

BOUNDS FOR RADII OF CONVEXITY OF SOME q -BESSEL FUNCTIONS

İBRAHİM AKTAŞ AND HALİT ORHAN

ABSTRACT. In the present investigation, by applying two different normalizations of the Jackson's second and third q -Bessel functions tight lower and upper bounds for the radii of convexity of the same functions are obtained. In addition, it was shown that these radii obtained are solutions of some transcendental equations. The known Euler-Rayleigh inequalities are intensively used in the proof of main results. Also, the Laguerre-Pólya class of real entire functions plays an important role in this work.

1. Introduction

Let \mathbb{D}_r be the open disk $\{z \in \mathbb{C} : |z| < r\}$ with radius $r > 0$ and $\mathbb{D}_1 = \mathbb{D}$. Let \mathcal{A} denote the class of analytic functions $f : \mathbb{D}_r \rightarrow \mathbb{C}$,

$$f(z) = z + \sum_{n \geq 2} a_n z^n,$$

which satisfy the normalization conditions $f(0) = f'(0) - 1 = 0$. By \mathcal{S} we mean the class of functions belonging to \mathcal{A} which are univalent in \mathbb{D}_r . The class of convex functions is defined by

$$\mathcal{K} = \left\{ f \in \mathcal{S} : \Re \left(1 + \frac{z f''(z)}{f'(z)} \right) > 0 \text{ for all } z \in \mathbb{D}_r \right\}.$$

It is known that the convex functions do not need to be normalized, namely, the definition of \mathcal{K} is also valid non-normalized analytic function $f : \mathbb{D} \rightarrow \mathbb{C}$ which has the property $f'(0) \neq 0$. The radius of convexity of an analytic locally univalent function $f : \mathbb{C} \rightarrow \mathbb{C}$ is defined by

$$r^c(f) = \sup \left\{ r > 0 : \Re \left(1 + \frac{z f''(z)}{f'(z)} \right) > 0 \text{ for all } z \in \mathbb{D}_r \right\}.$$

Received March 4, 2019; Accepted August 19, 2019.

2010 *Mathematics Subject Classification.* Primary 30C45, 30C15, 33C10.

Key words and phrases. convex functions, radius of convexity, Mittag-Leffler expansions, q -Bessel functions, zeros of q -Bessel functions, Laguerre-Pólya class of entire functions.

Note that $r^c(f)$ is the largest radius for which the image domain $f(\mathbb{D}_{r^c(f)})$ is a convex domain in \mathbb{C} . For more information about convex functions we refer to Duren's book [17] and to the references therein.

The Jackson's second and third q -Bessel functions are defined as follows:

$$J_\nu^{(2)}(z; q) = \frac{(q^{\nu+1}; q)_\infty}{(q; q)_\infty} \sum_{n \geq 0} \frac{(-1)^n \left(\frac{z}{2}\right)^{2n+\nu}}{(q; q)_n (q^{\nu+1}; q)_n} q^{n(n+\nu)}$$

and

$$J_\nu^{(3)}(z; q) = \frac{(q^{\nu+1}; q)_\infty}{(q; q)_\infty} \sum_{n \geq 0} \frac{(-1)^n z^{2n+\nu}}{(q; q)_n (q^{\nu+1}; q)_n} q^{\frac{1}{2}n(n+1)},$$

where $z \in \mathbb{C}, \nu > -1, q \in (0, 1)$ and

$$(a; q)_0 = 1, (a; q)_n = \prod_{k=1}^n (1 - aq^{k-1}), (a, q)_\infty = \prod_{k \geq 1} (1 - aq^{k-1}).$$

It is known that the Jackson's second and third q -Bessel functions are q -extensions of the classical Bessel function of the first kind J_ν . Clearly, for fixed z we have

$$J_\nu^{(2)}((1-z)q; q) \rightarrow J_\nu(z)$$

and

$$J_\nu^{(3)}((1-z)q; q) \rightarrow J_\nu(2z)$$

as $q \nearrow 1$. The readers can find the properties of Jackson's second and third q -Bessel functions in [18, 19, 21, 22] and also comprehensive information on the Bessel function of the first kind can be found in Watson's treatise [25]. Recently, the geometric properties of some special functions (like Bessel, Struve, Lommel and Wright functions) have been investigated by many authors (see [2–6, 8–16, 20, 23, 26]). Also, the authors in [1, 7] have studied the radii of starlikeness and convexity of some q -Bessel functions. In particular, tight lower and upper bounds for the radii of starlikeness of some q -Bessel functions were obtained in [1]. Most of above papers benefited from some properties of the positive zeros of some special functions. Also, the Laguerre-Pólya class \mathcal{LP} of real entire functions, which consist of uniform limits of real polynomials whose zeros are all real, was used intensively (for more details on the Laguerre-Pólya class of entire functions we refer to [7] and to the references therein). A real entire function q belongs to the Laguerre-Pólya class \mathcal{LP} if it can be represented in the form

