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ON THE FIRST GENERALIZED HILBERT COEFFICIENT AND DEPTH OF ASSOCIATED GRADED RINGS

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Abstract. Let $(R,\mathfrak{m})$ be a d-dimensional Cohen-Macaulay local ring with infinite residue field. Let I be an ideal of R that has analytic spread $\ell(I)=d$, satisfies the G_d condition, the weak Artin-Nagata property AN^-_{d-2} and $\mathfrak{m}$ is not an associated prime of R/I. In this paper, we show that if $j_1(I)=\lambda(I/J)+\lambda[R/(J_{d-1}:_R I+(J_{d-2}:_R I+I):_R \mathfrak{m}^\infty)]+1$, then I has almost minimal j-multiplicity, $G(I)$ is Cohen-Macaulay and $r_J(I)$ is at most 2, where $J=(x_1,\ldots,x_d)$ is a general minimal reduction of I and $J_i=(x_1,\ldots,x_i)$. In addition, the last theorem is in the spirit of a result of Sally who has studied the depth of associated graded rings and minimal reductions for $\mathfrak{m}$-primary ideals.

1. Introduction

Let $(R,\mathfrak{m},k)$ be a Noetherian Cohen-Macaulay local ring of dimension d with maximal ideal $\mathfrak{m}$ and infinite residue field k. For an ideal I of R, we will denote by $G(I)=\oplus_{n=0}^{\infty}I^n/I^{n+1}$ the associated graded algebra of I, and by $\mathcal{F}(I)=\oplus_{n=0}^{\infty}I^n/\mathfrak{m}I^n$ the fiber cone of I, that reflect various algebraic and geometric properties of the ideal I. The dimension of $G(I)$ is always d and the dimension of $\mathcal{F}(I)$ is called the analytic spread of I and is denoted by $\ell(I)$. The associated graded rings are indeed interesting since they encode many useful properties. Provided that both the ideal I and the ring R satisfy some good properties, the associated graded ring of I reveals itself as a key tool in the study of many invariants such as the depth or the regularity mainly by means of the Hilbert polynomial of I, and its Hilbert coefficients. For classes of ideals that are not necessarily $\mathfrak{m}$-primary, the concept of the j-multiplicity was introduced as a generalization of the Hilbert multiplicity in 1993 by Achilles and Manaresi [1] and further this topic was studied in [2], [3] and [4]. Polini and Xie in [14] investigate the depth of the associated graded ring of an arbitrary ideal using the j-multiplicity. On the other hand, the numerical information on the Hilbert coefficient has been used to obtain information on the depth

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of associated graded ring. For instance, Northcott in [11] showed that the bound $e_1(I) \geq e_0(I) - \lambda(R/I) = \lambda(R/J) - \lambda(R/I) = \lambda(I/J)$, always holds for any minimal reduction J of I. When equality holds, the ideal I enjoys nice properties. Indeed, it was shown that $e_1(I) = \lambda(I/J)$ if and only if the reduction number of I is at most 1, and when this is the case, the associated graded ring of I is Cohen-Macaulay (see [8] and [13]). Xie in [19] generalized Northcott's inequality to R-ideal as follows:

$$j_1(I) \geq \lambda(I/J) + \lambda[R/(J_{d-1} :_R I + (J_{d-2} :_R I + I) :_R \mathfrak{m}^\infty)]$$

and she proved that $j_1(I) = \lambda(I/J) + \lambda[R/(J_{d-1} :_R I + (J_{d-2} :_R I + I) :_R \mathfrak{m}^\infty)]$ if and only if $r(I) \leq 1$ and when this is the case, the associated graded ring of I is Cohen-Macaulay. Sally in [16] studied $\mathfrak{m}$-primary ideal I satisfying the condition $e_1(I) = \lambda(I/J) + 1$, with $e_2(I) > 0$ if $d \geq 2$ and she proved that if $e_1(I) = \lambda(I/J) + 1$, then $\operatorname{depth}(G(I)) \geq d - 1$ and $r_J(I) \leq 2$ for any minimal reduction J of I. This paper generalizes the above results to ideals of maximal analytic spread. In Section 1, we recall the definition of generalized Hilbert-samuel function of I and residual intersections. In Section 2, we prove that if $(R, \mathfrak{m})$ is a Cohen-Macaulay local ring, I is a non $\mathfrak{m}$-primary R-ideal which satisfies $\ell(I) = d$, the G_d condition, the AN^-_{d-2} and $\operatorname{depth}(R/I) \geq \min\{1, \dim R/I\}$, then for a general minimal reduction $J = (x_1, \ldots, x_d)$ of I and $J_i = (x_1, \ldots, x_i)$, one has that $j_1(I) \geq \lambda(I/J) + \lambda(I^2/JI) + \lambda[R/(J_{d-1} :_R I + (J_{d-2} :_R I + I) :_R \mathfrak{m}^\infty)]$. When $j_1(I) = \lambda(I/J) + \lambda[R/(J_{d-1} :_R I + (J_{d-2} :_R I + I) :_R \mathfrak{m}^\infty)] + 1$ then $\operatorname{depth}(G(I)) \geq d - 1$ and $r_J(I) \leq 2$ for any general minimal reduction $J = (x_1, \ldots, x_d)$ of I.

