

MULTIPLIERS OF DIRICHLET-TYPE SUBSPACES OF BLOCH SPACE

SONGXIAO LI, ZENGJIAN LOU, AND CONGHUI SHEN

ABSTRACT. Let $M(X, Y)$ denote the space of multipliers from X to Y , where X and Y are analytic function spaces. As we known, for Dirichlet-type spaces \mathcal{D}_α^p , $M(\mathcal{D}_{p-1}^p, \mathcal{D}_{q-1}^q) = \{0\}$, if $p \neq q$, $0 < p, q < \infty$. If $0 < p, q < \infty$, $p \neq q$, $0 < s < 1$ such that $p + s, q + s > 1$, then $M(\mathcal{D}_{p-2+s}^p, \mathcal{D}_{q-2+s}^q) = \{0\}$. However, $X \cap \mathcal{D}_{p-1}^p \subseteq X \cap \mathcal{D}_{q-1}^q$ and $X \cap \mathcal{D}_{p-2+s}^p \subseteq X \cap \mathcal{D}_{q-2+s}^q$ whenever X is a subspace of the Bloch space \mathcal{B} and $0 < p \leq q < \infty$. This says that the set of multipliers $M(X \cap \mathcal{D}_{p-2+s}^p, X \cap \mathcal{D}_{q-2+s}^q)$ is nontrivial. In this paper, we study the multipliers $M(X \cap \mathcal{D}_{p-2+s}^p, X \cap \mathcal{D}_{q-2+s}^q)$ for distinct classical subspaces X of the Bloch space \mathcal{B} , where $X = \mathcal{B}, BMOA$ or \mathcal{H}^∞ .

1. Introduction

Let \mathbb{D} denote the unit disk of the complex plane \mathbb{C} and $\partial\mathbb{D}$ be the boundary of \mathbb{D} , the unit circle. Denote by $\mathcal{H}(\mathbb{D})$ the space of all analytic functions in \mathbb{D} . The Bloch space \mathcal{B} , consists of those $f \in \mathcal{H}(\mathbb{D})$ for which

$$\|f\|_{\mathcal{B}} = |f(0)| + \sup_{z \in \mathbb{D}} (1 - |z|^2) |f'(z)| < \infty.$$

Let $f \in \mathcal{H}(\mathbb{D})$. For $0 < p < \infty$, $0 < r < 1$, set

$$M_p^p(r, f) = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta$$

and

$$M_\infty(r, f) = \sup_{|z|=r} |f(z)|.$$

The Hardy space \mathcal{H}^p ($0 < p \leq \infty$) is defined as the space of $f \in \mathcal{H}(\mathbb{D})$ such that

$$\|f\|_{\mathcal{H}^p} = \sup_{0 < r < 1} M_p(r, f) < \infty.$$

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For the theory about the Hardy space \mathcal{H}^p , we refer the readers to [6]. The *BMOA* space is the set of those $f \in \mathcal{H}^1$ whose boundary values have bounded mean oscillation on the unit circle $\partial\mathbb{D}$ [10]. It is well known that *BMOA* is contained in the Bloch space \mathcal{B} continuously.

The weighted Dirichlet-type space \mathcal{D}_α^p ($0 < p < \infty, \alpha > -1$) is the class of all $f \in \mathcal{H}(\mathbb{D})$ such that

$$\|f\|_{\mathcal{D}_\alpha^p}^p = |f(0)|^p + \int_{\mathbb{D}} |f'(z)|^p dA_\alpha(z) < \infty,$$

here $dA_\alpha(z) = (\alpha + 1)(1 - |z|^2)^\alpha dA(z)$ and $dA(z) = \frac{1}{\pi} dx dy$ is the normalized Lebesgue area measure. It is well known that when $p < \alpha + 1$, $\mathcal{D}_\alpha^p = A_{\alpha-p}^p$, the Bergman space [7]. If $p > \alpha + 2$, then $\mathcal{D}_\alpha^p \subseteq \mathcal{H}^\infty$. Therefore, when $\alpha + 1 \leq p \leq \alpha + 2$, \mathcal{D}_α^p is a proper Dirichlet-type space. The spaces \mathcal{D}_{p-1}^p are closely related with Hardy spaces. In fact, $\mathcal{D}_1^2 = \mathcal{H}^2$. Notice that when $0 < p \leq 2$, $\mathcal{D}_{p-1}^p \subseteq \mathcal{H}^p$ [7]. When $2 \leq p < \infty$, $\mathcal{H}^p \subseteq \mathcal{D}_{p-1}^p$ [14].

For $g \in \mathcal{H}(\mathbb{D})$, the multiplication operator M_g is defined by

$$M_g f(z) = g(z)f(z), \quad z \in \mathbb{D}, \quad f \in \mathcal{H}(\mathbb{D}).$$

Let X, Y be the norm spaces of analytic functions in \mathbb{D} . We denote by $M(X, Y)$ the space of multipliers from X to Y , in other words,

$$M(X, Y) = \{g \in \mathcal{H}(\mathbb{D}) : fg \in Y, \forall f \in X\}.$$

For convenience, we write $M(X) := M(X, X)$. Denote the norm of the multiplication operator M_g by $\|M_g\|$. From [2, 3], we see that

$$(1) \quad M(\mathcal{B}) = \mathcal{H}^\infty \cap \mathcal{B}_{\log}.$$

Here \mathcal{B}_{\log} is the logarithmic Bloch space, consists of those $f \in \mathcal{H}(\mathbb{D})$ for which

$$\|f\|_{\mathcal{B}_{\log}} = |f(0)| + \sup_{z \in \mathbb{D}} (1 - |z|^2) |f'(z)| \left(\log \frac{2}{1 - |z|^2} \right) < \infty.$$