$$q(x) = cx^m e^{-\alpha x^2 + \beta x} \prod_{n \geq 1} \left(1 + \frac{x}{x_n}\right) e^{-\frac{x}{x_n}},$$

where c, β, x_n are real numbers, $\alpha \geq 0$, m is a natural number or zero, and $\sum_{n \geq 1} x_n^{-2}$ converges. Motivated by the earlier works, in this work our main aim is to give some lower and upper bounds for the radii of convexity of some normalized q -Bessel functions. The results presented in this paper complement the results of [7] about the radii of convexity and extend the known results

from [2] on classical Bessel functions of the first kind to q -Bessel functions. In this study we consider two different normalized forms of Jackson's second and third q -Bessel functions which are analytic in the unit disk of the complex plane. Because the functions $J_\nu^{(2)}(\cdot; q)$ and $J_\nu^{(3)}(\cdot; q)$ do not belong to \mathcal{A} , first we consider the following four normalized forms as in [7]. For $\nu > -1$,

$$\begin{aligned} g_\nu^{(2)}(z; q) &= 2^\nu c_\nu(q) z^{1-\nu} J_\nu^{(2)}(z; q) = \sum_{n \geq 0} \frac{(-1)^n q^{n(n+\nu)}}{4^n (q; q)_n (q^{\nu+1}; q)_n} z^{2n+1}, \\ h_\nu^{(2)}(z; q) &= 2^\nu c_\nu(q) z^{1-\frac{\nu}{2}} J_\nu^{(2)}(\sqrt{z}; q) = \sum_{n \geq 0} \frac{(-1)^n q^{n(n+\nu)}}{4^n (q; q)_n (q^{\nu+1}; q)_n} z^{n+1}, \\ g_\nu^{(3)}(z; q) &= c_\nu(q) z^{1-\nu} J_\nu^{(3)}(z; q) = \sum_{n \geq 0} \frac{(-1)^n q^{\frac{1}{2}n(n+1)}}{(q; q)_n (q^{\nu+1}; q)_n} z^{2n+1}, \\ h_\nu^{(3)}(z; q) &= c_\nu(q) z^{1-\frac{\nu}{2}} J_\nu^{(3)}(\sqrt{z}; q) = \sum_{n \geq 0} \frac{(-1)^n q^{\frac{1}{2}n(n+1)}}{(q; q)_n (q^{\nu+1}; q)_n} z^{n+1}, \end{aligned}$$

where $c_\nu(q) = (q; q)_\infty / (q^{\nu+1}; q)_\infty$. As a result of the above normalizations, all of the above functions belong to the class \mathcal{A} .

2. Bounds for the radii of convexity of some normalized q -Bessel functions

In this section we give some tight lower and upper bounds for the radii of convexity of the above mentioned four normalized forms of the Jackson's second and third q -Bessel functions. Also, we show that the radii of convexity of the above functions are solutions of some transcendental equations.

Theorem 2.1. *Let $\nu > -1$. Then the radius of convexity $r^c \left(g_\nu^{(2)}(z; q) \right)$ of the function*

$$z \mapsto g_\nu^{(2)}(z; q) = 2^\nu c_\nu(q) z^{1-\nu} J_\nu^{(2)}(z; q)$$

is the smallest positive root of the equation

$$(1-\nu)^2 J_\nu^{(2)}(r; q) + (3-2\nu)rdJ_\nu^{(2)}(r; q)/dr + r^2 d^2 J_\nu^{(2)}(r; q)/dr^2 = 0$$

and satisfies the following inequalities

$$\begin{aligned} \sqrt{\frac{4(1-q^{\nu+1})(1-q)}{9q^{\nu+1}}} &< r^c \left(g_\nu^{(2)}(z; q) \right) < \sqrt{\frac{36(q^2-1)(1-q^{\nu+1})(1-q^{\nu+2})}{q^{\nu+1}S_\nu(q)}}, \\ 2\sqrt[4]{\frac{(1+q)(1-q)^2(1-q^{\nu+1})^2(q^{\nu+2}-1)}{q^{2(\nu+1)}S_\nu(q)}} &< r^c \left(g_\nu^{(2)}(z; q) \right) \\ &< \sqrt{\frac{4(1-q^{\nu+1})(1-q^{\nu+3})T(q)S_\nu(q)}{q^{\nu+1}(1+q)(P_\nu(q)+R_\nu(q))}}, \end{aligned}$$

where

$$\begin{aligned} P_\nu(q) &= 1458q - 729q^{\nu+2} - 1512q^{\nu+3} - 2241q^{\nu+4} - 837q^{\nu+5} - 54q^{\nu+6} \\ &\quad + 479q^{\nu+7} + 729q^{2\nu+5}, \\ R_\nu(q) &= 783q^{2\nu+6} + 783q^{2\nu+7} + 152q^{2\nu+8} + 98q^6 - 675q^4 + 54q^3 \\ &\quad + 783q^2 + 729, \\ S_\nu(q) &= 31q^{\nu+3} + 81q^{\nu+2} + 50q^2 - 81q - 81 \end{aligned}$$