2. Preliminary

In this paper, we always assume that $(R, \mathfrak{m}, k)$ is a Noetherian local ring of dimension d with maximal ideal $\mathfrak{m}$ and infinite residue field k. Northcott and Rees [12] defined a minimal reduction of I to be a d-generated ideal $J \subseteq I$ of R such that $JI^n = I^{n+1}$ for some nonnegative integers n. We denote $r_J(I) = \min\{n \in \mathbb{Z} : JI^n = I^{n+1}\}$. The reduction number $r(I)$ defined as $r(I) = \min\{r_J(I) : J \text{ is minimal reduction of } I\}$. The reduction number $r(I)$ is said to be independent if $r(I) = r_J(I)$ for all minimal reduction J of I.

Let $I = (a_1, \ldots, a_t)$ and write $x_i = \sum_{j=1}^{t} \lambda_{ij} a_j$ for $1 \leq i \leq s$ and $\lambda_{ij} \in R^{st}$. The elements $x_1, \ldots, x_s$ form a sequence of general elements in I (equivalently $x_1, \ldots, x_s$ are general in I) if there exists a Zariski dense open subset U of k^{st} such that the image $(\overline{\lambda_{ij}}) \in U$. When $s = 1$, $x = x_1$ is said to be general in I. Furthermore, general $\ell(I)$ elements in I form a minimal reduction J whose $r_J(I)$ coincides with the reduction number $r(I)$ (see [9] or [17]). One says that J is a general minimal reduction of I if it is generated by $\ell(I)$ general elements in I.

Let $G(I) = \oplus_{n=0}^{\infty} I^n/I^{n+1}$ be the associated graded ring of I. Since the homogeneous components of $G(I)$ may not have finite length, so we consider the ideal of $G(I)$ of elements supported on $\mathfrak{m}$ and defined as follow:

$$W = \{\xi \in G(I) \,|\, \exists s > 0 \text{ such that } \xi.\mathfrak{m}^s = 0\} = H^0_{\mathfrak{m}}(G) = \oplus_{n=0}^{\infty} H^0_{\mathfrak{m}}(I^n/I^{n+1}).$$

Since W is annihilated by a large power of $\mathfrak{m}$, it is a finite graded module over $G(I) \otimes_R R/\mathfrak{m}^t$ for some $t \geq 0$, hence its Hilbert-Samuel function

$$H_I(n) = H_W(n) = \sum_{j=0}^{n} \lambda(H^0_m(I^j/I^{j+1}))$$

and Hilbert-Samuel polynomial

$$P_I(n) = \sum_{i=0}^{d} (-1)^i j_i(I) \begin{pmatrix} n+d-i \\ d-i \end{pmatrix}$$

are well defined. Polini and Xie defined $P_I(n)$ to be the generalized Hilbert-Samuel polynomial of I and $j_i(I)$, $0 \leq i \leq d$, the generalized Hilbert coefficients of I. The normalized leading coefficient $j_0(I)$ is called the j-multiplicity of I (see [1] or [15]). The next normalized coefficient $j_1(I)$ is called the generalized first Hilbert coefficient. Xie provided a formula relating the length $\lambda(I^{n+1}/JI^n)$ to the difference $P_I(n) - H_I(n)$, where I is an R-ideal (generalization of fundamental Lemma of Huneke [8, Lemma 2.4] and extension of this Lemma by Huckaba [6, Theorem 2.4]) and found a formula for the first generalized Hilbert coefficient (see [19, Theorem 3.2, Corollary 3.3]). If I is $\mathfrak{m}$-primary, each homogeneous component of $G(I)$ has finite length, thus $W = G(I)$ and the generalized Hilbert-Samuel function coincides with the usual Hilbert-Samuel function; in particular, the generalized Hilbert coefficients $j_i(I)$, $0 \leq i \leq d$, coincide with the usual Hilbert coefficients $e_i(I)$. We now recall some definitions and facts from the theory of residual intersections which will be used frequently in the rest of the paper.

