldim}(R) = 2$.
- (2) $w\text{-cfD}(R') = \text{wGldim}(R') = 0$ and $w\text{-wdim}(R') = \text{wdim}(R') = \infty$.
- (3) $w\text{-cfD}(R \times R') = 1 < \text{wGldim}(R) = 2$ and $w\text{-wdim}(R) = \text{wdim}(R) = \infty$.

Proof. (1) It is known that R is a Krull domain, and so a $PvMD$. Thus by [25, Theorem 3.5] and [21, Theorem 4.4], $w\text{-wdim}(R) = 1$. Hence by using Proposition 3.4, $w\text{-cfD}(R) = 1$. Moreover, $\text{wGldim}(R) = IFD(R) = \text{wdim}(R) = 2$ since $\text{wdim}(R) = 2 < \infty$ (by [5, Corollary 3.3]).

(2) By [5, Corollary 3.3], $\text{wdim}(R') = \infty$ and $\text{wGldim}(R') = IFD(R') = 0$. On the other hand, $w\text{-cfD}(R') = \text{wGldim}(R') = 0$ and $w\text{-wdim}(R') = \text{wdim}(R') = \infty$. Now by Corollary 3.8, $w\text{-cfD}(R) = 0$ and R is DW , and then $w\text{-wdim}(R') = \infty$.

(3) This follows from Proposition 3.13 and [17, Theorem 3.1]. \square

4. Change of rings theorems for the w -copure flat dimension

In this section, we study change of rings theorems for the w -copure flat dimension in various contexts. Although some results of this section are analogous to those in [28], we need a new concept to give their proofs as follows.

Definition 4.1. (1) Let $\phi : R \rightarrow T$ be a ring homomorphism. Then T is said to *have property* (B_ϕ) if the following property is satisfied:

(B_ϕ) Let N be a T -module. If N is a GV -torsion R -module, then N is also a GV -torsion T -module.

- (2) Let $R \subseteq T$ be a ring extension. Use $\phi : R \rightarrow T$ to denote the embedding map. Then this ring extension is said to *have property* (B) if T has property (B_ϕ) .
- (3) Let $a \in R$. Use $\phi : R \rightarrow R/aR$ to denote the natural homomorphism. Then R is said to *have property* (B_a) if R/aR has property (B_ϕ) .

Proposition 4.2. Let $\phi : R \rightarrow T$ be a ring homomorphism such that T has property B_ϕ and let N be a T -module. Then

- (1) If N is a GV -torsion-free T -module, then N is a GV -torsion-free R -module.
- (2) If N is an injective w -module over T , then N is a strong w -module over R .

Proof. (1) Set $A := \{x \in N \mid \text{there exists } J \in GV(R) \text{ such that } Jx = 0\}$. Then it is easy to see that A is a T -submodule of N . Since A is a GV -torsion R -module, it follows from property (B_ϕ) that A is a GV -torsion T -module. Since N is a GV -torsion-free T -module, we have $A = 0$. Therefore N is also a GV -torsion-free R -module.

(2) Let C be a GV -torsion R -module. Then $T \otimes_R C$ is a GV -torsion R -module. By hypothesis, $T \otimes_R C$ is a GV -torsion T -module. By [23, Theorem 3.4.11], $\text{Ext}_R^k(C, N) \cong \text{Hom}_T(\text{Tor}_k^R(T, C), E) = 0$. Thus N is a strong w -module over R . \square

Recall that a ring extension $R \subseteq T$ is said to be w -linked if T as an R -module is a w -module. For example, the polynomial extension $R \subset R[X]$ is w -linked. Let $a \in R$. Write $\bar{R} = R/aR$. Let J be an ideal of R . Write $\bar{J} = \{\bar{b} \mid b \in J\}$, i.e., $\bar{J} = (J + aR)/aR$.

Proposition 4.3. (1) *If a ring extension $R \subseteq T$ is w -linked, then T has property (B).*
 (2) *Let S be a multiplicative subset of a ring R . Use $\phi : R \rightarrow R_S$ to denote the natural homomorphism. Then R_S has property (B_ϕ) .*
 (3) *Let $a \in R$. If $\bar{J} \in GV(\bar{R})$ for any $J \in GV(R)$, then R has property (B_a) .*

Proof. (1) Let N be a T -module, which is also a GV -torsion R -module. Let $x \in N$. Then there exists $J \in GV(R)$ such that $Jx = 0$. Hence $JTx = 0$. By [27, Lemma 3.3], $JT \in GV(T)$. Then N is a GV -torsion T -module. Therefore T has property (B).