In [15], we have that

$$(2) \quad M(\mathcal{BMOA}) = \mathcal{BMOA}_{\log} \cap \mathcal{H}^\infty,$$

where \mathcal{BMOA}_{\log} is the space of those functions $f \in \mathcal{H}^1$ such that the positive Borel measure $(1 - |z|^2) |f'(z)|^2 dA(z)$ is a 2-logarithmic Carleson measure. In other words, $f \in \mathcal{BMOA}_{\log}$ if and only if $f \in \mathcal{H}^1$ such that

$$\sup_{a \in \mathbb{D}} \left(\log \frac{2}{1 - |a|} \right)^2 \int_{\mathbb{D}} |f'(z)|^2 (1 - |\varphi_a(z)|^2) dA(z) < \infty,$$

where φ_a is the disk automorphism which interchange the origin and a , that is

$$(3) \quad \varphi_a(z) = \frac{a - z}{1 - \bar{a}z}, \quad z \in \mathbb{D}.$$

The multipliers of Dirichlet-type space \mathcal{D}_α^p have been studied in [8, 9, 11, 12]. In [8], the authors proved that for $1 < p \leq q < \infty$, a function $g \in \mathcal{H}(\mathbb{D})$ belongs to $M(\mathcal{D}_{p-2}^p, \mathcal{D}_{q-2}^q)$ if and only if $g \in \mathcal{H}^\infty$ and the positive Borel measure μ

defined by $d\mu(z) = |g'(z)|^q(1 - |z|^2)^{q-2}dA(z)$ is a q -Carleson measure for \mathcal{D}_{q-2}^q . If $1 < q < p < \infty$, then $M(\mathcal{D}_{p-2}^p, \mathcal{D}_{q-2}^q) = \{0\}$.

It is standard that if $0 < p, q < \infty$ and $p \neq q$, then we have

$$M(\mathcal{D}_{p-1}^p, \mathcal{D}_{q-1}^q) = \{0\}.$$

Let X be a non-zero subspace of the Bloch space \mathcal{B} . The space $X \cap \mathcal{D}_\alpha^p$ is equipped with the norm

$$\|f\|_{X \cap \mathcal{D}_\alpha^p} = \|f\|_X + \|f\|_{\mathcal{D}_\alpha^p}.$$

Lemma 1 in [5] says that if $0 < p \leq q < \infty$, then $X \cap \mathcal{D}_{p-1}^p \subseteq X \cap \mathcal{D}_{q-1}^q$. It follows that the set of multipliers $M(X \cap \mathcal{D}_{p-1}^p, X \cap \mathcal{D}_{q-1}^q)$ is nontrivial.

By Corollary 1 in [12] and Theorem 2 in [9], for all $p \neq q$ and $0 < s < 1$,

$$M(\mathcal{D}_{p-2+s}^p, \mathcal{D}_{q-2+s}^q) = \{0\}.$$

But when $0 < p \leq q < \infty$, if $f \in X \cap \mathcal{D}_{p-2+s}^p$, then

$$\begin{aligned} \int_{\mathbb{D}} |f'(z)|^q(1 - |z|^2)^{q-2+s}dA(z) &\leq \|f\|_{\mathcal{B}}^{q-p} \int_{\mathbb{D}} |f'(z)|^p(1 - |z|^2)^{p-2+s}dA(z) \\ &\leq \|f\|_{\mathcal{B}}^{q-p} \|f\|_{\mathcal{D}_{p-2+s}^p}^p \\ &\leq C \|f\|_X^{q-p} \|f\|_{\mathcal{D}_{p-2+s}^p}^p \\ &\leq C \|f\|_{X \cap \mathcal{D}_{p-2+s}^p}^q. \end{aligned}$$

Hence $f \in X \cap \mathcal{D}_{q-2+s}^q$ and $\|f\|_{X \cap \mathcal{D}_{q-2+s}^q} \leq C \|f\|_{X \cap \mathcal{D}_{p-2+s}^p}$. In other words, $X \cap \mathcal{D}_{p-2+s}^p \subseteq X \cap \mathcal{D}_{q-2+s}^q$. So the set of multipliers $M(X \cap \mathcal{D}_{p-2+s}^p, X \cap \mathcal{D}_{q-2+s}^q)$ is also nontrivial.

From [5], we see that if $q > 1$ and $0 < p \leq q < \infty$, then

$$M(\mathcal{B} \cap \mathcal{D}_{p-1}^p, \mathcal{B} \cap \mathcal{D}_{q-1}^q) = M(\mathcal{B})$$

and

$$M(BMOA \cap \mathcal{D}_{p-1}^p, BMOA \cap \mathcal{D}_{q-1}^q) = M(BMOA).$$

If $0 < p \leq q < \infty$, then

$$M(\mathcal{H}^\infty \cap \mathcal{D}_{p-1}^p, \mathcal{H}^\infty \cap \mathcal{D}_{q-1}^q) = \mathcal{H}^\infty \cap \mathcal{D}_{q-1}^q.$$

Motivated by [8] and [5], it is natural to ask what is the set of multipliers $M(X \cap \mathcal{D}_{p-2+s}^p, X \cap \mathcal{D}_{q-2+s}^q)$ when $0 < s < 1$. In this paper, we characterize the multipliers $M(X \cap \mathcal{D}_{p-2+s}^p, X \cap \mathcal{D}_{q-2+s}^q)$ when $0 < s < 1$, $X = \mathcal{B}$, $X = BMOA$ or $X = \mathcal{H}^\infty$, respectively. Our main results are stated as follows.