(i) The ideal I is said to satisfy the G_s condition if for every $\mathfrak{p} \in V(I)$ with height$\mathfrak{p} = i < s$, the ideal $I_{\mathfrak{p}}$ is generated by i elements, i.e., $I_{\mathfrak{p}} = (x_1, \ldots, x_i)_{\mathfrak{p}}$ for some $x_1, \ldots, x_i$ in I.
(ii) Let $J_s = (x_1, \ldots, x_s)$, where $x_1, \ldots, x_s$ are elements in I. Then $J_s : I$ is called a s-residual intersection of I if $I_{\mathfrak{p}} = (x_1, \ldots, x_s)_{\mathfrak{p}}$ for every $\mathfrak{p} \in \operatorname{Spec}(R)$ with $\dim R_{\mathfrak{p}} \leq s-1$.
(iii) A s-residual intersection $J_s : I$ is called geometric if $I_{\mathfrak{p}} = (x_1, \ldots, x_s)_{\mathfrak{p}}$ for every $\mathfrak{p} \in V(I)$ with $\dim R_{\mathfrak{p}}$ at most s.
(iv) Let R be Cohen-Macaulay, the ideal I has the Artin-Nagata property AN_s^- if, for every $0 \leq i \leq s$ and every geometric i-residual intersection $J_i : I$ of I, one has that $R/J_i : I$ is Cohen-Macaulay.

It was shown that if I satisfies the G_s condition, then for general elements $x_1, \ldots, x_s$ in I and each $0 \leq i < s$, the ideal $J_i : I$ is a geometric i-residual intersection of I, and $J_s : I$ is a s-residual intersection of I (see [18] and [14]). It is well-known that the G_d condition and the Artin-Nagata property AN_{d-2}^- are automatically satisfied by any $\mathfrak{m}$-primary ideal in a Cohen-Macaulay local ring. From now on, we will assume I has maximal analytic spread $\ell(I) = d$, satisfies the G_d condition and $J = (x_1, \dots, x_d)$ is a general minimal reduction of I, where $x_1, \dots, x_d$ are general elements in I. We write $R^i = R/(J_i : I^\infty)$, where $J_i : I^\infty = \{r \in R \,|\, \exists n > 0 \text{ such that } rI^n \subseteq J_i\}$ and $J_i = (x_1, \dots, x_i)$ (with the convention $J_i = (0)$ if $i \leq 0$) for $i = 0, 1, \dots, d-1$. By the above hypothesis, R^{d-1} is a 1-dimensional Cohen-Macaulay local ring and IR^{d-1} is primary to the maximal ideal $\mathfrak{m}R^{d-1}$. Therefore, one has the Hilbert-Samuel function of I on R^{d-1}:

$$H_{IR^{d-1}}(n) = \lambda(I^n R^{d-1}/I^{n+1}R^{d-1}) \text{ for } n \geq 0.$$

Hence the generalized Hilbert-Samuel function $H_{IR^{d-1}}(n)$ and the generalized Hilbert-Samuel polynomial $P_{IR^{d-1}}(n)$ are respectively the usual Hilbert-Samuel function and the usual Hilbert-Samuel polynomial of IR^{d-1}. Let I be an R-ideal with analytic spread $\ell(I) = d$. We recall that I has minimal j-multiplicity if $j(I) = \lambda(IR^{d-1}/I^2R^{d-1})$, I has almost minimal j-multiplicity if $j(I) = \lambda(IR^{d-1}/I^2R^{d-1}) + 1$ and the index of nilpotency of an R-ideal I with respect to a reduction J is defined to be the integer $s_J(I) = \min\{n \,|\, I^{n+1} \subseteq J\}$ [14, Definition 2.4 and Definition 4.1].

3. Main result

In this section we find a lower bound for the first generalized Hilbert coefficient $j_1(I)$, when I is not an $\mathfrak{m}$-primary ideal with $\ell(I) = d$, and satisfies the G_d condition and the AN^-_{d-2}, J is a general minimal reduction of I (see Theorem 3.6). Also, we prove that if $j_1(I) = \lambda(I/J) + \lambda[R/(J_{d-1} :_R I + (J_{d-2} :_R I + I) :_R \mathfrak{m}^\infty)] + 1$, then $G(I)$ is almost Cohen-Macaulay (see Theorem 3.7).

