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HELMHOLTZ DECOMPOSITION AND SEMIGROUP THEORY TO THE FLUID AROUND A MOVING BODY

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Abstract. To understand the interaction of a fluid and a rigid body, we use the concept of B-evolution. Then in a similar way to the usual Navier-Stokes system, we obtain a Helmholtz type decomposition. Using B-evolution theory and the decomposition, we work on the semigroup to analyze the linear part of the system.

1. Introduction

We consider the interaction of a rigid body and a fluid, which describes the motion of a fluid $\mathcal{L}$ around a rigid body $\mathcal{B}$ moving through the liquid $\mathcal{L}$ that fills the whole space. This phenomenon is well described in [4], and we follow similar notations for the formulation. The rigid body $\mathcal{B}$ is assumed to be an open, connected and bounded set in $\mathbb{R}^3$. The region occupied by the rigid body $\mathcal{B}$ at time t is denoted by $\mathcal{B}_t$ with $\mathcal{B}_0 = \mathcal{B}$ by convention. Then $\Omega(t) := \mathbb{R}^3 \setminus \overline{\mathcal{B}}_t$ is the region occupied by the fluid $\mathcal{L}$ at time t.

The velocity field of the motion of $\mathcal{B}$ with respect to an inertial frame $\mathfrak{I}$ is denoted by $V = V(x,t)$:

$$V(x,t) = \eta(t) + w(t) \times \big(x - x_c(t)\big),$$

where $\eta(t) = \dot{x}_c(t)$, $x_c(t)$ is the position of the center of $\mathcal{B}$ at time t, and $w(t)$ is the angular velocity of $\mathcal{B}$. The Eulerian velocity and pressure fields of the liquid $\mathcal{L}$ in $\mathfrak{I}$ are denoted by $v = v(x,t)$ and $q = q(x,t)$. The equations of conservation of linear momentum and mass of $\mathcal{L}$ with respect to $\mathfrak{I}$ are given by

$$\text{(1)}\qquad \begin{aligned} &\rho\frac{dv}{dt} = \nabla\cdot\mathcal{T}(v,q) + \rho\mathcal{F}(x,t), \\ &\nabla\cdot v = 0 \end{aligned} \qquad \text{for } (x,t) \in \bigcup_{t>0} \Omega(t)\times\mathbb{R},$$

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where ρ is the density of $\mathcal{L}$, d/dt is the material derivative, $\mathcal{F}$ is the body force acting on $\mathcal{L}$, for example the gravity, and $\mathcal{T}(v,q)$ is the Cauchy stress tensor with shear viscosity μ:

$$\mathcal{T}(v,q) = -q\mathbf{1} + 2\mu D(v), \qquad D(v) = \frac{1}{2}(\nabla v + (\nabla v)^\top),$$

where $\mathbf{1}$ is the identity tensor.

The liquid is assumed to be at rest at infinity:

$$\lim_{|x|\to\infty} v(x,t) = 0,$$

and also satisfy

$$v(x,t) = V(x,t) \qquad (x,t) \in \bigcup_{t>0} \Sigma(t) \times \{t\},$$

where $\Sigma(t)$ is the boundary of $\Omega(t)$.

There are other similar equations describing the motion of rigid body-fluid, and there are also several results on the existence of solutions for the motion of rigid bodies in a viscous fluid. For example, [1, 2, 4, 5, 8, 11, 14].

In this paper, we approach to this problem from the perspective of the semigroup theory. The final aims of the subject is to analyze the full Navier-Stokes equations, however, as a first step we consider linear part, the Stokes equations, and fractional derivatives of the prime term of the equations for the future analysis to the full Navier-Stokes equations. This article consists of three parts. The first result is the Helmholtz type decomposition, and the second result is to study mathematical theory of the linear parts. Our final destination of this approach is to analyze the Navier-Stokes problem interacted with the rigid body, which is our next subject. For that, in this article as a third result we provide several properties of the linear operator including fractional powers.

In this point of view, there are several results. Grobbelaar-Van Dalsen and Sauer [7] worked on a symmetric rigid body performing a rotation in a fluid. In their model the body does not translate, instead it only rotates depending on time t. They obtained the existence of strong solutions using B-evolution theory in [12]. A similar problem is treated in [9]; the case that a rotating body with constant speed, but no translation. In [9], it is shown that the linear operator $\Delta + (\omega \times x) \cdot \nabla$ generates a C_0 semigroup contraction on L^2, which is not analytic.

In Section 2, we review the derivation of the problem in [4]. In Section 3, we obtain a Helmholtz type decomposition, which is one of our main results. In Section 4, we apply the B-evolution theory developed in [12, 13] to the problem to find its solution. In Section 5 we provide several properties of the operators for the future study of the full interaction problem of the rigid body and the Navier-Stokes fluid.

2. Formulations of the problem

In this section we review the formulation of the problem from (1) in [4]. Denote by m and $\rho_{\mathcal{B}}$ the mass and the density of the rigid body $\mathcal{B}$, respectively. The inertia tensor I of $\mathcal{B}$ with respect to the center of $\mathcal{B}$ is defined by the relation

$$a \cdot I \cdot b = \int_{\mathcal{B}_t} \rho_{\mathcal{B}} [a \times (x - x_c)] \cdot [b \times (x - x_c)]$$

for all $a, b \in \mathbb{R}^3$. The equations of motion of $\mathcal{B}$ in the frame $\mathfrak{I}$ are given by

$$\begin{aligned} m \frac{d\eta}{dt} &= F - \int_{\Sigma(t)} (\mathcal{T}(v, q) \cdot N - \rho v (v - V) \cdot N), \\ \frac{d(I \cdot w)}{dt} &= M - \int_{\Sigma(t)} (x - x_c) \times (\mathcal{T} \cdot N - \rho v (v - V) \cdot N), \end{aligned} \tag{2}$$

where N is the inward unit normal vector to $\Sigma(t)$, and F and M are total external force and external torque with respect to the center, acting on $\mathcal{B}$.

