

SOME NEW CHARACTERIZATIONS OF QUASI-FROBENIUS RINGS BY USING PURE-INJECTIVITY

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ABSTRACT. A ring R is called right pure-injective if it is injective with respect to pure exact sequences. According to a well known result of L. Melkersson, every commutative Artinian ring is pure-injective, but the converse is not true, even if R is a commutative Noetherian local ring. In this paper, a series of conditions under which right pure-injective rings are either right Artinian rings or quasi-Frobenius rings are given. Also, some of our results extend previously known results for quasi-Frobenius rings.

1. Introduction

According to a well known result of L. Melkersson [11, Corollary 4.2], every commutative Artinian ring is pure-injective. But, Example 2.2 shows that the converse is not true, even if R is a commutative Noetherian local ring. In this paper, we determine several classes of rings over which every right pure-injective ring is right Artinian (see Propositions 2.3 and 2.9 and Theorems 2.11 and 2.14). Also, it is well known that over a commutative ring, every Noetherian module with essential socle is Artinian. But a right Noetherian ring with essential right socle need not be right Artinian as shown by an example due to Faith and Menal [6]. However a result of Ginn and Moss [8, Theorem] asserts that if R is a two-sided Noetherian ring such that the right socle of R is either left or right essential, then R is right and left Artinian. Recently, Chen, Ding and Yousif [1] obtained a onesided version of this theorem by showing that if R is a right Noetherian ring whose right socle is essential as a right ideal and is contained in the left socle, then R is right Artinian. In this paper, it is shown that every right pure-injective right Noetherian ring R is a right Artinian left Kasch ring if its left socle is an essential right ideal in R (see Theorem 2.11).

Recall that a ring R is called *right Johns* if R is right Noetherian and every right ideal is an annihilator. Faith and Menal [6] gave a counter example

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to show that right Johns rings need not be right artinian. Later in [7] they defined *strongly right Johns ring* (i.e., the matrix ring $M_n(R)$ is right Johns for all $n \geq 1$) and characterized such rings as right Noetherian and left FP-injective rings. In this direction, there is an open conjecture due to Faith-Menal [7]: “*every strongly right Johns ring is quasi-Frobenius*”. It is shown that the Faith-Menal conjecture is true when R is right pure-injective (see Corollary 2.6). In fact, we show that a ring R is quasi-Frobenius if and only if R is a right pure-injective right Noetherian left 2-injective ring (see Theorem 2.5).

Also, we give some new characterizations of quasi-Frobenius rings and semisimple rings by using pure-injectivity and RD -injectivity, respectively. For instance, it is shown that a ring R is quasi-Frobenius if and only if R is a left and right Johns left and right pure-injective ring, if and only if, R is a right Noetherian right P-injective left mininjective ring, if and only if, R is a right semi-artinian right Kasch IF ring, if and only if, R is a right pure-injective right Johns left min-CS ring (i.e., if every minimal left ideal is essential in a direct summand of ${}_R R$). In commutative case, it is shown that a ring R is quasi-Frobenius if and only if R is a Noetherian RD -injective ring with T-nilpotent Jacobson, if and only if, R is a Noetherian pure-injective ring with T-nilpotent Jacobson radical and $E(R)$ is an RD -projective R -module, if and only if, every projective R -module is RD -injective and either R is coherent or $J(R)$ is finitely generated (see Theorem 2.7, Proposition 2.13 and Corollary 2.15). Finally, it is shown that a ring R is semisimple if and only if R is a right pure-injective right hereditary right p-injective ring (see Corollary 2.21). In commutative case, it is shown that a ring R is semisimple if and only if R is Noetherian pure-injective with T-nilpotent projective Jacobson radical (see Corollary 2.8).

Throughout the paper, R will denote an arbitrary ring with identity, $J(R)$ will denote its Jacobson radical and all modules will be assumed to be unitary. For a ring R , we denote by $\text{Soc}({}_R R)$ and $\text{Soc}(R_R)$ for the left socle and the right socle, respectively. For any nonempty set X , the right annihilator of X in R will be denoted by $\text{Ann}_r(X)$. We use $K \subseteq^{ess} N$ to indicate that K is an essential submodule of N . Also, a ring R is *semilocal* if $R/J(R)$ is a semisimple Artinian ring. A ring R is *right perfect* if R is semilocal and $J(R)$ is right T-nilpotent. A ring R is said to be *right CS ring* if every right ideal of R is essential in a direct summand of ${}_R R$. A ring R is called *left Kasch* if every simple left R -module can be embedded in ${}_R R$. A ring R is called *right coherent* if every finitely generated right ideal of R is finitely presented.