Remark 3.1. Assume R is Cohen-Macaulay. Let I be an R-ideal which satisfies $\ell(I) = d$, the G_d condition and AN^-_{d-2}. Let $J = (x_1, \dots, x_d)$ be a general minimal reduction of I and $J_i = (x_1, \dots, x_i)$. Since I satisfies $\ell(I) = d$ and the G_d condition, then by [14, Lemma 3.1], $J_i : I$ is a geometric i-residual intersection of I for $0 \leq i \leq d-1$. By the weak Artin-Nagata property AN^-_{d-2}, one has that $J_i : I^\infty = J_i : I = J_i : x_{i+1}$ and $(J_i : I^\infty) \cap I = (J_i : I) \cap I = J_i$ for $0 \leq i \leq d-1$ [14, Lemma 3.2] and [18]. Therefore

$$\begin{aligned} I^{n+1} \cap (J_i : I^\infty) &= I^{n+1} \cap (J_i : I^\infty) \cap I \\ &= I^{n+1} \cap J_i \end{aligned}$$

and by similar argument we have $JI^n \cap (J_i : I^\infty) = JI^n \cap J_i$ for $n \geq 0$ and $0 \leq i \leq d-1$.

Lemma 3.2. *Assume R is Cohen-Macaulay. Let I be a non $\mathfrak{m}$-primary R-ideal which satisfies $\ell(I) = d$, the G_d condition, the AN^-_{d-2}, $J = (x_1, \dots, x_d)$ general minimal reduction of I and $\mathfrak{m}$ is not an associated prime of R/I. Then $I^2 \cap J_i = J_iI$ for any $0 \leq i \leq d-1$.*

Proof. Since $\operatorname{depth}(R/I) \geq 1$, by [14, Lemma 3.2(f)], we have $J_{d-1} :_{I^2} I^\infty = J_{d-1}I$. Then by Remark 3.1, $I^2 \cap J_{d-1} = J_{d-1}I$. Now we show that $I^2 \cap J_{d-2} = J_{d-2}I$. Indeed,

$$I^2 \cap J_{d-2} \subseteq I^2 \cap J_{d-1} = J_{d-1}I = J_{d-2}I + x_{d-1}I.$$

Intersecting both sides with J_{d-2}, we have

$$\begin{aligned} I^2 \cap J_{d-2} &= (J_{d-2}I + x_{d-1}I) \cap J_{d-2} \\ &= J_{d-2}I + (J_{d-2} \cap x_{d-1}I) \\ &= J_{d-2}I + (J_{d-2} \cap x_{d-1}R \cap x_{d-1}I) \\ &= J_{d-2}I + x_{d-1}(J_{d-2} : x_{d-1} \cap I) \\ &= J_{d-2}I + x_{d-1}J_{d-2} \\ &= J_{d-2}I. \end{aligned}$$

By the similar argument we have $I^2 \cap J_i = J_iI$ for any $0 \leq i \leq d-3$. □

Lemma 3.3. *Assume R is Cohen-Macaulay. Let I be an R-ideal which satisfies $\ell(I) = d \geq 3$, the G_d condition, the AN_{d-2}^-, $J = (x_1, \ldots, x_d)$ a general minimal reduction of I and $\operatorname{depth}(R/I) \geq \min\{1, \dim R/I\}$. Then $\operatorname{depth}(R^1/IR^1) \geq \min\{1, \dim(R^1/IR^1)\}$, where $R^1 = R/J_1 : I$ and IR^1 is the image of I in the quotient ring R^1.*

Proof. If I is an $\mathfrak{m}$-primary ideal the assertion is clear, so we assume that $\dim R/I \geq 1$. Set $R^0 = R/0 : I$. Then R^0 is Cohen-Macaulay since I satisfies AN_{d-2}^-. Note that $\dim R^0 = \dim R = d$, $\operatorname{grade}(IR^0) \geq 1$, IR^0 still satisfies the G_d condition, AN_{d-2}^- , $\ell(IR^0) = \ell(I) = d$ (see for instance [18]). Hence $\operatorname{depth}(R^0/x_1R^0) = d-1$ or $\operatorname{depth}(R/J_1 + 0 : I) = d-1$. By the exact sequence

$$0 \longrightarrow (J_1 : I)/J_1 + (0 : I) \longrightarrow R/J_1 + (0 : I) \longrightarrow R/(J_1 : I) \longrightarrow 0$$

one has $\operatorname{depth}((J_1 : I)/J_1 + (0 : I)) \geq d-1$. Since

$$\begin{aligned} (J_1 : I) + I/(0 : I) + I &= (J_1 : I) + I + (0 : I)/(0 : I) + I \\ &\cong (J_1 : I)/((0 : I) + I) \cap (J_1 : I) \\ &= (J_1 : I)/(0 : I) + J_1 \end{aligned}$$