The motions of $\mathcal{B}$ and $\mathcal{L}$ will be determined by the above equations, once the initial conditions on v and V are prescribed. The region $\Omega(t)$ occupied by $\mathcal{L}$ is an unknown function of t. In [4], the problem is reformulated in a frame $\mathcal{S}$ attached to $\mathcal{B}$, where this region remains the same at all times. It is done in the following transformations:

$$x = Q(t) \cdot y + x_c(t), \qquad Q(0) = \mathbf{1}, \qquad x_c(0) = 0,$$

where $\mathbf{1}$ is the identity tensor, and $Q(t)$ is the orthogonal linear transformation satisfying

$$\frac{d}{dt}(Q(t) \cdot a) = w(t) \times (Q(t) \cdot a) \quad \text{for all } a \in \mathbb{R}^3.$$

Then, new variables and notations are also introduced in the following ways: for $\mathcal{B}$,

$$\xi(t) = Q^\top(t) \cdot \eta(t), \qquad \omega(t) = Q^\top(t) \cdot w(t),$$
$$\mathcal{I} = Q^\top \cdot I \cdot Q, \quad G(t) = Q^\top \cdot g,$$

and for $\mathcal{L}$,

$$u(y, t) = Q^\top(t) \cdot v(Q(t) \cdot y + x_c(t), t), \qquad p(y, t) = q(Q(t) \cdot y + x_c(t), t),$$
$$T(u, p) = Q^\top \cdot \mathcal{T}(Q \cdot u, p) \cdot Q.$$

Notice that $\mathcal{I}$ is independent of time, since

$$a \cdot \mathcal{I} \cdot b = \int_{\mathcal{B}} \rho_{\mathcal{B}} (a \times y) \cdot (b \times y) \qquad \text{for all } a, b \in \mathbb{R}^3.$$

For the Navier-Stokes liquid, the Cauchy tensor is given by

$$\mathcal{T}(v, p) \equiv \mathcal{T}_{NS}(v, p) = -p\mathbf{1} + 2\mu D(v),$$

where μ is the shear viscosity coefficient (assumed to be 1). Then, the following system is obtained from (1) in [4];

$$\begin{aligned}
&\frac{\partial u}{\partial t} + R_e(u - u_\Sigma)\cdot\nabla u + \omega\times u = \Delta u - \nabla p + G \quad \text{in } \Omega\times(0,\infty),\\
&\nabla\cdot u = 0 \qquad \text{in } \Omega\times(0,\infty),\\
&\lim_{|x|\to\infty} u(x,t) = 0,\\
&u(x,t) = u_\Sigma(x,t) \qquad (x,t)\in\Sigma\times(0,\infty),
\end{aligned}$$

where $u_\Sigma(x,t) = \xi + \omega\times x$, and G is obtained from $\mathcal{F}$. Here $\Omega := \Omega(0) = \mathbb{R}^3\setminus\overline{\mathcal{B}}$ is a fixed region in $\mathbb{R}^3$, Σ is the boundary of Ω and R_e is the Reynolds number.

From (2), the following equations are obtained: in $(0,\infty)$,

$$\begin{aligned}
m\frac{d\xi}{dt} + R_e m\omega\times\xi &= Q^\top\cdot F - \int_\Sigma T(u,p)\cdot n,\\
\mathcal{I}\frac{d\omega}{dt} + R_e\omega\times(\mathcal{I}\omega) &= Q^\top\cdot M - \int_\Sigma x\times(T(u,p)\cdot n),
\end{aligned}$$

where $n = Q^T\cdot N$ is the unit normal to Σ directed toward $\mathcal{B}$. The initial conditions are given by

$$u(x,0) = u_0, \quad \xi(0) = \xi_0, \quad \omega(0) = \omega_0. \tag{3}$$

Let $\mathcal{J}_1$, $\mathcal{J}_2$, $\mathcal{J}_3$ and $\mathcal{J}_4$ be the boundary operators defined by

$$\begin{aligned}
\mathcal{J}_1 p &\equiv -m^{-1/2}\int_\Sigma pn, & \mathcal{J}_2 u &\equiv -m^{-1/2}\int_\Sigma D(u)\cdot n,\\
\mathcal{J}_3 p &\equiv -\mathcal{I}^{-1/2}\int_\Sigma x\times(pn), & \mathcal{J}_4 u &\equiv -\mathcal{I}^{-1/2}\int_\Sigma x\times(D(u)\cdot n).
\end{aligned}$$

We finally obtain the following system:

$$\begin{cases}
u_t + \nabla p = \Delta u - R_e(u - u_\Sigma)\cdot\nabla u - \omega\times u + G,\\
\nabla\cdot u = 0,\\
m^{1/2}\xi_t + \mathcal{J}_1 p = \mathcal{J}_2 u - R_e m^{1/2}\omega\times\xi + m^{1/2}Q^\top\cdot F,\\
\mathcal{I}^{1/2}\omega_t + \mathcal{J}_3 p = \mathcal{J}_4 u - R_e\mathcal{I}^{-1/2}\omega\times(\mathcal{I}\omega) + \mathcal{I}^{-1/2}Q^\top\cdot M
\end{cases} \tag{4}$$

with the initial condition (3), and the boundary condition is

$$\lim_{|x|\to\infty} u(x,t) = 0, \quad u(x,t) = u_\Sigma(x,t) \quad (x,t)\in\Sigma\times(0,\infty), \tag{5}$$

where $u_0 = \xi_0 + \omega_0\times x$ on Σ, and $u_\Sigma(x,t) = \xi + \omega\times x$. Here, $\mathcal{I}^{1/2}\mathcal{I}^{1/2} = \mathcal{I}$ and $\mathcal{I}^{-1/2}$ is its inverse.

Like in many articles, for example [2, 7], we also ignore the external force terms. Since in this article we are interested in the semigroup theory, we consider the linear parts of the system with the same initial and boundary conditions (3), (5):

$$(6) \qquad \begin{cases} u_t + \nabla p = \Delta u, & \text{in } \Omega \times (0, \infty), \\ \nabla \cdot u = 0, & \text{in } \Omega \times (0, \infty), \\ m^{1/2}\xi_t + \mathcal{J}_1 p = \mathcal{J}_2 u, & \text{in } (0, \infty), \\ \mathcal{I}^{1/2}\omega_t + \mathcal{J}_3 p = \mathcal{J}_4 u, & \text{in } (0, \infty). \end{cases}$$

3. Helmholtz type decomposition

For the study of the Navier-Stokes equations, people, in many cases, try to remove the pressure term and obtain a system consisting of velocity alone using the Helmholtz decomposition and the Leray projection. The system (4) contains the pressure terms. In order to remove these terms we obtain a Helmholtz type decomposition in this section. We need several definitions of spaces and operators.