Theorem 1.1. *Suppose that $g \in \mathcal{H}(\mathbb{D})$, $0 < p \leq q < \infty$, $0 < s < 1$ satisfying $p + s > 1$. Define the positive Borel measure μ by $d\mu(z) = |g'(z)|^q(1 - |z|^2)^{q-2+s}dA(z)$, then*

- (i) $g \in M(\mathcal{B} \cap \mathcal{D}_{p-2+s}^p, \mathcal{B} \cap \mathcal{D}_{q-2+s}^q)$ if and only if $g \in M(\mathcal{B})$ and μ is a q -Carleson measure for $\mathcal{B} \cap \mathcal{D}_{p-2+s}^p$.

- (ii) $g \in M(BMOA \cap \mathcal{D}_{p-2+s}^p, BMOA \cap \mathcal{D}_{q-2+s}^q)$ if and only if $g \in M(BMOA)$ and μ is a q -Carleson measure for $BMOA \cap \mathcal{D}_{p-2+s}^p$.
- (iii) $M(\mathcal{H}^\infty \cap \mathcal{D}_{p-2+s}^p, \mathcal{H}^\infty \cap \mathcal{D}_{q-2+s}^q) = \mathcal{H}^\infty \cap \mathcal{D}_{q-2+s}^q$.

Theorem 1.2. *Suppose $0 < q < p < \infty$, $0 < s < 1$ with $q + s > 1$. Then*

- (i) $M(\mathcal{B} \cap \mathcal{D}_{p-2+s}^p, \mathcal{B} \cap \mathcal{D}_{q-2+s}^q) = \{0\}$.
- (ii) $M(BMOA \cap \mathcal{D}_{p-2+s}^p, BMOA \cap \mathcal{D}_{q-2+s}^q) = \{0\}$.
- (iii) $M(\mathcal{H}^\infty \cap \mathcal{D}_{p-2+s}^p, \mathcal{H}^\infty \cap \mathcal{D}_{q-2+s}^q) = \{0\}$.

Throughout this paper, C denotes a positive constant depending only on indexes p, q, s, \dots , it is not necessary to be the same from one line to another. Let f and g be two positive functions. For convenience, we write $f \preceq g$, if $f \preceq Cg$ holds, where C is a positive constant independent of f and g . If $f \preceq g$ and $g \preceq f$, then we say $f \asymp g$.

2. Preliminary

In this section, we state some definitions and lemmas which will be used in the paper. Let I be an arc of $\partial\mathbb{D}$. Denote the normalized Lebesgue measure of I by $|I|$, that is, $|I| = \frac{1}{2\pi} \int_I |d\xi|$. For an arc $I \subseteq \partial\mathbb{D}$, the Carleson square based on I is defined by

$$S(I) := \left\{ z \in \mathbb{D} : 1 - |I| \leq |z| < 1, \frac{z}{|z|} \in I \right\}.$$

If $I = \partial\mathbb{D}$, then we set $S(I) = \mathbb{D}$. Let μ be a positive Borel measure on \mathbb{D} . For $0 \leq \alpha < \infty$, $0 < s < \infty$, we say that μ is an α -logarithmic s -Carleson measure if there exists a constant $C > 0$ such that for all arcs $I \subseteq \partial\mathbb{D}$,

$$\mu(S(I)) \leq C \frac{|I|^s}{(\log \frac{2}{|I|})^\alpha}.$$

If $\alpha = 0$, then μ is called an s -Carleson measure. If $\alpha = 0$ and $s = 1$, then μ is said to be a Carleson measure. Recall that an $f \in \mathcal{H}^1$ belongs to the space $BMOA$ if and only if the positive Borel measure $|f'(z)|^2(1 - |z|^2)dA(z)$ is a Carleson measure.

Let $(X, \|\cdot\|_X)$ be a normed space of analytic functions. Then a positive Borel measure μ on \mathbb{D} is said to be an s -Carleson measure for X , if there exists a constant $C > 0$ such that for all $f \in X$,

$$\int_{\mathbb{D}} |f(z)|^s d\mu(z) \leq C \|f\|_X^s.$$

The following lemma can be found in Theorem 2 of [17], which plays an important role in the proofs of theorems.

Lemma 2.1. *Suppose that $0 \leq \alpha < \infty$ and $0 < s < \infty$. Then a positive Borel measure μ on \mathbb{D} is an α -logarithmic s -Carleson measure if and only if*

$$\sup_{a \in \mathbb{D}} \left(\log \frac{2}{1 - |a|} \right)^\alpha \int_{\mathbb{D}} \left(\frac{1 - |a|^2}{|1 - \bar{a}z|^2} \right)^s d\mu(z) < \infty.$$

We will make use of the lacunary power series (also called power series with Hadamard gaps) of a function $f \in \mathcal{H}(\mathbb{D})$, that is, f is of the form

$$f(z) = \sum_{k=0}^{\infty} a_k z^{n_k}, \quad z \in \mathbb{D},$$

with $\frac{n_{k+1}}{n_k} \geq \lambda > 1$ for all k . Several known results on lacunary power series will be used in this paper. We put them together in the following statement, see [1, 4, 5, 13, 19].