2. Results

An exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ of right R -modules is said to be *pure exact* (resp. *RD -exact*) if the induced homomorphism $\text{Hom}_R(M, B) \rightarrow \text{Hom}_R(M, C)$ is surjective for any finitely presented (resp. cyclically presented) right R -module M . A right R -module M is said to be *pure-injective* (resp. *RD -injective*) if it is injective with respect to pure exact

(resp. RD -exact) sequences. Also, a right R -module M is said to be *pure-projective* (resp. *RD -projective*) if it is projective with respect to pure exact (resp. RD -exact) sequences. In model theory, pure-injective modules are more useful than the injective modules. Also, a left R -module is called *left FP-injective* (or *left absolutely pure*) if it is pure in every module that contains it.

Recall that a ring R is *semiperfect* if R is semilocal and idempotents of $R/J(R)$ can be lifted to R .

Lemma 2.1. *For a ring R , the following conditions hold:*

- (a) *Every right pure-injective semilocal ring is semiperfect.*
- (b) *Every right pure-injective right Noetherian ring is semiperfect.*
- (c) *Every right pure-injective ring is semiperfect whenever is a direct sum of indecomposable right ideals.*

Proof. (a) follows from [17, Theorem 9(1)].

(c) follows from [17, Theorem 9(3)].

(b) follows from (c). □

By [11, Corollary 4.2], we know that every commutative Artinian ring is pure-injective. But the following example shows that the converse is not true, in general.

Example 2.2. Let R be a commutative Noetherian local domain with maximal ideal \mathfrak{m} , complete in its \mathfrak{m} -adic topology. By [16], R is a pure-injective R -module, but R is not an Artinian ring.

Recall that a ring R is called *left n -injective* (*mininjective*) if, for any n -generated (minimal) left ideal I of R , every R -homomorphism from I to R extends to an R -homomorphism from R to R . Left 1-injective rings are called *left P -injective*. Now, we determine several classes of rings over which every right pure-injective ring is right Artinian (Proposition 2.3 and 2.9, and Theorems 2.11 and 2.14).

Proposition 2.3. *For a ring R , the following conditions hold:*

- (a) *Every right pure-injective right Noetherian ring with right or left T-nilpotent Jacobson radical is a right Artinian ring.*
- (b) *Every right pure-injective right Noetherian right P -injective ring is a two-sided Artinian ring.*
- (c) *Every right pure-injective right Noetherian left P -injective ring is a right Artinian ring.*

Proof. (a) Assume that R is a right pure-injective right Noetherian ring with right or left T-nilpotent Jacobson radical. Thus, by Lemma 2.1, R is a right or left perfect ring and so R is a right Artinian ring.

(b) Assume that R is a right pure-injective right Noetherian right P -injective ring. Thus, by [13, Proposition 5.15], R is left Artinian and so by (a), R is also a right Artinian ring.

(c) Assume that R is a right pure-injective right Noetherian left P-injective ring. Thus, by [13, Lemma 8.6], $J(R)$ is nilpotent and so by (a), R is a right Artinian ring. \square

Recall that a right R -module M is called \sum -pure-injective if all direct sums of copies of M are pure-injective.

Corollary 2.4. *For a commutative ring R , the following statements are equivalent:*

- (1) R is an Artinian ring.
- (2) R is a Noetherian pure-injective ring with T -nilpotent Jacobson radical.
- (3) R is a \sum -pure-injective ring and $J(R)$ is finitely generated.

Proof. (1) \Rightarrow (2) is clear by [11, Corollary 4.2].

(2) \Rightarrow (3) follows from Proposition 2.3(a) and [3, Example 1.41].

(3) \Rightarrow (1) Assume that R is a \sum -pure-injective ring and $J(R)$ is finitely generated. We know that any \sum -pure-injective ring is semiprimary (i.e., $R/J(R)$ is semisimple and $J(R)$ is nilpotent) and so R is semiprimary. Also, $J(R)^n$ is finitely generated for each $n \in \mathbb{N}$. Hence, the finitely generated semisimple R -module $J(R)^n/J(R)^{n+1}$ is Artinian for each $n \in \mathbb{N}$. Also, $J(R)$ is nilpotent and so it follows that R is an Artinian ring. \square

A famous conjectures on quasi-Frobenius rings is the Faith-Menal conjecture: “every strongly right Johns ring is quasi-Frobenius?”