∘ Space $\mathbb{Y}_0$:

We consider the Hilbert space

$$\mathbb{Y}_0 := L^2(\Omega) \times \mathbb{R}^3 \times \mathbb{R}^3 = \left\{ [v, \zeta, \theta] : v \in L^2(\Omega),\ \zeta \in \mathbb{R}^3,\ \theta \in \mathbb{R}^3 \right\}$$

equipped with the usual norm

$$\left\| [v, \zeta, \theta] \right\|_{\mathbb{Y}_0} := \left(\|v\|^2_{L^2(\Omega)} + |\zeta|^2 + |\theta|^2 \right)^{1/2}$$

and the inner product

$$\left\langle [v^1, \zeta^1, \theta^1], [v^2, \zeta^2, \theta^2] \right\rangle_{\mathbb{Y}_0} := \left\langle v^1, v^2 \right\rangle_{L^2(\Omega)} + \zeta^1 \cdot \zeta^2 + \theta^1 \cdot \theta^2.$$

∘ Space $\mathbf{\Phi}_\sigma$ and Space $\mathbb{X}$

Following Grobberlaar-Van Dalsen and Sauer [7], we also introduce the space $\mathbf{\Phi}_\sigma$ of all vector fields $\phi \in C^\infty_{0,\sigma}(\mathbb{R}^3)$ such that $\phi = \xi + \omega \times x$ in a neighborhood of Ω^c for some constant vectors ξ, ω, where $C^\infty_{0,\sigma}(\mathbb{R}^3)$ is the set of infinitely smooth, divergence free functions with compact support. It is quite obvious that for each $\phi \in \mathbf{\Phi}_\sigma$ there exist unique vectors $\xi(\phi), \omega(\phi)$ such that $\phi = \xi(\phi) + \omega(\phi) \times x$ in a neighborhood of Ω^c. Moreover, the mapping $\phi \mapsto [\xi(\phi), \omega(\phi)]$ is linear from $\mathbf{\Phi}_\sigma$ into $\mathbb{R}^3 \times \mathbb{R}^3$.

The space $\mathbb{X}$ denotes the closure of $\mathbf{\Phi}_\sigma$ in $L^2(\Omega)$.

∘ Mapping K and Space $\mathbb{Y}$

Define $K : \mathbf{\Phi}_\sigma \to \mathbb{Y}_0$ by

$$K\phi := [\phi, m^{1/2}\xi(\phi), \mathcal{I}^{1/2}\omega(\phi)] \quad \text{for all } \phi \in \mathbf{\Phi}_\sigma.$$

We denote by $\mathbb{Y}$ the closure of $K(\mathbf{\Phi}_\sigma)$ in $\mathbb{Y}_0$. Note that $K : \mathbf{\Phi}_\sigma \to \mathbb{Y}_0$ is linear. Hence, $\mathbb{Y}$ is a closed subspace of the Hilbert space $\mathbb{Y}_0$. It follows from the projection theorem in the Hilbert space theory that $\mathbb{Y}_0 = \mathbb{Y} \oplus \mathbb{Y}^\perp$.

∘ Projection $\mathcal{P}$

Let $\mathcal{P}$ be the orthogonal projection of $\mathbb{Y}_0$ onto $\mathbb{Y}$, so that

$$\operatorname{Ker}\mathcal{P} = \mathbb{Y}^\perp \quad \text{and} \quad \mathcal{P}K\phi = K\phi \quad \text{for all } \phi \in \mathbf{\Phi}_\sigma. \tag{7}$$

The goal of this section is to prove the following characterization of $\mathbb{Y}^\perp$, which provides a Helmholtz type decomposition of $\mathbb{Y}_0$. This is one of the main theorems.

Theorem 3.1.

$$\mathbb{Y}^\perp = \left\{[\nabla p, \mathcal{J}_1 p, \mathcal{J}_3 p] : p \in D^{1,2}(\Omega)\right\},$$

where $D^{1,2}(\Omega) := \{\pi \in L^2_{loc}(\overline{\Omega}) : \nabla\pi \in L^2(\Omega)\}$ *is the homogeneous Sobolev space.*

Proof. Define $\mathbb{G} := \left\{[\nabla p, \mathcal{J}_1 p, \mathcal{J}_3 p] : p \in D^{1,2}(\Omega)\right\}$.

We first show that $\mathbb{G} \subset \mathbb{Y}^\perp$. To do this, let $\phi \in \mathbf{\Phi}_\sigma$ and $p \in D^{1,2}(\Omega)$ be given. Then setting $[\xi, \omega] = [\xi(\phi), \omega(\phi)]$, we have

$$\begin{aligned}
&\langle K\phi, [\nabla p, \mathcal{J}_1 p, \mathcal{J}_3 p]\rangle_{\mathbb{Y}_0} \\
&= \int_\Omega \phi \cdot \nabla p\, dx + m^{1/2}\xi \cdot \mathcal{J}_1 p + \mathcal{I}^{1/2}\omega \cdot \mathcal{J}_3 p \\
&= -\int_\Omega p\nabla \cdot \phi\, dx + \int_\Sigma p\phi \cdot n\, ds - \xi \cdot \int_\Sigma pn\, ds - \omega \cdot \int_\Sigma x \times (pn)\, ds \\
&= \int_\Sigma p\,(\phi - \xi(\phi) - \omega(\phi) \times x) \cdot n\, ds = 0.
\end{aligned}$$

This proves that $\mathbb{G} \perp K(\mathbf{\Phi}_\sigma)$ (*orthogonal*). Since $K(\mathbf{\Phi}_\sigma)$ is dense in $\mathbb{Y}$, it follows that $\mathbb{G} \perp \mathbb{Y}$ and so $\mathbb{G} \subset \mathbb{Y}^\perp$.

To prove that $\mathbb{Y}^\perp \subset \mathbb{G}$, suppose that $[v, \zeta, \theta] \in \mathbb{Y}^\perp$. Then for all $\phi \in \mathbf{\Phi}_\sigma$, we have

$$\langle v, \phi\rangle_{L^2(\Omega)} + m^{1/2}\zeta \cdot \xi(\phi) + \mathcal{I}^{1/2}\theta \cdot \omega(\phi) = \langle [v, \zeta, \theta], K\phi\rangle_{\mathbb{Y}_0} = 0. \tag{8}$$

Noticing that if $\phi \in C^\infty_{0,\sigma}(\Omega)$ then $[\xi(\phi), \omega(\phi)] = [0, 0]$, we deduce that

$$\langle v, \phi\rangle_{L^2(\Omega)} = \langle [v, \zeta, \theta], K\phi\rangle_{\mathbb{Y}_0} = 0 \quad \text{for all } \phi \in C^\infty_{0,\sigma}(\Omega).$$

Hence by the famous De Rham theorem(see [3, Chapter III] e.g.), there exists a scalar $p \in D^{1,2}(\Omega)$ such that $v = \nabla p$ in Ω.