Lemma 2.2. *Suppose that $0 < p < \infty$, $\alpha > -1$. $f \in \mathcal{H}(\mathbb{D})$ which is given by a lacunary power series, $f(z) = \sum_{k=0}^{\infty} a_k z^{n_k}$, $z \in \mathbb{D}$. Then*

(i) $f \in \mathcal{D}_{\alpha}^p$ if and only if $\sum_{k=0}^{\infty} n_k^{p-\alpha-1} |a_k|^p < \infty$, and

$$\|f - f(0)\|_{\mathcal{D}_{\alpha}^p}^p \asymp \sum_{k=0}^{\infty} n_k^{p-\alpha-1} |a_k|^p.$$

(ii) $f \in \mathcal{H}^{\infty}$ if and only if $\sum_{k=0}^{\infty} |a_k| < \infty$, and

$$\|f\|_{\mathcal{H}^{\infty}} \asymp \sum_{k=0}^{\infty} |a_k|.$$

(iii) $f \in \mathcal{B}$ if and only if $\sup_k |a_k| < \infty$, and

$$\|f\|_{\mathcal{B}} \asymp \sup_k |a_k|.$$

The following estimate can be found in [13].

Lemma 2.3. *Suppose that $\beta > -1$, $s > 0$ and $f \in \mathcal{H}(\mathbb{D})$ with $f(z) = \sum_{k=1}^{\infty} a_k z^{n_k}$, $z \in \mathbb{D}$. Then*

$$\sum_{k=1}^{\infty} n_k^{-(\beta+1)} |a_k|^s \asymp \int_0^1 (1-r)^{\beta} |f(re^{i\theta})|^s dr$$

for all $\theta \in \mathbb{R}$.

The following lemma is useful in theory of analytic function spaces and operator theory, see [18].

Lemma 2.4. *Suppose that $z \in \mathbb{D}$, c is real, $t > -1$, and*

$$I_{c,t}(z) = \int_{\mathbb{D}} \frac{(1-|w|^2)^t}{|1-\bar{w}z|^{2+t+c}} dA(w).$$

(i) *If $c < 0$, then as a function of z , $I_{c,t}$ is bounded on \mathbb{D} .*

(ii) *If $c = 0$, then*

$$I_{c,t}(z) \asymp \log \frac{1}{1-|z|^2} \quad \text{as } |z| \rightarrow 1^-.$$

(iii) *If $c > 0$, then*

$$I_{c,t}(z) \asymp \frac{1}{(1-|z|^2)^c} \quad \text{as } |z| \rightarrow 1^-.$$

We will use the following estimate to prove our results, which can be found in [16].

Lemma 2.5. *For $s > -1$, $r, t > 0$ with $0 < r + t - s - 2 < r$, there exists a constant $C > 0$ such that for any $a, b \in \mathbb{D}$,*

$$\int_{\mathbb{D}} \frac{(1 - |z|^2)^s}{|1 - \bar{a}z|^r |1 - \bar{b}z|^t} dA(z) \leq \frac{C}{(1 - |a|^2)^{r+t-s-2}}.$$

3. Proof of main results

Proof of Theorem 1.1. (i) First suppose that $g \in M(\mathcal{B} \cap \mathcal{D}_{p-2+s}^p, \mathcal{B} \cap \mathcal{D}_{q-2+s}^q)$. For any $a \in \mathbb{D}$, let φ_a be defined by (3) and f_a be defined by

$$f_a(z) = \log \frac{1}{1 - \bar{a}z}, \quad z \in \mathbb{D}.$$

A simple computation shows that $\sup_{a \in \mathbb{D}} \|\varphi_a\|_{\mathcal{B}} < \infty$ and $\sup_{a \in \mathbb{D}} \|\varphi_a\|_{\mathcal{D}_{p-2+s}^p} < \infty$. This implies that $\varphi_a \in \mathcal{B} \cap \mathcal{D}_{p-2+s}^p$ and $\sup_{a \in \mathbb{D}} \|\varphi_a\|_{\mathcal{D}_{p-2+s}^p \cap \mathcal{B}} < \infty$. We have $g\varphi_a \in \mathcal{B} \cap \mathcal{D}_{q-2+s}^q$ and

$$\begin{aligned} (1 - |z|^2)|(g\varphi_a)'(z)| &\leq \|g\varphi_a\|_{\mathcal{B}} \\ &\leq \|g\varphi_a\|_{\mathcal{B} \cap \mathcal{D}_{q-2+s}^q} \\ &\leq \|M_g\| \|\varphi_a\|_{\mathcal{B} \cap \mathcal{D}_{p-2+s}^p} \leq C \|M_g\|, \end{aligned}$$

that is,

$$(1 - |z|^2)|g'(z)\varphi_a(z) + g(z)\varphi_a'(z)| \leq C \|M_g\|.$$

Taking $z = a$, using the fact that $\varphi_a(a) = 0$ and $|\varphi_a'(a)| = \frac{1}{1 - |a|^2}$ we get

$$|g(a)| \leq C \|M_g\|,$$

which implies that $g \in \mathcal{H}^\infty$.