Faith and Menal [6] gave a counter example to show that right Johns rings may not be right artinian. They characterized strongly right Johns rings as right noetherian and left FP-injective rings (see [7, Theorem 1.1]). Now, we apply Rutter’s result ([15, Corollary 3]) to show that Faith-Menal conjecture is true when R is right pure-injective.

Theorem 2.5. *A ring R is quasi-Frobenius if and only if R is a right pure-injective right Noetherian left 2-injective ring.*

Proof. Assume that R is a left 2-injective, right Noetherian and right pure-injective ring. Thus, by Proposition 2.3(c), R is a right Artinian ring and so R has ascending chain condition (ACC) on left annihilators. Also, Rutter in [15, Corollary 3] proved that a left 2-injective ring with ACC on left annihilators is quasi-Frobenius. Therefore, R is a quasi-Frobenius ring. The converse is clear. \square

Since every left FP-injective ring is left 2-injective, the following result follows from Theorem 2.5.

Corollary 2.6. *A ring R is quasi-Frobenius if and only if R is a right pure-injective strongly right Johns ring.*

The following theorem generalizes the result of Couchot [2, Proposition I.2].

Theorem 2.7. *For a commutative ring R , the following statements are equivalent:*

- (1) R is a quasi-Frobenius ring.
- (2) R is a Noetherian RD -injective ring with T -nilpotent Jacobson radical.
- (3) R is a Noetherian pure-injective ring with T -nilpotent Jacobson radical and $E(R)$ is an RD -projective R -module.

Proof. (1) \Rightarrow (2) and (1) \Rightarrow (3) are clear.

(2) \Rightarrow (1) Assume that R is a Noetherian RD -injective ring with T -nilpotent Jacobson radical. Thus, by Corollary 2.4, R is Artinian, since every RD -injective module is pure-injective. Therefore, [2, Proposition I.2] allows us to conclude.

(3) \Rightarrow (1) Assume that R is a Noetherian pure-injective ring with T -nilpotent Jacobson radical and $E(R)$ is an RD -projective R -module. Thus, by Corollary 2.4, R is Artinian. Without loss of generality, we can assume that R is a local ring. So, by [16, Corollary 2], $E(R)$ is a direct sum of cyclically presented R -modules. Since R is Artinian, $E(R)$ is a finite direct sum of indecomposable cyclic R -modules. Assume that Rx is an indecomposable cyclic direct summand of $E(R)$ where $x \in E(R)$. So, Rx has a simple R -submodule, since Rx is an Artinian R -module. Also, since any indecomposable injective module is the injective envelope of each its submodule, we conclude that $Rx \cong E(R/\mathcal{M})$ where \mathcal{M} is the maximal ideal of R . Now, $\text{Ann}(x) = 0$, since $E(R/\mathcal{M})$ is faithful. Thus, $Rx \cong R$ and so R is self-injective. Therefore, R is a quasi-Frobenius ring. \square

Recall that a ring R is said to be *right hereditary* if every right ideal of R is projective.

Corollary 2.8. *For a commutative ring R , the following statements are equivalent:*

- (1) R is a semisimple ring.
- (2) R is Noetherian pure-injective with T -nilpotent projective Jacobson radical.
- (3) R is hereditary Noetherian pure-injective with T -nilpotent Jacobson radical.

Proof. (1) \Rightarrow (2) is clear.

(2) \Rightarrow (3) follows from Corollary 2.4 and Auslander's Theorem [10, Theorem 5.72].

(3) \Rightarrow (1) Assume that R is a hereditary Noetherian pure-injective ring with T -nilpotent Jacobson radical. Thus, by Corollary 2.4, R is Artinian and so $E(R)$ is finitely generated by [10, Theorem 3.64]. So, $E(R)$ is finitely presented and so it is pure-projective. Thus, by [12, Proposition 2.11], $E(R)$ is RD -projective and so Theorem 2.7 allows us to conclude. \square

Faith and Menal [6] gave a counter example to show that right Johns rings need not be right artinian. Also, there is a two-sided Artinian right Johns

ring which is not quasi-Frobenius as shown by Rutter [15, Example 1]. The following result shows that every right Johns right pure-injective ring is right Artinian.