To show that $[v, \zeta, \theta] \in \mathbb{G}$, it now remains to prove that $[\zeta, \theta] = [\mathcal{J}_1 p, \mathcal{J}_3 p]$. To do this, let $\xi, \omega \in \mathbb{R}^3$ be given. For all $x \in \mathbb{R}^3$, we define

$$\phi(x) = \frac{1}{2}\nabla \times \left[\rho(x)\left(\xi \times x - |x|^2\omega\right)\right],$$

where $\rho \in C^\infty_0(\mathbf{R}^3)$ is a cut-off function with $\rho = 1$ near Ω^c. Then it is easy to show that $\phi \in \mathbf{\Phi}_\sigma$ and $[\xi, \omega] = [\xi(\phi), \omega(\phi)]$. Hence by (8),

$$\begin{aligned}
0 = \langle [v, \zeta, \theta], K\phi\rangle_{\mathbb{Y}_0} &= \int_\Omega \nabla p \cdot \phi\, dx + m^{1/2}\zeta \cdot \xi + \mathcal{I}^{1/2}\theta \cdot \omega \\
&= \int_\Sigma p\phi \cdot n\, ds + m^{1/2}\zeta \cdot \xi + \mathcal{I}^{1/2}\theta \cdot \omega
\end{aligned}$$

$$= \int_\Sigma p(\xi + \omega \times x) \cdot n \, ds + m^{1/2}\zeta \cdot \xi + \mathcal{I}^{1/2}\theta \cdot \omega$$
$$= \xi \cdot \left(\int_\Sigma pn \, ds + m^{1/2}\zeta \right) + \omega \cdot \left(\int_\Sigma p(x \times n) \, ds + \mathcal{I}^{1/2}\theta \right)$$
$$= m^{1/2}\xi \cdot (-\mathcal{J}_1 p + \zeta) + \mathcal{I}^{1/2}\omega \cdot (-\mathcal{J}_3 p + \theta) .$$

Since $\xi, \omega \in \mathbb{R}^3$ are arbitrary, it follows that $-\mathcal{J}_1 p + \zeta = -\mathcal{J}_3 p + \theta = 0$. This proves that $\mathbb{Y}^\perp \subset \mathbb{G}$. The proof of Theorem 3.1 is completed. □

4. Abstract B-evolution equation

4.1. Review of B-evolutions

In [12] and [13], Sauer developed the B-evolution theory to study the Sobolev-Galpern type problem:

$$\text{(9)} \qquad \begin{cases} \dfrac{d}{dt}[Bu](t) = Au(t) & \text{for } t > 0, \\ Bu(0) = y, \end{cases}$$

where A and B are linear elliptic operators. In this subsection we review the theory and apply it to our problem in the following subsections.

Let X and Y be Banach spaces and let $B : X \to Y$ be a linear operator with domain $\mathcal{D}(B) \subset X$ and range $\mathcal{R}(B) \subset Y$. Then by a *B-evolution*, we mean a family $\{S(t)\}_{t>0}$ of bounded linear operators from Y into X such that

$$S(t)[Y] \subset \mathcal{D}(B) \quad \text{for all } t > 0$$

and

$$\text{(10)} \qquad S(t+s) = S(s)BS(t) \quad \text{for all } s, t > 0.$$

Let $\{S(t)\}_{t>0}$ be a B-evolution. For each $t > 0$, we define

$$\text{(11)} \qquad E(t) := BS(t) : Y \to Y.$$

Then $\{E(t)\}_{t>0}$ is a semigroup of (possibly unbounded) linear operators on Y, which is called the *semigroup associated with the B-evolution* $\{S(t)\}_{t>0}$. It follows from (10) and (11) that

$$\text{(12)} \qquad S(t+s) = S(s)E(t) = S(t)E(s) \quad \text{for all } s, t > 0.$$

If $E(t)$ is a C_0 semigroup, then $S(t)$ is called *strongly continuous*.

The (*infinitesimal*) *generator* A of a B-evolution $S(t)$ with *domain* $\mathcal{D}(A)$ is defined in the following way: denote by $\mathcal{D}(A)$, the set of all $x \in \mathcal{D}(B)$ such that the limit

$$Ax := \lim_{h \to 0^+} \frac{BS(h)Bx - Bx}{h}$$

exists. Then $\mathcal{D}(A)$ is a subspace of X and $A : \mathcal{D}(A) \to Y$ is a linear operator.

Suppose that $\{E(t)\}_{t>0}$ is a C_0-semigroup of bounded linear operators on Y with generator A_Y. Then it can be easily shown that for every $y \in Y$ such that $E(\cdot)y$ is differentiable on $(0, \infty)$, the function $u = S(\cdot)y$ is a solution of

(9); in particular, if $\{E(t)\}_{t>0}$ is an analytic semigroup, then the problem (9) is solvable for every $y \in Y$. It is proved in [12, Theorem 2.1] that when $S(t)$ is a strongly continuous B-evolution,

$$x \in \mathcal{D}(A) \text{ if and only if } Bx \in \mathcal{D}(A_Y) \text{ and } Ax = A_Y Bx \text{ for such } x. \tag{13}$$

The following useful result is much inspired by Theorem 5.1 in [12].

Theorem 4.1. *Let A and B be linear operators from X into Y such that B has a bounded inverse and AB^{-1} generates a C_0 semigroup $\{E(t)\}_{t\geq 0}$. Then there exists a unique B-evolution $\{S(t)\}_{t>0}$ whose associated semigroup is $\{E(t)\}_{t>0}$. Moreover, A is the generator of $\{S(t)\}_{t>0}$. Finally, for every $y \in Y$, $u = S(\cdot)y$ is the unique solution of the abstract problem* (9)*.*

Proof. Let $C : \mathcal{R}(B) \to \mathcal{D}(B)$ be the bounded inverse of B, so that

$$BCy = y \quad \text{for all } y \in \mathcal{R}(B) \quad \text{and} \quad CBx = x \quad \text{for all } x \in \mathcal{D}(B). \tag{14}$$

Since AB^{-1} is the generator of a C_0-semigroup $\{E(t)\}_{t\geq 0}$, by the Hille-Yosida theorem its domain $\mathcal{D}(AB^{-1})$ is dense in Y, therefore, so is $\mathcal{R}(B)$ since $\mathcal{D}(AB^{-1}) \subset \mathcal{R}(B)$ clearly. Hence C can be extended uniquely to a bounded linear operator from Y into X, denoted again by C. For each $t > 0$, define

$$S(t) = CE(t).$$

Then $S(t) : Y \to X$ is obviously bounded. Moreover, it follows $E(t)[Y] \subset \mathcal{R}(B)$ and $BCE(t) = E(t)$ for each $t > 0$. This enables us to deduce that

$$S(t)[Y] \subset \mathcal{D}(B), \quad BS(t) = BCE(t) = E(t)$$

and

$$S(t+s) = CE(t+s) = CE(t)E(s) = S(t)\,[BCE(s)] = S(t)BS(s)$$

for all $s, t > 0$. Hence $\{S(t)\}_{t>0}$ is a B-evolution and its associated semigroup is $\{E(t)\}_{t>0}$.