It is obvious that $f_a'(z) = \frac{\bar{a}}{1 - \bar{a}z}$ and $\sup_{a \in \mathbb{D}} \|f_a\|_{\mathcal{B}} < \infty$. By Lemma 2.4, there is a constant $C > 0$ independent of a such that

$$\begin{aligned} \int_{\mathbb{D}} |f_a'(z)|^p (1 - |z|^2)^{p-2+s} dA(z) &\leq \int_{\mathbb{D}} \frac{(1 - |z|^2)^{p-2+s}}{|1 - \bar{a}z|^p} dA(z) \\ &= \int_{\mathbb{D}} \frac{(1 - |z|^2)^{p-2+s}}{|1 - \bar{a}z|^{2+p-2+s-s}} dA(z) \\ &\leq C. \end{aligned}$$

This implies that $\sup_{a \in \mathbb{D}} \|f_a\|_{\mathcal{D}_{p-2+s}^p} < \infty$. Hence, we have $f_a \in \mathcal{B} \cap \mathcal{D}_{p-2+s}^p$ and $\sup_{a \in \mathbb{D}} \|f_a\|_{\mathcal{B} \cap \mathcal{D}_{p-2+s}^p} < \infty$. So $gf_a \in \mathcal{B} \cap \mathcal{D}_{q-2+s}^q$ and

$$(4) \quad (1 - |z|^2)|(gf_a)'(z)| \leq \|gf_a\|_{\mathcal{B} \cap \mathcal{D}_{q-2+s}^q} \leq \|M_g\| \|f_a\|_{\mathcal{B} \cap \mathcal{D}_{p-2+s}^p} \leq C \|M_g\|.$$

On the other hand, since $g \in \mathcal{H}^\infty$,

$$(5) \quad (1 - |z|^2)|g(z)f'_a(z)| \leq \|g\|_{\mathcal{H}^\infty} \|f_a\|_{\mathcal{B}} \leq C\|g\|_{\mathcal{H}^\infty}.$$

Combining (4) and (5) we deduce that

$$(1 - |z|^2)|g'(z)f_a(z)| \leq C(\|M_g\| + \|g\|_{\mathcal{H}^\infty}).$$

Taking $z = a$ we obtain

$$(1 - |a|^2)|g'(a)| \log \frac{1}{1 - |a|^2} \leq C,$$

which shows that $g \in \mathcal{B}_{\log}$. From (1) we see that $g \in M(\mathcal{B})$.

We next show that $d\mu(z) = |g'(z)|^q(1 - |z|^2)^{q-2+s}dA(z)$ is a q -Carleson measure for $\mathcal{B} \cap \mathcal{D}_{p-2+s}^p$. Let $f \in \mathcal{B} \cap \mathcal{D}_{p-2+s}^p$. Since $g \in \mathcal{H}^\infty$, we have

$$(6) \quad \begin{aligned} \int_{\mathbb{D}} |g(z)|^q |f'(z)|^q (1 - |z|^2)^{q-2+s} dA(z) &\leq \|g\|_{\mathcal{H}^\infty}^q \|f\|_{\mathcal{B}}^{q-p} \|f\|_{\mathcal{D}_{p-2+s}^p}^p \\ &\leq \|g\|_{\mathcal{H}^\infty}^q \|f\|_{\mathcal{B} \cap \mathcal{D}_{p-2+s}^p}^q. \end{aligned}$$

Note that $gf \in \mathcal{B} \cap \mathcal{D}_{q-2+s}^q$,

$$(7) \quad \begin{aligned} \int_{\mathbb{D}} |(gf)'(z)|^q (1 - |z|^2)^{q-2+s} dA(z) &\leq \|gf\|_{\mathcal{B} \cap \mathcal{D}_{q-2+s}^q}^q \\ &\leq \|M_g\|^q \|f\|_{\mathcal{B} \cap \mathcal{D}_{p-2+s}^p}^q. \end{aligned}$$

Combining (6) and (7) implies

$$\int_{\mathbb{D}} |f(z)|^q |g'(z)|^q (1 - |z|^2)^{q-2+s} dA(z) \leq C(\|g\|_{\mathcal{H}^\infty}^q + \|M_g\|^q) \|f\|_{\mathcal{B} \cap \mathcal{D}_{p-2+s}^p}^q.$$

That is, $d\mu(z) = |g'(z)|^q(1 - |z|^2)^{q-2+s}dA(z)$ is a q -Carleson measure for $\mathcal{B} \cap \mathcal{D}_{p-2+s}^p$.

Suppose that $g \in M(\mathcal{B})$ and $d\mu(z) = |g'(z)|^q(1 - |z|^2)^{q-2+s}dA(z)$ is a q -Carleson measure for $\mathcal{B} \cap \mathcal{D}_{p-2+s}^p$, we prove that $g \in M(\mathcal{B} \cap \mathcal{D}_{p-2+s}^p, \mathcal{B} \cap \mathcal{D}_{q-2+s}^q)$. For any $f \in \mathcal{B} \cap \mathcal{D}_{p-2+s}^p$, we have $gf \in \mathcal{B}$. It remains to prove that $gf \in \mathcal{D}_{q-2+s}^q$. Since $d\mu(z) = |g'(z)|^q(1 - |z|^2)^{q-2+s}dA(z)$ is a q -Carleson measure for $\mathcal{B} \cap \mathcal{D}_{p-2+s}^p$, there is a constant $C > 0$ independent of f such that

$$(8) \quad \int_{\mathbb{D}} |f(z)|^q |g'(z)|^q (1 - |z|^2)^{q-2+s} dA(z) \leq C \|f\|_{\mathcal{B} \cap \mathcal{D}_{p-2+s}^p}^q.$$

Combining (6) and (8) we see that

$$\int_{\mathbb{D}} |(gf)'(z)|^q (1 - |z|^2)^{q-2+s} dA(z) \leq C \|f\|_{\mathcal{B} \cap \mathcal{D}_{p-2+s}^p}^q,$$

which implies that $gf \in \mathcal{D}_{q-2+s}^q$.