(2) Let N be an R_S -module, which is also a GV -torsion R -module. Let $x \in N$. Then there exists $J \in GV(R)$ such that $Jx = 0$. By [23, Theorem 6.8.31], $J_S \in GV(R_S)$. Hence $J_S x = 0$. Then N is a GV -torsion R_S -module. Therefore R_S has property (B_ϕ) .

(3) Let N be an \bar{R} -module, which is also a GV -torsion R -module. Let $x \in N$. Then there exists $J \in GV(R)$ such that $Jx = 0$. Hence $\bar{J}x = 0$. By hypothesis, $\bar{J} \in GV(\bar{R})$. Then N is a GV -torsion \bar{R} -module. Therefore R has property (B_a) . \square

Next we state the main theorem of this section.

Theorem 4.4. *Let $\phi : R \rightarrow T$ be a ring homomorphism such that $\text{fd}_R T < \infty$ and T has property (B_ϕ) . If M is a strongly w -copure flat R -module and $\text{Tor}_k^R(T, M) = 0$ for any $k \geq 1$, then $T \otimes_R M$ is a strongly w -copure flat T -module.*

Proof. Let N be an injective w -module over T and let $0 \rightarrow A \rightarrow F \rightarrow M \rightarrow 0$ be an R -exact sequence, where F is a flat R -module. Since $\text{Tor}_1^R(T, M) = 0$, we have the following exact sequence:

$$0 \rightarrow T \otimes_R A \rightarrow T \otimes_R F \rightarrow T \otimes_R M \rightarrow 0.$$

Thus we have the following commutative diagram with two exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & X & \longrightarrow & \text{Tor}_1^R(N, M) & \longrightarrow & N \otimes_R A \longrightarrow N \otimes_R F \\ & & & & \downarrow \theta & & \downarrow & \downarrow \\ 0 & \longrightarrow & \text{Tor}_1^T(N, T \otimes_R M) & \longrightarrow & N \otimes_T (T \otimes_R A) & \longrightarrow & N \otimes_T (T \otimes_R F) \end{array}$$

Thus θ is an epimorphism. By [23, Exercise 3.15], $\text{id}_R N < \infty$. By Proposition 4.2, N is a strong w -module. Thus it follows from Proposition 2.7 that $\text{Tor}_1^R(N, M)$ is a GV -torsion R -module. Now it follows from property (B_ϕ) that

$\text{Tor}_1^T(N, T \otimes_R M)$ is a GV -torsion T -module. Note that A is also a strongly w -pure flat R -module by Proposition 2.3. By “dimension shifting”, we have $\text{Tor}_k^T(N, T \otimes_R M)$ is a GV -torsion T -module for any $k > 1$. Therefore $T \otimes_R M$ is a strongly w -copure flat T -module. \square

Let R be a ring, M be an R -module, and $a \in R$. Then we say that M is a -torsion-free if $am = 0$ for $m \in M$ implies that $m = 0$.

Corollary 4.5. *Let M be a strongly w -copure flat R -module. Then*

- (1) $M[X]$ is a strongly w -copure flat $R[X]$ -module.
- (2) If S is a multiplicative subset of R , then M_S is a strongly w -copure flat R_S -module.
- (3) Let $a \in R$, which is not a zero-divisor of both R and M . If R has property (B_a) , then M/aM is a strongly w -copure flat R/aR -module.

Proof. (1) and (2) follow by setting $T := R[X]$ and $T := R_S$ respectively in Theorem 4.4.

(3) Since M is a -torsion-free, we get that $\text{Tor}_1^R(R/aR, M) = 0$ by [28, Lemma 1]. Now the assertion follows by applying Theorem 4.4. \square

Set $N_w := \{f \in R[X] \mid c(f)_w = R\}$, where $c(f)$ denotes the content of f . Let $R\{X\} := R[X]_{N_w}$ and $M\{X\} := M[X]_{N_w}$ be the w -Nagata ring of R and the w -Nagata module of M respectively.

Proposition 4.6. *Let M be a strongly w -copure flat R -module. Then*

- (1) $M\{X\}$ is a strongly copure flat $R\{X\}$ -module.
- (2) $M_{\mathfrak{m}}$ is a strongly copure flat $R_{\mathfrak{m}}$ -module for any maximal w -ideal \mathfrak{m} of R .

Proof. (1) Let M be a strongly w -copure flat R -module. By Corollary 4.5(1), $M[X]$ is a strongly w -copure flat $R[X]$ -module. By Corollary 4.5(2), $M\{X\}$ is a strongly w -copure flat $R\{X\}$ -module. Since $R\{X\}$ is a DW-ring, $M\{X\}$ is a strongly copure flat $R\{X\}$ -module.