If $\{\overline{S}(t)\}_{t>0}$ is a B-evolution and its associated semigroup is $\{E(t)\}_{t>0}$, then by (14), we have

$$\overline{S}(t) = CB\overline{S}(t) = CE(t) = S(t) \quad \text{for all } t > 0,$$

which proves the uniqueness.

Next, let $\overline{A}$ be the generator of $\{S(t)\}_{t>0}$. Then from the property of generators (13), it follows that $x \in \mathcal{D}(\overline{A})$ if and only if $x \in \mathcal{D}(B)$ and $Bx \in \mathcal{D}(AB^{-1}) \subset \mathcal{R}(B)$. This implies that if $x \in \mathcal{D}(\overline{A})$ then $Bx \in \mathcal{D}(AB^{-1})$ and

$$\overline{A}x = \lim_{h\to 0^+} \frac{E(h)Bx - Bx}{h} = (AB^{-1})Bx = Ax,$$

therefore, $x \in \mathcal{D}(A)$, that is, $\mathcal{D}(\overline{A}) \subset \mathcal{D}(A)$.

For $x \in \mathcal{D}(A)$, one has that $Bx \in \mathcal{D}(AB^{-1})$ and

$$Ax = (AB^{-1})Bx = \lim_{h\to 0^+} \frac{E(h)Bx - Bx}{h},$$

which is $\overline{A}x$ since the left side value converges. This implies that $\mathcal{D}(\overline{A}) = \mathcal{D}(A)$, therefore, A is the generator of $S(t)$.

To complete the proof, let $y \in Y$ be given. Then obviously $u = S(\cdot)y$ is a solution of (9). Let $\overline{u} : (0,\infty) \to \mathcal{D}(B) \subset X$ be another solution of (9). Define $v : (0,\infty) \to \mathcal{R}(B) \subset Y$ by $v(t) = B\overline{u}(t)$ for $t > 0$. Then it follows from (14) that $Cv(t) = CB\overline{u}(t) = \overline{u}(t)$ for $t > 0$. Hence v satisfies

$$\begin{cases} \dfrac{d}{dt}v(t) = (AB^{-1})v(t) \quad \text{for } t > 0 \\ \quad v(t) \to y \quad \text{as } t \to 0^+. \end{cases}$$

Since AB^{-1} is the generator of a C_0-semigroup $\{E(t)\}_{t\geq 0}$, it follows from the semigroup theory that $v(t) = E(t)y$ and so $\overline{u}(t) = Cv(t) = CE(t)y = S(t)y = u(t)$ for all $t > 0$. The proof of Theorem 4.1 is complete. $\square$

In the above, if $\mathcal{D}(A) = \mathcal{D}(B)$, then $\mathcal{D}(AB^{-1}) = \mathcal{R}(B)$.

4.2. Abstract formulation of our problem

Recall that $[\xi(\cdot), \omega(\cdot)]$ is a linear operator from $\mathbf{\Phi}_\sigma$ into $\mathbb{R}^3 \times \mathbb{R}^3$ such that $\phi = \xi(\phi) + \omega(\phi) \times x$ in a neighborhood of Ω^c. It was shown in [4, Lemma 4.9] that there is a constant $C = C(\mathcal{B}) > 0$ such that

$$|\xi(\phi)| + |\omega(\phi)| \leq C\|D(\phi)\|_{L^2} \tag{15}$$

for all $\phi \in \mathbf{\Phi}_\sigma$. Let $\mathcal{H}^1_\sigma(\Omega)$ be the closure of $\mathbf{\Phi}_\sigma$ in the standard Sobolev space $W^{1,2}_{loc}(\overline{\Omega})$ in the norm $\|D(\cdot)\|_{L^2}$. Then by the inequality (15), $[\xi(\cdot), \omega(\cdot)]$ can be extended uniquely to a bounded linear operator from $\mathcal{H}^1_\sigma(\Omega)$ into $\mathbb{R}^3 \times \mathbb{R}^3$, denoted still by $[\xi(\cdot), \omega(\cdot)]$. Let us also extend K to $\mathcal{H}^1_\sigma(\Omega)$ by defining

$$Ku = [u, m^{1/2}\xi(u), \mathcal{I}^{1/2}\omega(u)] \quad \text{for all } u \in \mathcal{H}^1_\sigma(\Omega).$$

Since $K : \mathcal{H}^1_\sigma(\Omega) \to \mathbb{Y}_0 = L^2(\Omega) \times \mathbb{R}^3 \times \mathbb{R}^3$ is bounded and $\mathbb{Y}$ is the closure of $K(\mathbf{\Phi}_\sigma)$ in $\mathbb{Y}_0$, it follows that $K\left[\mathcal{H}^1_\sigma(\Omega)\right] \subset \mathbb{Y}$.

Lemma 4.2.

$$\mathcal{H}^1_\sigma(\Omega) = \Big\{ u \in W^{1,2}_{loc}(\overline{\Omega}) \cap L^6(\Omega) : D(u) \in L^2(\Omega),\ \nabla \cdot u = 0 \ \mathit{in}\ \Omega, \\ u|_\Sigma = \xi + \omega \times x \ \mathit{for\ some}\ \xi, \omega \in \mathbb{R}^3 \Big\}.$$

Proof. For the proof, refer to the proof of [4, Lemma 4.11]. $\square$

Now let A and B be linear operators with common domain

$$\mathcal{D} := \mathcal{H}^1_\sigma(\Omega) \cap W^{2,2}(\Omega) \subset \mathbb{X} \subset L^2(\Omega)$$

with values in $\mathbb{Y}$, defined by

$$Av := \mathcal{P}[\Delta v, \mathcal{J}_2 v, \mathcal{J}_4 v] \quad \text{and} \quad Bv := Kv \quad \text{for } v \in \mathcal{D},$$

where the orthogonal projection $\mathcal{P}$ is defined in Section 3. Then, for $u(t) \in \mathcal{D} = \mathcal{H}^1_\sigma(\Omega) \cap W^{2,2}(\Omega)$, the problem (5)–(6) can be rewritten in the abstract form

$$\begin{aligned} \partial_t[Bu] &= Au, \\ Bu(0) &= Bu_0, \end{aligned} \tag{16}$$

where $u_0(x) = \xi_0 + \omega_0 \times x$ for $x \in \Sigma$.

We shall show that Sauer's theory of B-evolutions is applicable.