The idea of proofs of (ii) and (iii) is similar to that of (i). For the completeness of the paper, we give their proofs briefly below.

(ii) Assume that $g \in M(BMOA \cap \mathcal{D}_{p-2+s}^p, BMOA \cap \mathcal{D}_{q-2+s}^q)$. For any $a \in \mathbb{D}$, let φ_a and f_a be defined as in the proof of (i). An easy computation shows

that $\sup_{a \in \mathbb{D}} \|f_a\|_{\mathcal{D}_{p-2+s}^p} < \infty$. Since $\frac{1}{2\pi} \int_0^{2\pi} |\log \frac{1}{1-\bar{a}e^{i\theta}}| d\theta < \infty$, we have $f_a \in \mathcal{H}^1$. Since $f'_a(z) = \frac{\bar{a}}{1-\bar{a}z}$, by Lemma 2.5, there exists a constant $C > 0$ such that

$$\begin{aligned} \int_{\mathbb{D}} |f'_a(z)|^2 (1 - |\varphi_b(z)|^2) dA(z) &= \int_{\mathbb{D}} \frac{|a|^2}{|1-\bar{a}z|^2} \frac{(1-|b|^2)(1-|z|^2)}{|1-\bar{b}z|^2} dA(z) \\ &\leq (1-|b|^2) \int_{\mathbb{D}} \frac{1-|z|^2}{|1-\bar{a}z|^2 |1-\bar{b}z|^2} dA(z) \\ &\leq C. \end{aligned}$$

Hence, the Borel measure $|f'_a(z)|^2 (1-|z|^2) dA(z)$ is a Carleson measure by Lemma 2.1, so $f_a \in BMOA$. Since C is independent of a , we deduce that $\sup_{a \in \mathbb{D}} \|f_a\|_{BMOA} < \infty$. Hence, $f_a \in BMOA \cap \mathcal{D}_{p-2+s}^p$ and $\sup_{a \in \mathbb{D}} \|f_a\|_{BMOA \cap \mathcal{D}_{p-2+s}^p} < \infty$. In addition, a similar argument implies $g \in \mathcal{H}^\infty$. So $gf_a \in BMOA \cap \mathcal{D}_{q-2+s}^q$. Hence, there exists a constant $C > 0$ such that for any arc I ,

$$(9) \quad \int_{S(I)} |(gf_a)'(z)|^2 (1-|z|^2) dA(z) \leq C|I|$$

and

$$(10) \quad \int_{S(I)} |f'_a(z)|^2 (1-|z|^2) dA(z) \leq C|I|.$$

Then by $g \in \mathcal{H}^\infty$, (9) and (10) we obtain

$$(11) \quad \int_{S(I)} |g'(z)|^2 |f_a(z)|^2 (1-|z|^2) dA(z) \leq C|I|.$$

Take $a = (1-|I|)e^{i\theta}$, where $e^{i\theta}$ is the center of I , then for any $z \in S(I)$,

$$|1-\bar{a}z| \asymp 1-|a| = |I|, \quad |f_a(z)| \asymp \log \frac{1}{|I|}.$$

Thus (11) implies that

$$\left(\log \frac{1}{|I|}\right)^2 \int_{S(I)} |g'(z)|^2 (1-|z|^2) dA(z) \leq C|I|,$$

in other words, $g \in BMOA_{\log}$. Therefore $g \in M(BMOA)$ from (2).

We turn to show that $|g'(z)|^q (1-|z|^2)^{q-2+s} dA(z)$ is a q -Carleson measure for $BMOA \cap \mathcal{D}_{p-2+s}^p$. For every $f \in BMOA \cap \mathcal{D}_{p-2+s}^p$, we have $gf \in BMOA \cap \mathcal{D}_{q-2+s}^q$ and

$$\begin{aligned} \int_{\mathbb{D}} |(gf)'(z)|^q (1-|z|^2)^{q-2+s} dA(z) &\leq \|gf\|_{\mathcal{D}_{q-2+s}^q}^q \\ &\leq \|gf\|_{BMOA \cap \mathcal{D}_{q-2+s}^q}^q \\ &\leq \|M_g\|^q \|f\|_{BMOA \cap \mathcal{D}_{p-2+s}^p}^q. \end{aligned} \tag{12}$$

A similar argument as in the proof of (i) shows that

$$(13) \quad \int_{\mathbb{D}} |g(z)|^q |f'(z)|^q (1 - |z|^2)^{q-2+s} dA(z) \leq \|g\|_{\mathcal{H}^\infty}^q \|f\|_{BMOA \cap \mathcal{D}_{p-2+s}^p}^q.$$

Combining (12) and (13) yields

$$\int_{\mathbb{D}} |f(z)|^q |g'(z)|^q (1 - |z|^2)^{q-2+s} dA(z) \leq C(\|g\|_{\mathcal{H}^\infty}^q + \|M_g\|^q) \|f\|_{BMOA \cap \mathcal{D}_{p-2+s}^p}^q.$$

We conclude that $d\mu(z) = |g'(z)|^q (1 - |z|^2)^{q-2+s} dA(z)$ is a q -Carleson measure for $BMOA \cap \mathcal{D}_{p-2+s}^p$.