(2) Let E be an injective $R_{\mathfrak{m}}$ -module. Then E is an injective w -module over R . Thus $\text{Tor}_i^R(E, M)$ is a GV -torsion R -module for any $i \geq 1$. Since $\text{Tor}_i^R(E, M)$ is also an $R_{\mathfrak{m}}$ -module, it follows from [23, Proposition 6.2.8] that

$$\text{Tor}_i^{R_{\mathfrak{m}}}(E_{\mathfrak{m}}, M_{\mathfrak{m}}) \cong \text{Tor}_i^R(E, M)_{\mathfrak{m}} \cong \text{Tor}_i^R(E, M) = 0.$$

Now the assertion follows. \square

Proposition 4.7. *Let S be a multiplicative subset of a ring R . Then $w\text{-cfD}(R_S) \leq w\text{-cfD}(R)$.*

Proof. Without loss of generality, we may assume that $m := w\text{-cfD}(R) < \infty$. Let M be an R_S -module. Then by Proposition 2.10, there is an exact sequence

$$0 \rightarrow P_m \rightarrow P_{m-1} \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0,$$

where each P_i is a strongly w -copure flat R -module. Thus we have an exact sequence

$$0 \rightarrow (P_m)_S \rightarrow (P_{m-1})_S \rightarrow \cdots \rightarrow (P_1)_S \rightarrow (P_0)_S \rightarrow M \rightarrow 0,$$

where each $(P_i)_S$ is a strongly w -copure flat R_S -module by Corollary 4.5(2). Therefore, $w\text{-cfD}(R_S) \leq m$. \square

Acknowledgements. The authors would like to express their gratitude to an anonymous reviewer of this paper for his insightful comments and suggestions which have greatly improved the paper. This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (2017R1D1A3B0303342).

References

- [1] F. A. A. Almahdi, M. Tamekkante, and R. A. K. Assaad, *On the right orthogonal complement of the class of w -flat modules*, J. Ramanujan Math. Soc. **33** (2018), no. 2, 159–175.
- [2] D. Bennis and N. Mahdou, *Global Gorenstein dimensions*, Proc. Amer. Math. Soc. **138** (2010), no. 2, 461–465. <https://doi.org/10.1090/S0002-9939-09-10099-0>
- [3] H. Cartan and S. Eilenberg, *Homological Algebra*, Princeton University Press, Princeton, NJ, 1956.
- [4] R. R. Colby, *Rings which have flat injective modules*, J. Algebra **35** (1975), 239–252. [https://doi.org/10.1016/0021-8693\(75\)90049-6](https://doi.org/10.1016/0021-8693(75)90049-6)
- [5] N. Ding and J. L. Chen, *The flat dimensions of injective modules*, Manuscripta Math. **78** (1993), no. 2, 165–177. <https://doi.org/10.1007/BF02599307>
- [6] ———, *On copure flat modules and flat resolvents*, Comm. Algebra **24** (1996), no. 3, 1071–1081. <https://doi.org/10.1080/00927879608825623>
- [7] E. E. Enochs and O. M. G. Jenda, *Copure injective resolutions, flat resolvents and dimensions*, Comment. Math. Univ. Carolin. **34** (1993), no. 2, 203–211.
- [8] ———, *On Gorenstein injective modules*, Comm. Algebra **21** (1993), no. 10, 3489–3501. <https://doi.org/10.1080/00927879308824744>
- [9] ———, *Gorenstein injective and projective modules*, Math. Z. **220** (1995), no. 4, 611–633. <https://doi.org/10.1007/BF02572634>
- [10] E. E. Enochs, O. M. G. Jenda, and B. Torrecillas, *Gorenstein flat modules*, Nanjing Daxue Xuebao Shuxue Bannian Kan **10** (1993), no. 1, 1–9.
- [11] X. Fu and N. Ding, *On strongly copure flat modules and copure flat dimensions*, Comm. Algebra **38** (2010), no. 12, 4531–4544. <https://doi.org/10.1080/00927870903428262>
- [12] S. Glaz, *Commutative Coherent Rings*, Lecture Notes in Mathematics, **1371**, Springer-Verlag, Berlin, 1989. <https://doi.org/10.1007/BFb0084570>
- [13] S. Glaz and W. V. Vasconcelos, *Flat ideals. II*, Manuscripta Math. **22** (1977), no. 4, 325–341. <https://doi.org/10.1007/BF01168220>
- [14] H. Holm, *Gorenstein homological dimensions*, J. Pure Appl. Algebra **189** (2004), no. 1-3, 167–193. <https://doi.org/10.1016/j.jpaa.2003.11.007>
- [15] K. Hu, F. G. Wang, and L. Xu, *A note on Gorenstein Prüfer domains*, Bull. Korean Math. Soc. **53** (2016), no. 5, 1447–1455. <https://doi.org/10.4134/BKMS.b150760>
- [16] H. Kim and F. G. Wang, *On LCM-stable modules*, J. Algebra Appl. **13** (2014), no. 4, 1350133, 18 pp. <https://doi.org/10.1142/S0219498813501338>
- [17] N. Mahdou and M. Tamekkante, *On (weak) Gorenstein global dimensions*, Acta Math. Univ. Comenian. (N.S.) **82** (2013), no. 2, 285–296.