Lemma 4.3. *For all $u, v \in \mathcal{H}^1_\sigma(\Omega)$, we have*

$$2\langle D(u), D(v)\rangle_{L^2} = \langle \nabla u, \nabla v\rangle_{L^2} + 4|\mathcal{B}|\omega(u)\cdot\omega(v)$$

Proof. By density, we may assume that $u, v \in \mathbf{\Phi}_\sigma$. Then by a direct calculation,

$$\begin{aligned} 2\langle D(u), D(v)\rangle_{L^2} &= \langle \nabla u, \nabla v\rangle_{L^2} + 2\int_\Omega \partial_i(u_j\partial_j v_i)dx \\ &= \langle \nabla u, \nabla v\rangle_{L^2} + 2\int_\Sigma (u_j\partial_j v_i)n_i ds. \end{aligned}$$

But since $u = u_\Sigma = \xi(u) + \omega(u) \times x$ and $v = v_\Sigma = \xi(v) + \omega(v) \times x$ near $\mathcal{B}$, we have

$$\int_\Sigma (u_j\partial_j v_i)n_i ds = -\int_{\mathcal{B}} \partial_i(u_\Sigma)_j\partial_j(v_\Sigma)_i dx = 2|\mathcal{B}|\omega(u)\cdot\omega(v).$$

This completes the proof of Lemma 4.3. □

Lemma 4.4. *For all $u \in \mathcal{H}^1_\sigma(\Omega) \cap W^{2,2}(\Omega)$ and $v \in \mathcal{H}^1_\sigma(\Omega)$, we have*

$$\langle -Au, Bv\rangle_{\mathbb{Y}} = \langle D(u), D(v)\rangle_{L^2(\Omega)}.$$

Proof. Since $B = K : \mathcal{H}^1_\sigma(\Omega) \to \mathbb{Y}$, it follows that

$$\begin{aligned} \langle -Au, Bv\rangle_{\mathbb{Y}} &= \langle -\mathcal{P}[\Delta u, \mathcal{J}_2 u, \mathcal{J}_4 u], Kv\rangle_{\mathbb{Y}_0} \\ &= \langle -[\Delta u, \mathcal{J}_2 u, \mathcal{J}_4 u], [v, m^{1/2}\xi(v), \mathcal{I}^{1/2}\omega(v)]\rangle_{\mathbb{Y}_0} \\ &= -\int_\Omega \Delta u\cdot v + \xi(v)\cdot\int_\Sigma D(u)\cdot n + \omega(v)\cdot\int_\Sigma x\times(D(u)\cdot n) \\ &= -\int_\Omega \nabla\cdot D(u)\cdot v + \int_\Sigma (\xi(v) + \omega(v)\times x)\cdot(D(u)\cdot n). \end{aligned}$$

By the divergence theorem, we thus have

$$\begin{aligned} \langle -Au, Bv\rangle_{\mathbb{Y}} = \langle D(u), D(v)\rangle_{L^2(\Omega)} &- \int_\Sigma v\cdot(D(u)\cdot n) \\ &+ \int_\Sigma (\xi(v) + \omega(v)\times x)\cdot(D(u)\cdot n) \\ = \langle D(u), D(v)\rangle_{L^2(\Omega)}&. \end{aligned}$$

The proof of Lemma 4.4 is completed. □

Lemma 4.5. *The range $\mathcal{R}(B)$ of B is dense in $\mathbb{Y}$, B has a bounded inverse, and A is a closed operator.*

Proof. Since $K[\mathbf{\Phi}_\sigma] \subset K[\mathcal{H}^1_\sigma(\Omega)] \subset \mathcal{R}(B)$ and $K[\mathbf{\Phi}_\sigma]$ is dense in $\mathbb{Y}$, it follows immediately that $\mathcal{R}(B)$ is dense in $\mathbb{Y}$. Moreover, B is obviously injective and satisfies

$$\|Bu\|_{\mathbb{Y}} = \|u\|_{L^2} + m^{1/2}|\xi(u)| + \mathcal{I}^{1/2}|\omega(u)| \geq \|u\|_{L^2}$$

for all $u \in \mathcal{H}^1_\sigma(\Omega) \cap W^{2,2}(\Omega)$. Hence, B has a bounded inverse. To show that A is closed, it suffices to show that

$$\|u\|_{W^{1,2}(\Omega)} \leq C\left(\|Au\|_{\mathbb{Y}} + \|u\|_{L^2(\Omega)}\right) \tag{17}$$

for all $u \in \mathcal{H}^1_\sigma(\Omega) \cap W^{2,2}(\Omega)$. Let $u \in \mathcal{H}^1_\sigma(\Omega) \cap W^{2,2}(\Omega)$ be given. Then by Lemmas 4.3 and 4.4, we have

$$\|\nabla u\|^2_{L^2(\Omega)} \leq \langle -Au, Bu\rangle_{\mathbb{Y}} \leq \|Au\|_{\mathbb{Y}}\|Bu\|_{\mathbb{Y}}.$$

Since B maps $\mathcal{H}^1_\sigma(\Omega)$ into $\mathbb{Y}$ boundedly, we also have

$$\|\nabla u\|^2_{L^2(\Omega)} \leq C\|Au\|_{\mathbb{Y}}\|u\|_{W^{1,2}(\Omega)}$$

and so

$$\|u\|_{W^{1,2}(\Omega)} \leq C\left(\|Au\|_{\mathbb{Y}} + \|u\|_{L^2(\Omega)}\right).$$

Moreover, by Theorem 3.1, there exists $p \in D^{1,2}(\Omega)$ such that

$$-[\Delta u, \mathcal{J}_2 u, \mathcal{J}_4 u] = [w, \zeta, \theta] + [\nabla p, \mathcal{J}_1 p, \mathcal{J}_3 p],$$

where $[w, \zeta, \theta] = -Au \in \mathbb{Y}$. This implies in particular that $(u,p) \in W^{2,2}(\Omega) \times W^{1,2}(\Omega)$ is a strong solution of the exterior Stokes problem:

$$-\Delta u - \nabla p = w, \quad \nabla\cdot u = 0 \quad \text{in } \Omega, \quad u = u_\Sigma \quad \text{on } \Sigma,$$

where $u_\Sigma(x) = \xi(u) + \omega(u) \times x$. Hence using the standard estimate (see [3] e.g.), we obtain

$$\|\nabla^2 u\|_{L^2(\Omega)} \leq C\left(\|w\|_{L^2(\Omega)} + \|u_\Sigma\|_{W^{3/2,2}(\Sigma)}\right).$$

Finally, by (15), we have

$$\|u_\Sigma\|_{W^{3/2,2}(\Sigma)} \leq C\left(|\xi(u)| + |\omega(u)|\right) \leq C\|u\|_{W^{1,2}(\Omega)},$$

where C depends on $|\mathcal{B}|$. which proves (17). The proof of Lemma 4.5 is completed. □

Lemma 4.6. *The operator AB^{-1} generates an analytic semigroup $\{E(t)\}_{t\geq 0}$ on $\mathbb{Y}$.*