we have $\operatorname{depth}((J_1 : I) + I/(0 : I) + I)) \geq d-1$. Furthermore by the exact sequence

$$0 \longrightarrow 0 : I \longrightarrow R/I \longrightarrow R/((0 : I) + I) \longrightarrow 0,$$

and since $\operatorname{depth}(0 : I) = d$ (see [18]), we have $\operatorname{depth}(R/((0 : I) + I)) \geq 1$. Now by the following exact sequence

$$0 \longrightarrow (J_1 : I) + I/(0 : I) + I \longrightarrow R/(0 : I) + I \longrightarrow R/(J_1 : I) + I \longrightarrow 0$$

we have $\operatorname{depth}(R/(J_1 : I) + I) \geq 1$. Therefore

$$\operatorname{depth}(R^1/IR^1) \geq \min\{1, \dim(R^1/IR^1)\}.$$ □

Remark 3.4. Assume R is Cohen-Macaulay of dimension d. Let I be an $\mathfrak{m}$-primary ideal of R and $I^2 \cap J = JI$ for a minimal reduction J of I. By formula of Huckaba and Marley [7, Theorem 4.7], we have $e_1(I) \geq \lambda(I/J) + \lambda(I^2/JI)$, and if $e_1(I) = \lambda(I/J) + 1$, then $\operatorname{depth}(G(I)) \geq d - 1$.

Theorem 3.5. *Assume R is Cohen-Macaulay of dimension 2. Let I be a non $\mathfrak{m}$-primary R-ideal which satisfies $\ell(I) = 2$, the G_2 condition, the AN_0^- and $\mathfrak{m}$ is not an associated prime of R/I. Then for a general minimal reduction $J = (x_1, x_2)$ of I, one has that $j_1(I) \geq \lambda(I/J) + \lambda(I^2/JI) + \lambda[R/(J_1 :_R I + (0 :_R I + I) :_R \mathfrak{m}^\infty)]$.*

Proof. By [19, Corollary 3.3] and Remark 3.1, we have

$$\begin{aligned} j_1(I) = & \sum_{n=0}^{\infty} \lambda(I^{n+1}/JI^n) - \sum_{n=0}^{\infty} \lambda(I^{n+1} \cap J_1/JI^n \cap J_1) \\ & + \lambda(R/(I + J_1 : I)) - \lambda(H^0_{\mathfrak{m}}(R/I + 0 : I)) \\ & + \sum_{n=1}^{\infty} [\lambda(\widetilde{L^0_n}) - \lambda(L^0_n) + \lambda(N^0_n)], \end{aligned}$$

where

$$\widetilde{L^0_n} = I^n \cap J_1/[I^{n+1} \cap J_1 + x_1 I^{n-1}],$$

$$L^0_n = I^{n+1} :_{I^n \cap J_1} \mathfrak{m}^\infty/[I^{n+1} \cap J_1 + x_1(I^n :_{I^{n-1}} \mathfrak{m}^\infty)],$$

$$N^0_n = (I^n \cap J_1 + I^{n+1}) :_{I^n} \mathfrak{m}^\infty/[I^n \cap J_1 + (I^{n+1} :_{I^n} \mathfrak{m}^\infty)].$$

Recall $R^1 = R/(J_1 : I^\infty)$ and $IR^1 = (I + J_1 : I^\infty)/(J_1 : I^\infty)$. By the fact that $\lambda(I^{n+1}/JI^n) < \infty$ and $\lambda(I^{n+1} \cap J_1/JI^n \cap J_1) = \lambda(I^{n+1} \cap (JI^n + J_1 : I^\infty)/JI^n) < \infty$ (see [14, Lemma 4.6]), one has

$$\begin{aligned} \lambda(I^{n+1}R^1/JI^nR^1) &= \lambda((I^{n+1} + J_1 : I^\infty)/(JI^n + J_1 : I^\infty)) \\ &= \lambda(I^{n+1}/JI^n + (I^{n+1} \cap J_1 : I^\infty)) \\ &= \lambda(I^{n+1}/JI^n) - \lambda(I^{n+1} \cap J_1/JI^n \cap J_1). \end{aligned}$$