Conversely, for any $f \in BMOA \cap \mathcal{D}_{p-2+s}^p$, we have $gf \in BMOA$. We only need to prove $gf \in \mathcal{D}_{q-2+s}^q$. By hypothesis, there exists a constant $C > 0$ independent of f such that

$$(14) \quad \int_{\mathbb{D}} |f(z)|^q |g'(z)|^q (1 - |z|^2)^{q-2+s} dA(z) \leq C \|f\|_{BMOA \cap \mathcal{D}_{p-2+s}^p}^q.$$

By (13) and (14) we obtain

$$\int_{\mathbb{D}} |(gf)'(z)|^q (1 - |z|^2)^{q-2+s} dA(z) \leq C \|f\|_{BMOA \cap \mathcal{D}_{p-2+s}^p}^q.$$

That is, $gf \in \mathcal{D}_{q-2+s}^q$.

(iii) We only need to show

$$M(\mathcal{H}^\infty \cap \mathcal{D}_{p-2+s}^p, \mathcal{H}^\infty \cap \mathcal{D}_{q-2+s}^q) \supseteq \mathcal{H}^\infty \cap \mathcal{D}_{q-2+s}^q,$$

since the converse is obvious.

Let $g \in \mathcal{H}^\infty \cap \mathcal{D}_{q-2+s}^q$. For any $f \in \mathcal{H}^\infty \cap \mathcal{D}_{p-2+s}^p$, we have $gf \in \mathcal{H}^\infty$. It remains to prove that $gf \in \mathcal{D}_{q-2+s}^q$. These hypothesis imply

$$\begin{aligned} \int_{\mathbb{D}} |f(z)|^q |g'(z)|^q (1 - |z|^2)^{q-2+s} dA(z) &\leq \|f\|_{\mathcal{H}^\infty}^q \|g\|_{\mathcal{D}_{q-2+s}^q}^q \\ &\leq \|f\|_{\mathcal{H}^\infty \cap \mathcal{D}_{p-2+s}^p}^q \|g\|_{\mathcal{D}_{q-2+s}^q}^q \end{aligned}$$

and

$$\int_{\mathbb{D}} |g(z)|^q |f'(z)|^q (1 - |z|^2)^{q-2+s} dA(z) \leq \|g\|_{\mathcal{H}^\infty}^q \|f\|_{\mathcal{H}^\infty \cap \mathcal{D}_{p-2+s}^p}^q.$$

Hence

$$\int_{\mathbb{D}} |(gf)'(z)|^q (1 - |z|^2)^{q-2+s} dA(z) \leq C(\|g\|_{\mathcal{D}_{q-2+s}^q}^q + \|g\|_{\mathcal{H}^\infty}^q) \|f\|_{\mathcal{H}^\infty \cap \mathcal{D}_{p-2+s}^p}^q.$$

The proof is complete. \square

Proof of Theorem 1.2. (i) Suppose that $g \in M(\mathcal{B} \cap \mathcal{D}_{p-2+s}^p, \mathcal{B} \cap \mathcal{D}_{q-2+s}^q)$ and $g \neq 0$, then $g \in \mathcal{B} \cap \mathcal{D}_{q-2+s}^q$. Let

$$f(z) = \sum_{k=0}^{\infty} a_k z^{n_k}, \quad a_k = n_k^{\frac{s-1}{q}}, \quad z \in \mathbb{D},$$

with $\frac{n_{k+1}}{n_k} \geq \lambda > 1$ for all k . Since $\sum_{k=1}^{\infty} |a_k| < \infty$, by Lemma 2.2 we have $f \in \mathcal{H}^{\infty} \subseteq \mathcal{B}$. It is not difficult to see that $\sum_{k=0}^{\infty} n_k^{1-s} |a_k|^p < \infty$, Lemma 2.2 yields $f \in \mathcal{D}_{p-2+s}^p$. Hence $f \in \mathcal{B} \cap \mathcal{D}_{p-2+s}^p$ and $fg \in \mathcal{B} \cap \mathcal{D}_{q-2+s}^q$. We have

$$\int_{\mathbb{D}} (1 - |z|^2)^{q-2+s} |(gf)'(z)|^q dA(z) \leq \|gf\|_{\mathcal{D}_{q-2+s}^q}^q < \infty$$

and

$$\int_{\mathbb{D}} (1 - |z|^2)^{q-2+s} |g'(z)f(z)|^q dA(z) \leq \|f\|_{\mathcal{H}^{\infty}}^q \|g\|_{\mathcal{D}_{q-2+s}^q}^q < \infty.$$

These imply

$$(15) \quad \int_{\mathbb{D}} (1 - |z|^2)^{q-2+s} |g(z)f'(z)|^q dA(z) < \infty.$$

On the other hand, $f'(z) = \sum_{k=0}^{\infty} a_k n_k z^{n_k-1}$, by Lemma 2.3 we see that

$$\int_0^1 (1-r)^{q-2+s} |f'(re^{i\theta})|^q dr \asymp \sum_{k=0}^{\infty} n_k^{-(q+s-1)} |a_k n_k|^q = \infty.$$

Since $g \in \mathcal{D}_{q-2+s}^q \subseteq \mathcal{H}^q$ (see [9], p. 1877), g has a finite and nonzero radial limit almost everywhere on the boundary of \mathbb{D} . Thus

$$\int_0^1 (1-r)^{q-2+s} |f'(re^{i\theta})|^q |g(re^{i\theta})|^q dr = \infty$$

for almost all $\theta \in \mathbb{R}$ (see [9], p. 1878). This is in contradiction to (15).