Proposition 2.9. *Every right Johns right pure-injective ring is a right Artinian right CS right CF ring*

Proof. Assume that R is a right Johns right pure-injective ring. Thus, by Proposition 2.3(c) and since every right Johns ring is left P-injective, R is a right Artinian ring. Also, by [13, Theorem 8.9], R is a right CS right CF ring. \square

From Proposition 2.9 and [13, Theorem 7.1], we have:

Corollary 2.10. *A ring R is quasi-Frobenius if and only if R is a left and right Johns left and right pure-injective ring.*

A right Noetherian ring with essential right socle need not be right Artinian as shown by Faith-Menal example [6]. The following theorem shows that the pure-injectivity of R is strong enough to force a right Noetherian ring with right essential left socle to be right Artinian. Also, it may be viewed as a onesided version of a result of Ginn and Moss on two-sided Noetherian rings with essential socle [8].

Theorem 2.11. *If R is a right pure-injective right Noetherian ring and the left socle of R is right essential, then R is a right Artinian left Kasch ring.*

Proof. Assume that R is a right pure-injective right Noetherian ring and $\text{Soc}({}_R R) \subseteq^{ess} R_R$. First, we show that R is a right Artinian ring. To prove, by Proposition 2.3(a), it is enough to show that $J(R)$ is right T-nilpotent. Suppose that a_1, a_2, a_3, \dots is a sequence in $J(R)$. Now consider

$$\text{Ann}_r(a_1) \subseteq \text{Ann}_r(a_2 a_1) \subseteq \text{Ann}_r(a_3 a_2 a_1) \subseteq \dots$$

Since R is right Noetherian, there exists $n \in \mathbb{N}$ such that

$$\text{Ann}_r(a_n a_{n-1} \cdots a_1) = \text{Ann}_r(a_{n+1} a_n \cdots a_1).$$

This implies that

$$(a_n a_{n-1} \cdots a_1)R \cap \text{Ann}_r(a_{n+1}) = 0.$$

Also, since $J(R)\text{Soc}({}_R R) = 0$, we have $a_{n+1}\text{Soc}({}_R R) = 0$. Thus $\text{Soc}({}_R R) \subseteq \text{Ann}_r(a_{n+1})$ and so $\text{Ann}_r(a_{n+1}) \subseteq^{ess} R_R$, since $\text{Soc}({}_R R) \subseteq^{ess} R_R$. Therefore, $a_n a_{n-1} \cdots a_1 = 0$, as required. Moreover, by [13, Lemma 1.48], R is a left Kasch ring. \square

The following example shows that there exist two-sided Artinian, left and right CS rings R that are not quasi-Frobenius.

Example 2.12. Assume that $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$, where F is a field. Then R is a left and right artinian, left and right CS ring. But R is not quasi-Frobenius, because the R -module $e_2 R$ is not injective.

We know that a ring R is quasi-Frobenius if and only if every projective R -module is injective. We have the following result, which generalizes this fact in the cases R is a commutative coherent ring or R is a commutative ring with finitely generated Jacobson radical.

Proposition 2.13. *Let R be a ring. Consider the following conditions:*

- (1) R is a quasi-Frobenius ring.
- (2) R is a right Noetherian right P -injective left mininjective ring.
- (3) R is a right pure-injective right Johns left min-CS ring.
- (4) R is a right Noetherian left P -injective right pure-injective and left min-CS ring with $\text{Soc}({}_R R) \subseteq^{ess} R_R$.
- (5) R is a right Noetherian left P -injective right pure-injective and left min-CS ring with $\text{Soc}(R_R) \subseteq \text{Soc}({}_R R)$.
- (6) R is a coherent ring such that every projective R -module is RD -injective.
- (7) $J(R)$ is finitely generated and every projective R -module is RD -injective.

Then:

- (a) Conditions (1)-(5) are equivalent and imply conditions (6) and (7).
- (b) When R is a commutative ring, the seven conditions are equivalent.

Proof. Clearly (1) implies the conditions (2)-(7).

(2) \Rightarrow (1) Assume that R is a right Noetherian right P -injective left mininjective ring. Thus, by [13, Proposition 5.15], R is a left Artinian ring. Also, since R is left and right mininjective, $\text{Soc}({}_R R) = \text{Soc}(R_R)$ [13, Theorem 2.21]. In particular, R is semiperfect and left mininjective, so $\text{Soc}({}_R R)$ is finite dimensional as a right R -module [13, Theorem 3.7]. Therefore, by [13, Lemma 3.30], R is right Artinian and so by Ikeda's theorem [13, Theorem 2.30], R is quasi-Frobenius.