Since $\dim(R/I) \geq 1$ (I is a non $\mathfrak{m}$-primary ideal) by hypothesis $\operatorname{depth}(R/I) \geq 1$, then by [14, Lemma 3.2(f)], $I^2 \cap J_1 : I^\infty = J_1 I$. Now by summing these equations over all $n \geq 1$ and using the fact that $\lambda(IR^1/JR^1) = \lambda(I/J)$, $\lambda(I^2R^1/JIR^1) = \lambda(I^2/(JI + (I^2 \cap J_1 : I^\infty))) = \lambda(I^2/JI)$, we see that

$$\begin{aligned} & \sum_{n=0}^{\infty} \lambda(I^{n+1}/JI^n) - \sum_{n=0}^{\infty} \lambda(I^{n+1} \cap J_1/JI^n \cap J_1) \\ = & \sum_{n=0}^{\infty} \lambda(I^{n+1}R^1/JI^nR^1) \\ = & \lambda(I/J) + \lambda(I^2/JI) + \sum_{n=2}^{\infty} \lambda(I^{n+1}/(JI^n + (I^{n+1} \cap J_1 : I^\infty))). \end{aligned}$$

Claim 1.

$$\lambda(R/(I+J_1:I))-\lambda(H^0_{\mathfrak{m}}(R/I+0:I)) \geq \lambda[R/(J_1:_R I+(0:_R I+I):_R \mathfrak{m}^\infty)].$$

Proof of Claim 1. We can write

$$\begin{aligned} &\lambda(R/(I+J_1:I)) - \lambda(H^0_{\mathfrak{m}}(R/I+0:I)) \\ &= \lambda(R/(I+J_1:I)) - \lambda(H^0_{\mathfrak{m}}(R/I)) + \lambda(H^0_{\mathfrak{m}}(R/I)) - \lambda(H^0_{\mathfrak{m}}(R/I+0:I)). \end{aligned}$$

By applying the long exact sequence of local cohomology on the following exact sequence

$$0 \longrightarrow 0:I \longrightarrow R/I \longrightarrow R/(0:I+I) \longrightarrow 0,$$

and since depth$(0:I)=2$ (see [18]), we have

$$\lambda(H^0_{\mathfrak{m}}(R/I)) - \lambda(H^0_{\mathfrak{m}}(R/I+0:I)) \geq 0.$$

To complete the proof of Claim 1, we need to show that

$$\lambda(R/(I+J_1:I)) - \lambda(H^0_{\mathfrak{m}}(R/I)) \geq \lambda[R/(J_1:_R I+(0:_R I+I):_R \mathfrak{m}^\infty)].$$

One has

$$\begin{aligned} &\lambda(R/(I+J_1:I)) - \lambda(H^0_{\mathfrak{m}}(R/I)) \\ &= [\lambda(R/(J+J_1:I)) - \lambda(I/J)] - [\lambda(J:\mathfrak{m}^\infty/J) - \lambda(I/J)] \\ &= \lambda(R/(J+J_1:I)) - \lambda(J:\mathfrak{m}^\infty/J) \\ &= \lambda(R/(J_1:I+J:\mathfrak{m}^\infty)) + \lambda(J_1:I+J:\mathfrak{m}^\infty/J+J_1:I) - \lambda(J:\mathfrak{m}^\infty/J) \\ &= \lambda(R/(J_1:I+J:\mathfrak{m}^\infty)) + \lambda(J:\mathfrak{m}^\infty/(J+J_1:I)\cap J:\mathfrak{m}^\infty) - \lambda(J:\mathfrak{m}^\infty/J) \\ &= \lambda(R/(J_1:I+J:\mathfrak{m}^\infty)) - \lambda((J+J_1:I)\cap J:\mathfrak{m}^\infty/J) \\ &= \lambda(R/(J_1:I+J:\mathfrak{m}^\infty)) - \lambda(J/J) \\ &= \lambda(R/(J_1:I+J:\mathfrak{m}^\infty)) \\ &\geq \lambda(R/(J_1:I+0:I+I:\mathfrak{m}^\infty)), \end{aligned}$$

where the first equality holds because

$$\begin{aligned} \lambda(R/(I+J_1:I)) &= \lambda(R/(J+J_1:I)) - \lambda(I+J_1:I/J+J_1:I) \\ &= \lambda(R/(J+J_1:I)) - \lambda(I/J+(J_1:I\cap I)) \\ &= \lambda(R/(J+J_1:I)) - \lambda(I/J) \end{aligned}$$

and because $\lambda(I/J) < \infty$, we also have

$$\lambda(H^0_{\mathfrak{m}}(R/I)) = \lambda(I:\mathfrak{m}^\infty/I) = \lambda(J:\mathfrak{m}^\infty/I) = \lambda(J:\mathfrak{m}^\infty/J) - \lambda(I/J)$$

the sixth equality follows by [19, (9)] that $(J_1:I)\cap(J:\mathfrak{m}^\infty) = J_1$. □