- [18] A. Mimouni, *Integral domains in which each ideal is a W -ideal*, *Comm. Algebra* **33** (2005), no. 5, 1345–1355. <https://doi.org/10.1081/AGB-200058369>
- [19] F. G. Wang, *On w -projective modules and w -flat modules*, *Algebra Colloq.* **4** (1997), no. 1, 111–120.
- [20] ———, *Finitely presented type modules and w -coherent rings*, *J. Sichuan Normal Univ.*, **33** (2010), 1–9.
- [21] F. G. Wang and H. Kim, *w -injective modules and w -semi-hereditary rings*, *J. Korean Math. Soc.* **51** (2014), no. 3, 509–525. <https://doi.org/10.4134/JKMS.2014.51.3.509>
- [22] ———, *Two generalizations of projective modules and their applications*, *J. Pure Appl. Algebra* **219** (2015), no. 6, 2099–2123. <https://doi.org/10.1016/j.jpaa.2014.07.025>
- [23] ———, *Foundations of Commutative Rings and Their Modules*, *Algebra and Applications*, **22**, Springer, Singapore, 2016. <https://doi.org/10.1007/978-981-10-3337-7>
- [24] F. G. Wang and R. L. McCasland, *On w -modules over strong Mori domains*, *Comm. Algebra* **25** (1997), no. 4, 1285–1306. <https://doi.org/10.1080/00927879708825920>
- [25] F. G. Wang and L. Qiao, *The w -weak global dimension of commutative rings*, *Bull. Korean Math. Soc.* **52** (2015), no. 4, 1327–1338. <https://doi.org/10.4134/BKMS.2015.52.4.1327>
- [26] ———, *A homological characterization of Krull domains II*, *Comm. Algebra* **47** (2019), no. 5, 1917–1929. <https://doi.org/10.1080/00927872.2018.1524007>
- [27] L. Xie, F. G. Wang, and Y. Tian, *On w -linked overrings*, *J. Math. Res. Exposition* **31** (2011), no. 2, 337–346.
- [28] T. Xiong, F. G. Wang, G. L. Xia, and X. W. Sun, *Change theorem of rings on the copure flat dimensions*, *J. Nat. Sci. Heilongjiang Univ.* **33** (2016), 435–437.
- [29] J. Xu, *Flat Covers of Modules*, *Lecture Notes in Mathematics*, **1634**, Springer-Verlag, Berlin, 1996. <https://doi.org/10.1007/BFb0094173>
- [30] H. Yin, F. G. Wang, X. Zhu, and Y. Chen, *w -modules over commutative rings*, *J. Korean Math. Soc.* **48** (2011), no. 1, 207–222. <https://doi.org/10.4134/JKMS.2011.48.1.207>
- [31] S. Q. Zhao, F. G. Wang, and H. L. Chen, *Flat modules over a commutative ring are w -modules*, *J. Sichuan Normal Univ.* **35** (2012), 364–366.

EL MEHDI BOUBA
 DEPARTMENT OF MATHEMATICS
 FACULTY OF SCIENCE
 UNIVERSITY MOULAY ISMAIL MEKNES
 BOX 11201, ZITOUNE, MOROCCO
Email address: mehdi8bouba@hotmail.fr

HWANKOO KIM
 DIVISION OF COMPUTER & INFORMATION ENGINEERING
 HOSEO UNIVERSITY
 ASAN 31499, KOREA
Email address: hkkim@hoseo.edu

MOHAMMED TAMEKKANTE
 DEPARTMENT OF MATHEMATICS
 FACULTY OF SCIENCE
 UNIVERSITY MOULAY ISMAIL MEKNES
 BOX 11201, ZITOUNE, MOROCCO
Email address: tamekkante@yahoo.fr