(3) \Rightarrow (4) Assume that R is a right Johns right pure-injective left min-CS ring. Thus, by Proposition 2.9, R is right Artinian. Therefore, [13, Theorem 8.9] allows us to conclude.

(4) \Rightarrow (5) Given (4), so by Theorem 2.11, R is a right Artinian left Kasch ring. Thus, by [13, Lemma 4.5], $\text{Soc}({}_R R) \subseteq^{ess} R_R$. Hence, R is a left GPF ring (i.e., R is left P -injective, semiperfect, and $\text{Soc}({}_R R) \subseteq^{ess} R_R$) and so $\text{Soc}({}_R R) = \text{Soc}(R_R)$ by [13, Theorem 5.31].

(5) \Rightarrow (1) Given (5), so by Proposition 2.3(c), R is right Artinian. Also, by [13, Theorem 2.21], $\text{Soc}({}_R R) \subseteq \text{Soc}(R_R)$, since R is left P -injective and so by hypothesis $\text{Soc}({}_R R) = \text{Soc}(R_R) \subseteq^{ess} R_R$. Thus, [13, Lemma 4.4] implies that R is right mininjective. Since a left P -injective ring is left mininjective, R is two-sided mininjective and right artinian. Therefore, by [13, Theorem 3.31], R is a quasi-Frobenius ring.

(6) \Rightarrow (1) Assume that R is a commutative coherent ring such that every projective R -module is RD -injective. Thus every projective R -module is pure-injective, since every RD -injective is pure-injective. So, R is Σ -pure-injective,

since any direct sum of projective module is projective. Thus, R is semiprimary and so R is a perfect ring. Therefore, R is Artinian and so by Theorem 2.7, R is quasi-Frobenius.

(7) \Rightarrow (1) Similar to the proof of (6) \Rightarrow (1), R is Σ -pure-injective and so by Corollary 2.4, R is Artinian. Therefore, Theorem 2.7 allows us to conclude. \square

According to a famous result of Faith and Walker, a ring R is quasi-Frobenius (i.e., R is left or right Artinian and left or right self-injective) if and only if every R -module embeds in a projective module. Recall that a ring R is called *right CF* if every cyclic right R -module embeds in a free right R -module. It is still open “*whether a right CF ring is right Artinian?*”. In [9, Theorem 2.6], Gómez Pardo proved that every right (almost-)coherent right CF ring is right Artinian. Now, we obtain the following theorem that is an analogue of Gómez Pardo’s theorem.

Recall that a module M_R is said to be *semi-artinian* if every nonzero factor module of M has nonzero socle, and that a ring R is right semi-artinian if R_R is a semi-artinian module.

Theorem 2.14. *Every right coherent right semi-artinian right Kasch ring is right Artinian.*

Proof. Assume that R is a right coherent right semi-artinian right Kasch ring. So, by hypothesis, every simple right R -module embeds in R_R . It follows that every simple right R -module is finitely presented, since R is right coherent [10, Corollary 4.52]. Now, suppose that \mathcal{M} is a maximal right ideal of R and so $R/\mathcal{M} \cong F/K$ where K is a finitely generated submodule of finitely generated free right R -module F . Consider the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \hookrightarrow & F & \longrightarrow & F/K & \longrightarrow & 0 \\ & & & & & & \wr \downarrow & & \\ 0 & \longrightarrow & \mathcal{M} & \hookrightarrow & R & \longrightarrow & R/\mathcal{M} & \longrightarrow & 0. \end{array}$$

By Schanuel’s Lemma, we have that $F \oplus \mathcal{M} \cong R \oplus K$. Thus, \mathcal{M} is finitely generated, since K is finitely generated. So, every maximal right ideal of R is finitely generated. Therefore, [14, Proposition 4.8(1)] allows us to conclude. \square

Recall that a ring R is said to be *right IF ring* if every injective right R -module is flat. Colby and Würfel proved that a ring R is right *IF* if and only if all finitely presented right R -modules embed in a free right R -module.

Corollary 2.15. *A ring R is quasi-Frobenius if and only if it is a right semi-artinian right Kasch IF ring.*

Proof. Assume that R is a right semi-artinian right Kasch *IF* ring. So, by Theorem 2.14 and since every *IF* ring is right coherent, R is right Artinian. Also, by Bass’s theorem, over a right perfect ring, any flat right R -module is projective, hence any injective right R -module is projective. Therefore, R is a quasi-Frobenius ring. The converse is clear. \square

A *uniform module* is a nonzero module M such that the intersection of any two nonzero submodules of M is nonzero, or, equivalently, such that every nonzero submodule of M is essential in M .