Claim 2.

$$\lambda(\widetilde{L^0_n}) - \lambda(L^0_n) \geq 0$$

Proof of Claim 2. There is a map

$$I^{n+1} :_{I^n\cap J_1} m^{\infty} \longrightarrow I^n \cap J_1/I^{n+1} \cap J_1 + x_1 I^{n-1}$$

with kernel

$$\begin{aligned} &[I^{n+1} :_{I^n\cap J_1} \mathfrak{m}^{\infty}] \cap [I^{n+1} \cap J_1 + J_1 I^{n-1}] \\ &= I^{n+1} \cap J_1 + [I^{n+1} :_{I^n\cap J_1} \mathfrak{m}^{\infty}] \cap J_1 I^{n-1} \\ &= I^{n+1} \cap J_1 + [J_1 I^{n} :_{I^n\cap J_1} \mathfrak{m}^{\infty}] \cap J_1 I^{n-1} \\ &= I^{n+1} \cap J_1 + x_1(I^n :_{I^{n-1}} \mathfrak{m}^{\infty}). \end{aligned}$$

We see that $\lambda(\widetilde{L_n^0}) - \lambda(L_n^0) \geq 0$. □

Then by using the formula for $j_1(I)$, Claim 1 and Claim 2 we have $j_1(I) \geq \lambda(I/J) + \lambda(I^2/JI) + \lambda[R/(J_1 :_R I + (0 :_R I + I) :_R \mathfrak{m}^{\infty})]$, as required. □

Theorem 3.6. *Assume R is Cohen-Macaulay of dimension $d \geq 2$. Let I be a non $\mathfrak{m}$-primary R-ideal which satisfies $\ell(I) = d$, the G_d condition, the AN_{d-2}^{-} and $\mathfrak{m}$ is not an associated prime of R/I. Then for a general minimal reduction $J = (x_1, \ldots, x_d)$ of I, one has that $j_1(I) \geq \lambda(I/J) + \lambda(I^2/JI) + \lambda[R/(J_{d-1} :_R I + (J_{d-2} :_R I + I) :_R \mathfrak{m}^{\infty})]$.*

Proof. We prove the theorem by induction on d. The case $d = 2$ has been proved in Theorem 3.5. Let $d \geq 3$ and assume that the theorem holds for $d-1$. Let $R^1 = R/J_1 : I$, where $J_1 = (x_1)$. Observe that R^1 is a Cohen-Macaulay ring of dimension $d-1$ and $\ell(IR^1) = d-1$. Also, $IR^1 = (I + J_1 : I)/J_1 : I \cong I/(x_1)$ satisfies G_{d-1} and AN_{d-3}^{-} (see [14, Lemma 3.2] and [18]). By Lemma 3.3, $\operatorname{depth}(R^1/IR^1) \geq 1$ and by [15, Propositon 2.5], $j_1(I) = j_1(IR^1)$ and so we have

$$\begin{aligned} j_1(I) &= j_1(IR^1) \\ &\geq \lambda(IR^1/JR^1) + \lambda(I^2R^1/JR^1IR^1) \\ &\quad + \lambda[R^1/(x_2, \ldots, x_{d-1})R^1 :_{R^1} IR^1 \\ &\quad + (x_2, \ldots, x_{d-2})R^1 :_{R^1} IR^1 + IR^1) :_{R^1} \mathfrak{m}^{\infty})] \\ &= \lambda(I/J) + \lambda(I^2/(JI + (I^2 \cap J_1 : I))) \\ &\quad + \lambda[R/(J_{d-1} :_R I + (J_{d-2} :_R I + I) :_R \mathfrak{m}^{\infty})] \\ &= \lambda(I/J) + \lambda(I^2/JI) + \lambda[R/(J_{d-1} :_R I + (J_{d-2} :_R I + I) :_R \mathfrak{m}^{\infty})], \end{aligned}$$

where the last equality follows by Lemma 3.2. □

Theorem 3.7. *Assume R is Cohen-Macaulay. Let I be a non $\mathfrak{m}$-primary R-ideal which satisfies $\ell(I) = d$, the G_d condition, the AN_{d-2}^{-} and $\mathfrak{m}$ is not an associated prime of R/I. Then for a general minimal reduction $J = (x_1, \ldots, x_d)$ of I, with $j_1(I) = \lambda(I/J) + \lambda[R/(J_{d-1} :_R I + (J_{d-2} :_R I + I) :_R \mathfrak{m}^{\infty})] + 1$, I has almost minimal j-multiplicity, the reduction number equals the index of nilpotency and is at most 2, and $G(I)$ is Cohen-Macaulay.*