Proof. By Lemma 4.5, AB^{-1} is a closed linear operator in $\mathbb{Y}$ with dense domain. Hence to prove the lemma, it remains to obtain the resolvent estimates for AB^{-1}. Let $\lambda \in \mathbb{C}$ be given. Then for $F \in \mathcal{R}(B)$, let us consider

$$Q = \langle(\lambda I_{\mathbb{Y}} - AB^{-1})F, F\rangle_{\mathbb{Y}} = \lambda\|F\|^2_{\mathbb{Y}} - \langle AB^{-1}F, F\rangle_{\mathbb{Y}}.$$

Setting $u = B^{-1}F \in \mathcal{H}^1_\sigma(\Omega) \cap W^{2,2}(\Omega)$ and using Lemma 4.4, we have

$$\langle -AB^{-1}F, F\rangle_{\mathbb{Y}} = \langle -Au, Bu\rangle_{\mathbb{Y}} = \|D(u)\|^2_{L^2(\Omega)} \geq 0.$$

Hence, following the proof of Proposition 7.3 in [12], we can complete the proof of Lemma 4.6. □

By combining lemmas in this subsection and Theorem 4.1, we conclude that the problem (16) has a unique analytic solution.

Theorem 4.7. *There exist a unique B-evolution $S(t)$, of which associated semigroup $E(t)$ is analytic. Furthermore, there is a unique solution to the problem* (16).

5. Fractional power of operators

In this section, for our future analysis on the interaction of Navier-Stokes fluid and the rigid body, we study the B-evolution $S(t)$ more explicitly and fractional powers of the operators A, B for future analysis of Navier-Stokes type system (16). For that, we again consider the linearized system of (16)

$$\partial_t Bv = Av, \qquad Bv(0) = y \tag{18}$$

for $y \in \mathbb{Y}$.

As in [12] and [7], since $E(t)$ is analytic, the operator pair $\{-A, B\}$ generates an analytic B-evolution $S(t)$ which has a contour integral representation

$$S(t)y = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}(\lambda B - A)^{-1} y\, d\lambda,$$

for $y \in \mathbb{Y}$, where Γ is any piecewise smooth curve going from $\infty e^{-i(\frac{\pi}{2}+\delta')}$ to $\infty e^{i(\frac{\pi}{2}+\delta')}$ for some $\delta' \in (0,\delta)$ in $\Lambda_{\frac{\pi}{2}+\delta} \equiv \{\lambda \in \mathbb{C} : |\arg\lambda| < \frac{\pi}{2}+\delta\} \setminus \{0\}$ for some $0 < \delta \leq \frac{\pi}{2}$, which is contained in the resolvent set $\rho(-AB^{-1})$. Refer also to Pazy [10]. Furthermore, $S(t)$ is uniformly bounded. The system (18) with initial data y is uniquely solvable for any $y \in \mathbb{Y}$, and the solution is represented as $v(t) = S(t)y$.

Like [6], for $0 < \alpha \leq 1$, $B^{-\alpha}$ is defined by

$$B^{-\alpha}y := \frac{1}{\Gamma(\alpha)}\lim_{n\to\infty}\int_0^\infty ne^{-ns}S(s)(ns)^{\alpha-1}y\,ds \qquad \text{for all } y \in \mathbb{Y},$$

where $\Gamma(\alpha)$ is the Gamma function. Let $\Lambda_\delta \subset \rho(-AB^{-1})$ be the union of the open sector $\{\lambda \in \mathbb{C} : 0 < \delta < |\arg\lambda| \leq \pi\}$ and a neighbourhood of zero, where $\rho(-A)$ is the resolvent set of $-A$ such that $\mathbb{R}_+ \subset \Lambda_\delta \subset \rho(-AB^{-1})$.

Define the fractional power of $-A$ in the way that for $0 < \alpha < 1$,

$$(-A)^{-\alpha}y := -\frac{1}{2\pi i}B^{-\alpha}\int_\Gamma \lambda^{-\alpha}B(\lambda B + A)^{-1}y\,d\lambda,$$

for all $y \in \mathbb{Y}$, where Γ is a piecewise smooth path in Λ_δ going from $\infty e^{-i\theta}$ to $\infty e^{i\theta}$ for some $\delta < \theta < \pi$, avoiding the negative real axis and the origin. Let a and θ be positive real numbers. Let Γ be the path consisting of the half-lines going from $\infty e^{-i\theta}$ to a and from a to $\infty e^{i\theta}$. Then,

$$(-A)^{-\alpha} y = -\frac{1}{2\pi i} B^{-\alpha} \Big[\int_0^\infty (se^{i\delta}+a)^{-\alpha} B\big((se^{i\delta}+a)B+A\big)^{-1} e^{i\delta} y ds - \int_0^\infty (se^{-i\delta}+a)^{-\alpha} B\big((se^{-i\delta}+a)B+A\big)^{-1} e^{-i\delta} y ds \Big].$$

Notice that symbolically,

$$\begin{aligned} &(se^{i\theta})^{-\alpha}\big(se^{i\theta}B+A\big)^{-1}e^{i\theta} - (se^{-i\theta})^{-\alpha}\big(se^{-i\theta}B+A\big)^{-1}e^{-i\theta} \\ &= s^{-\alpha}\big[sB(e^{-i\theta\alpha}-e^{i\theta\alpha})+A(e^{-i\theta\alpha+i\theta}-e^{i\theta\alpha-i\theta})\big]\big(se^{i\theta}B+A\big)^{-1}\big(se^{-i\theta}B+A\big)^{-1} \\ &\to s^{-\alpha}2i\sin(\pi\alpha)(-sB+A)\big(-sB+A\big)^{-1}\big(-sB+A\big)^{-1} \\ &= s^{-\alpha}2i\sin(\pi\alpha)\big(-sB+A\big)^{-1} \end{aligned}$$

as $\theta \to \pi$. Hence, taking $a \searrow 0$ and $\theta \nearrow \pi$, we obtain

$$(-A)^{-\alpha} y = \frac{\sin(\pi\alpha)}{\pi} B^{-\alpha} \int_0^\infty \lambda^{-\alpha} B(\lambda B - A)^{-1} y d\lambda,$$

or

$$A^{-\alpha} y = (-1)^\alpha \frac{\sin(\pi\alpha)}{\pi} B^{-\alpha} \int_0^\infty \lambda^{-\alpha} B(\lambda B - A)^{-1} y d\lambda,$$