Proposition 2.16. *Let R_R be uniform and $J(R)$ left T -nilpotent. Then R is right self-injective if and only if it is right pure-injective and $E(R_R)$ is RD -projective.*

Proof. Assume that R is a right pure-injective ring and $E(R_R)$ is RD -projective. Thus, R is a indecomposable right pure-injective ring, since R_R is uniform. So, by [3, Corollary 2.27], $R \cong \text{End}(R_R)$ is local and so by hypothesis R is a left perfect ring. Also, by [4, Theorem 4.6], over one-sided perfect ring, every RD -projective right (left) R -module is a direct sum of finitely presented cyclic modules. So, $E(R_R)$ is a direct sum of finitely presented cyclic modules. Also, $E(R_R)$ is indecomposable, since R_R is uniform. This implies that $E(R_R)$ is cyclic. Also, Faith [5, Lemma 2] proved every left perfect ring with cyclic right injective envelope is right self-injective. So, R is a right self-injective ring. The converse is clear. \square

Now, the following result follows from Proposition 2.16 and [13, Theorems 1.50].

Corollary 2.17. *A right uniform ring R is quasi-Frobenius if and only if it is right pure-injective with left T -nilpotent Jacobson radical such that R has ACC on left or right annihilators and $E(R_R)$ is RD -projective.*

Recall that a ring R is said to be *semiprimitive* if $J(R) = 0$.

Lemma 2.18. *Every semiprimitive right pure-injective ring is a right self-injective von Neumann regular ring.*

Proof. Assume that R is a semiprimitive right pure-injective ring. So, by [17, Theorem 9], $R/J(R)$ is a von Neumann regular ring. Thus, R is a von Neumann regular ring, since R is semiprimitive. Thus, every R -module is flat and so every exact sequence is pure exact. Therefore, this implies that R is right self-injective. \square

Proposition 2.19. *For a ring R with $\text{Soc}(R_R) \subseteq^{ess} R_R$, the following statements are equivalent:*

- (1) R is a right self-injective von Neumann regular ring.
- (2) R is a right pure-injective ring and $\text{Soc}(R_R)$ is FP -injective.
- (3) R is a right pure-injective ring and $\text{Soc}(R_R)$ is a pure submodule of R_R .

Proof. (1) \Rightarrow (2) and (2) \Rightarrow (3) are clear, since over a von Neumann regular ring every submodule of a module is pure submodule.

(3) \Rightarrow (1) Assume that R is a right pure-injective ring in which $\text{Soc}(R_R)$ is an essential pure submodule of R_R . Thus, by [16, Proposition 3], we have

$$\text{Soc}(R_R)J(R) = RJ(R) \cap \text{Soc}(R_R) = J(R) \cap \text{Soc}(R_R).$$

Since $\text{Soc}(R_R)$ is an essential submodule of R_R and $\text{Soc}(R_R)J(R) = 0$, we have $J(R) = 0$. Therefore, Lemma 2.18 allows us to conclude. \square

Assume that M is a right R -module. Then

$$Z(M) := \{m \in M \mid \text{Ann}_r(m) \text{ is essential in } R_R\}$$

is a singular submodule of M . A right R -module M is *singular* if $Z(M_R) = M$ and *non-singular* if $Z(M_R) = 0$. A ring R is called *right non-singular* if the right R -module R_R is non-singular.

Corollary 2.20. *A right non-singular ring R is right self-injective if and only if it is a right pure-injective right p -injective ring.*

Proof. Assume that R is a right pure-injective right p -injective ring. Thus, by [13, Theorem 5.14], $J(R) = Z(R_R)$ and so $J(R) = 0$. Therefore, Lemma 2.18 allows us to conclude. The converse is clear. \square

Corollary 2.21. *A ring R is semisimple if and only if R is a right pure-injective right hereditary right p -injective ring.*

Proof. Assume that R is a right pure-injective right hereditary right p -injective ring. Thus, by Corollary 2.20, R is a right self-injective ring, since every right hereditary ring is right non-singular. Also, we know that over a right hereditary ring, every non-zero factor module of injective modules is injective. Therefore, every cyclic right R -module is injective and so R is a semisimple ring. The converse is clear. \square

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