Proof. By Theorem 3.6 we have $j_1(I) \geq \lambda(I/J) + \lambda(I^2/JI) + \lambda[R/(J_{d-1} :_R I + (J_{d-2} :_R I + I) :_R \mathfrak{m}^\infty)]$. From the assumption one obtains that $\lambda(I^2/JI)$ is at most 1, so the length of I^2R^{d-1}/JIR^{d-1} is at most one, and thus the length of I^2R^{d-1}/x_dIR^{d-1} is at most 1. The ideal I then has almost minimal j-multiplicity, based on [14, Definition 4.1]. Thus [14, Corollary 4.9] implies depth$(G(I))$ is at least $d-1$. If $\lambda(I^2/JI) = 0$, then I has minimal j-multiplicity and the reduction number is 1; in particular $G(I)$ is Cohen-Macaulay by [14, Corollary 3.5]. If $\lambda(I^2/JI) = 1$, then the reduction number of I is at least 2, and the index of nilpotency of I is $K = \lambda(I^2/JI) + 1 = 2$. If $j_1(I) = \lambda(I/J) + \lambda(I^2/JI) + \lambda[R/(J_{d-1} :_R I + (J_{d-2} :_R I + I) :_R m^\infty)]$, then by the formula of $j_1(I)$, we have $\sum_{n=2}^{\infty} \lambda(I^{n+1}/(JI^n + (I^{n+1} \cap J_1 : I^\infty))) = 0$. If depth$(G(I)) \geq d-1$, then $I^{n+1} \cap J_i = J_iI^n$ for every $n \geq 0$ and $0 \leq i \leq d-1$. Therefore we have $\sum_{n=2}^{\infty} \lambda(I^{n+1}/JI^n) = 0$ and so the reduction number is at most 2, thus it equals K. Then $G(I)$ is Cohen-Macaulay by [10, Theorem 4.1]. □

Example 3.8. Let $R = k[\![x, y, z]\!]$ be a polynomial ring where k is a field and $\mathfrak{q} = (x^4, xz, yz)$. Set $S = R/\mathfrak{q}$ and let $I = (x, y)$ be an ideal of S. Then S is a 1-dimensional Cohen-Macaulay local ring and I is a Cohen-Macaulay prime ideal that has $\ell(I) = 1$, G_1 condition and AN_{-1}^{-}. By using Macaulay 2 [5], we have depth$(S/I) = 1$ and the generalized Hilbert-Samuel polynomial is $P_I(n) = 4(n+1) - 7$. Hence $j_0(I) = 4$ and $j_1(I) = 7$. $J = (y)$ is a minimal reduction of I (see [10, Example 2.2]) and again by Macaulay 2 we have $\lambda(I/J) = 3$, $\lambda(I^2/JI) = 2$ and $\lambda[R/(0 :_R I + I :_R \mathfrak{m}^\infty)] = 1$. Therefore

$$j_1(I) > \lambda(I/J) + \lambda(I^2/JI) + \lambda[R/(0 :_R I + I :_R \mathfrak{m}^\infty)].$$

Example 3.9. Let $R = k[\![x, y, z]\!]$ be polynomial ring where k is a field and let $\mathfrak{q} = (x^3, xz, yz)$. Set $S = R/\mathfrak{q}$ and let $I = (x, y)$ be an ideal of S. Then S is a 1-dimensional Cohen-Macaulay local ring and I is a Cohen-Macaulay prime ideal that has $\ell(I) = 1$, G_1 condition and AN_{d-2}^{-} (automatically satisfied since $d = 1$). By using Macaulay 2 [5], we have depth$(S/I) = 1$ and the generalized Hilbert-Samuel polynomial is $P_I(n) = 3(n+1) - 4$. Hence $j_0(I) = 3$ and $j_1(I) = 4$. $J = (y)$ is a minimal reduction of I (see [10, Example 2.2]) and again by Macaulay 2 we have $\lambda(I/J) = 2$, $\lambda(I^2/JI) = 1$ and $\lambda[R/(0 :_R I + I :_R \mathfrak{m}^\infty)] = 1$ then we have,

$$j_1(I) = \lambda(I/J) + \lambda(I^2/JI) + \lambda[R/(0 :_R I + I :_R \mathfrak{m}^\infty)].$$

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