(ii) Assume that $g \in M(BMOA \cap \mathcal{D}_{p-2+s}^p, BMOA \cap \mathcal{D}_{q-2+s}^q)$ and $g \neq 0$, then $g \in BMOA \cap \mathcal{D}_{q-2+s}^q$. Let $a_k = (2^k)^{\frac{s-1}{q}}$, $k = 1, 2, \dots$, and

$$f(z) = \sum_{k=0}^{\infty} a_k z^{2^k}, \quad z \in \mathbb{D}.$$

Then $f \in \mathcal{H}^{\infty} \cap \mathcal{D}_{p-2+s}^p$ by Lemma 2.2. Hence $f \in BMOA \cap \mathcal{D}_{p-2+s}^p$ and $fg \in BMOA \cap \mathcal{D}_{q-2+s}^q$. So

$$\int_{\mathbb{D}} (1 - |z|^2)^{q-2+s} |(gf)'(z)|^q dA(z) \leq \|gf\|_{\mathcal{D}_{q-2+s}^q}^q < \infty$$

and

$$\int_{\mathbb{D}} (1 - |z|^2)^{q-2+s} |g'(z)f(z)|^q dA(z) \leq \|f\|_{\mathcal{H}^{\infty}}^q \|g\|_{\mathcal{D}_{q-2+s}^q}^q < \infty.$$

We get

$$\int_{\mathbb{D}} (1 - |z|^2)^{q-2+s} |g(z)f'(z)|^q dA(z) < \infty.$$

Since $f'(z) = \sum_{k=0}^{\infty} 2^k a_k z^{2^k-1}$, from Lemma 2.3,

$$\int_0^1 (1-r)^{q-2+s} |f'(re^{i\theta})|^q dr \asymp \sum_{k=0}^{\infty} (2^k)^{-(q+s-1)} |a_k 2^k|^q = \infty.$$

Therefore, for almost all $\theta \in \mathbb{R}$,

$$\int_0^1 (1-r)^{q-2+s} |f'(re^{i\theta})|^q |g(re^{i\theta})|^q dr = \infty.$$

This is a contradiction.

(iii) Assume $g \in M(\mathcal{H}^\infty \cap \mathcal{D}_{p-2+s}^p, \mathcal{H}^\infty \cap \mathcal{D}_{q-2+s}^q)$ and $g \neq 0$, then $g \in \mathcal{H}^\infty \cap \mathcal{D}_{q-2+s}^q$. Let $f \in \mathcal{H}(\mathbb{D})$ be defined as in the proof of (i). The same argument as in the proof of (i) shows that $f \in \mathcal{H}^\infty \cap \mathcal{D}_{p-2+s}^p$. So $fg \in \mathcal{H}^\infty \cap \mathcal{D}_{q-2+s}^q$, i.e.,

$$\int_{\mathbb{D}} (1-|z|^2)^{q-2+s} |(gf)'(z)|^q dA(z) \leq \|gf\|_{\mathcal{D}_{q-2+s}^q}^q.$$

In addition,

$$\int_{\mathbb{D}} (1-|z|^2)^{q-2+s} |g'(z)f(z)|^q dA(z) \leq \|f\|_{\mathcal{H}^\infty}^q \|g\|_{\mathcal{D}_{q-2+s}^q}^q.$$

We have

$$\int_{\mathbb{D}} (1-|z|^2)^{q-2+s} |g(z)f'(z)|^q dA(z) < \infty.$$

On the other hand, by Lemma 2.3 we deduce that

$$\int_0^1 (1-r)^{q-2+s} |f'(re^{i\theta})|^q dr = \infty.$$

This together with $g \in \mathcal{D}_{q-2+s}^q \subseteq \mathcal{H}^q$ yields

$$\int_0^1 (1-r)^{q-2+s} |f'(re^{i\theta})|^q |g(re^{i\theta})|^q dr = \infty$$

for almost all $\theta \in \mathbb{R}$ ([9], p. 1878). We obtain a contradiction. This finishes the proof. \square

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SONGXIAO LI
 DEPARTMENT OF MATHEMATICS
 SHANTOU UNIVERSITY
 SHANTOU 515063, P. R. CHINA
 Email address: jyu1sx@163.com

ZENGJIAN LOU
 DEPARTMENT OF MATHEMATICS
 SHANTOU UNIVERSITY
 SHANTOU 515063, P. R. CHINA
 Email address: zjlou@stu.edu.cn

CONGHUI SHEN
DEPARTMENT OF MATHEMATICS
SHANTOU UNIVERSITY
SHANTOU 515063, P. R. CHINA
Email address: shenconghui2008@163.com

51YUX.CZDFbY