Since $B(\lambda B - A)^{-1} y = \int_0^\infty e^{-\lambda t} BS(t) y dt$ for all $y \in \mathbb{Y}$ (refer to [12]), we obtain

$$\begin{aligned} A^{-\alpha} y &= (-1)^\alpha \frac{\sin(\pi\alpha)}{\pi} B^{-\alpha} \int_0^\infty \lambda^{-\alpha} \Big(\int_0^\infty e^{-\lambda t} BS(t) y dt \Big) d\lambda \\ &= (-1)^\alpha \frac{\sin(\pi\alpha)}{\pi} B^{-\alpha} \int_0^\infty BS(t) y \Big(\int_0^\infty e^{-\lambda t} \lambda^{-\alpha} d\lambda \Big) dt \\ &= (-1)^\alpha \frac{\sin(\pi\alpha)}{\pi} B^{-\alpha} \int_0^\infty BS(t) y \Big(\int_0^\infty e^{-\lambda} \lambda^{-\alpha} d\lambda \Big) t^{\alpha-1} dt. \end{aligned}$$

Owing to Euler's reflection formula

$$\Gamma(1-\alpha) \equiv \int_0^\infty e^{-s} s^{-\alpha} ds = \frac{\pi}{\sin(\pi\alpha)} \frac{1}{\Gamma(\alpha)},$$

we have

$$A^{-\alpha} y = (-1)^\alpha \frac{1}{\Gamma(\alpha)} B^{-\alpha} \int_0^\infty t^{\alpha-1} BS(t) y dt.$$

Here, $\Gamma(\alpha)$ is the gamma function.

Therefore, we have the following result.

Proposition 5.1. *The fractional powers* $A^{-\alpha}$ $(0<\alpha\leq 1)$ *is defined by*

$$\begin{aligned} A^{-\alpha}y &= (-1)^{\alpha+1}\frac{1}{2\pi i}B^{-\alpha}\int_\Gamma \lambda^{-\alpha}B(\lambda B+A)^{-1}yd\lambda \\ &= (-1)^{\alpha}\frac{\sin(\pi\alpha)}{\pi}B^{-\alpha}\int_0^\infty \lambda^{-\alpha}B(\lambda B-A)^{-1}yd\lambda \\ &= (-1)^{\alpha}\frac{1}{\Gamma(\alpha)}B^{-\alpha}\int_0^\infty t^{\alpha-1}BS(t)ydt \end{aligned}$$

for all $y\in\mathbb{Y}$.

The partial fraction of A, A^α is defined by $(A^{-\alpha})^{-1}$. The closedness of the pairs $\{A^\alpha, B^\alpha\}$ and $\{A^\alpha B^{-\alpha}B, B\}$ can be proved in the same way in [7]. The following property can be shown in the same way in [7].

Proposition 5.2. *For* $u\in\mathcal{D}$,

$$\|A^{1/2}B^{-1/2}Bu\|^2 = \langle -Au, Bu\rangle = \frac{1}{2}\|\nabla u\|_{L^2} + 2|\mathcal{B}|\,|\omega(u)|^2.$$

Theorem 5.3. *For all* $y\in\mathbb{Y}$,

$$\|BS(t)y\|\leq c\|y\|$$

Proof. Setting $v=Sy$ for $y\in\mathbb{Y}$, we have

$$\begin{gathered} \langle\partial_t Bv, Bv\rangle = \langle Av, Bv\rangle, \\ \partial_t\|Bv\|^2 = -\|A^{1/2}B^{-1/2}Bv\|^2\leq 0, \end{gathered}$$

hence, we have $\|Bv\|\leq c\|y\|$. □

Proposition 5.4.

$$\|A^{1/2}B^{-1/2}BS(t)y\|\leq ct^{-1/2}\|y\|.$$

Proof. Since

$$\|A^{1/2}B^{-1/2}Bv\|^2 = \langle -Av, Bv\rangle_{\mathbb{Y}_0}\leq \|Av\|\,\|Bv\|,$$

we have

$$\begin{gathered} \langle\partial_t Bv, -Av\rangle - \langle Av, -Av\rangle = 0, \\ \langle\partial_t(-A)^{1/2}B^{-1/2}Bv, (-A)^{1/2}B^{-1/2}Bv\rangle + \langle Au, Au\rangle = 0, \\ \partial_t\|A^{1/2}B^{-1/2}Bv\|^2 + 2\|Bv\|^{-2}\,\|A^{1/2}B^{1-2/}Bv\|^4\leq 0, \end{gathered}$$

where $v=S(t)y$. Setting $Y=\|A^{1/2}B^{-1/2}Bv\|^2$, we have

$$\begin{gathered} \partial_t Y + c\|Bv\|^{-2}Y^2\leq 0, \\ \frac{dY}{Y^2}\leq -c\|Bv\|^{-2}dt, \\ -Y^{-1}(t)+Y^{-1}(0)\leq -c\|Bv\|^{-2}t, \\ \|A^{1/2}B^{-1/2}Bv(t)\|\leq c\|Bv\|t^{-1/2}\leq ct^{-1/2}\|y\|. \end{gathered}$$ □

Proposition 5.5.

$$\|AB^{-1}BS(t)y\| \le ct^{-1}\|y\|.$$

Proof. Since

$$\begin{aligned}\|Av\|^2 &= \langle A^{3/2}B^{-3/2}Bv, A^{1/2}B^{-1/2}Bv\rangle_{\mathbb{Y}_0}\\ &\le \|A^{3/2}B^{-3/2}Bv\|\,\|A^{1/2}B^{-1/2}Bv\|\\ &\le ct^{-1/2}\|y\|\,\|A^{3/2}B^{-3/2}Bv\|,\end{aligned}$$

we have

$$\langle \partial_t Bv, A^2B^{-2}Bv\rangle + \langle -Av, A^2B^{-2}Bv\rangle = 0,$$
$$2\langle \partial_t Av, Av\rangle + \langle (-A)^{3/2}B^{-3/2}Bu, (-A)^{3/2}B^{-3/2}Bu\rangle = 0,$$
$$\partial_t\|Av\|^2 + ct\|y\|^{-2}\,\|Av\|^4 \le 0,$$

where $v = S(t)y$. In the same way as above, we can complete the proof. □

Theorem 5.6. *Suppose that for* $0 < \alpha \le 1$,

$$\|A^\alpha B^{-\alpha}BS(t)y\| \le ct^{-\alpha}\|y\|.$$

Then we have that

$$\|A^{\alpha/2}B^{-\alpha/2}BS(t)y\| \le ct^{-\alpha/2}\|y\|.$$

Proof.

$$\|A^{\alpha/2}B^{-\alpha/2}Bv\|^2 = \langle (-A)^\alpha B^{-\alpha}Bv, Bv\rangle_{\mathbb{Y}_0} \le \|A^\alpha B^{-\alpha}Bv\|\,\|Bv\| \le ct^{-\alpha}\|y\|^2.$$

□

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