and

$$T(q) = (q-1)(q^3 + 2q^2 + 2q + 1).$$

Note that by multiplying by $(1-q)^{-1}$ both sides of the above inequalities and taking the limit as $q \nearrow 1$ for $\nu > -1$ we obtain the first two inequalities of [2, Theorem 6], namely:

$$(1) \quad \frac{2\sqrt{\nu+1}}{3} < r^c(g_\nu) < 6\sqrt{\frac{(\nu+1)(\nu+2)}{56\nu+137}}$$

and

$$(2) \quad 2\sqrt[4]{\frac{(\nu+1)^2(\nu+2)}{56\nu+137}} < r^c(g_\nu) < \sqrt{\frac{2(\nu+1)(\nu+3)(56\nu+137)}{208\nu^2+1172\nu+1693}},$$

where $r^c(g_\nu)$ stands for the radii of convexity of the normalized Bessel function

$$z \mapsto g_\nu(z) = 2^\nu \Gamma(\nu+1) z^{1-\nu} J_\nu(z).$$

Theorem 2.2. *Let $\nu > -1$. Then the radius of convexity $r^c(h_\nu^{(2)}(z; q))$ of the function*

$$z \mapsto h_\nu^{(2)}(z; q) = 2^\nu c_\nu(q) z^{1-\frac{\nu}{2}} J_\nu^{(2)}(\sqrt{z}; q)$$

is the smallest positive root of the equation

$$(2-\nu)^2 J_\nu^{(2)}(\sqrt{r}; q) + (5-2\nu) \sqrt{r} dJ_\nu^{(2)}(\sqrt{r}; q)/dr + rd^2 J_\nu^{(2)}(\sqrt{r}; q)/dr^2 = 0$$

and satisfies the following inequalities

$$\begin{aligned} \frac{(1-q)(1-q^{\nu+1})}{q^{\nu+1}} < r^c(h_\nu^{(2)}(z; q)) < \frac{8(q^{\nu+1}-1)(q^{\nu+2}-1)(1-q^2)}{q^{\nu+1}U_\nu(q)}, \\ \sqrt{\frac{8(1-q)^2(1+q)(1-q^{\nu+1})^2(1-q^{\nu+2})}{q^{2\nu+2}U_\nu(q)}} < r^c(h_\nu^{(2)}(z; q)) \\ < \frac{2(1-q^{\nu+1})(q^{\nu+3}-1)U_\nu(q)T(q)}{(1+q)q^{\nu+1}(M_\nu(q)+N_\nu(q))}, \end{aligned}$$

where

$$\begin{aligned} U_\nu(q) &= (8q - 8q^{\nu+2} + q^{\nu+3} - 9q^2 + 8), \\ M_\nu(q) &= 32q - 16q^{\nu+2} - 21q^{\nu+3} - 37q^{\nu+4} + 6q^{\nu+5} + 11q^{\nu+6} + 3q^{\nu+7} \end{aligned}$$

and

$$N_\nu(q) = 16q^{2\nu+5} + 5q^{2\nu+6} + 5q^{2\nu+7} + q^{2\nu+8} + 5q^2 - 11q^3 - 27q^4 + 12q^6 + 16.$$

Here we would like to emphasize that by multiplying by $(1-q)^{-2}$ both sides of the above inequalities and taking the limit as $q \nearrow 1$ for $\nu > -1$ we obtain the first two inequalities of [2, Theorem 7], namely:

$$(3) \quad \nu + 1 < r^c(h_\nu) < \frac{16(\nu+1)(\nu+2)}{7\nu+23}$$

and

$$(4) \quad \sqrt{\frac{16(\nu+1)^2(\nu+2)}{7\nu+23}} < r^c(h_\nu) < \frac{2(\nu+1)(\nu+3)(7\nu+23)}{9\nu^2+60\nu+115},$$

where $r^c(h_\nu)$ stands for the radii of convexity of the normalized Bessel function

$$z \mapsto h_\nu(z) = 2^\nu \Gamma(\nu+1) z^{1-\frac{\nu}{2}} J_\nu(\sqrt{z}).$$

Theorem 2.3. *Let $\nu > -1$. Then the radius of convexity $r^c(g_\nu^{(3)}(z; q))$ of the function*

$$z \mapsto g_\nu^{(3)}(z; q) = c_\nu(q) z^{1-\nu} J_\nu^{(3)}(z; q)$$

is the smallest positive root of the equation

$$(1-\nu)^2 J_\nu^{(3)}(r; q) + (3-2\nu) r dJ_\nu^{(3)}(r; q)/dr + r^2 d^2 J_\nu^{(3)}(r; q)/dr^2 = 0$$

and satisfies the following inequalities

$$\begin{aligned} \sqrt{\frac{(1-q^{\nu+1})(1-q)}{9q}} < r^c(g_\nu^{(3)}(z; q)) < \sqrt{\frac{9(q^2-1)(1-q^{\nu+1})(1-q^{\nu+2})}{qY_\nu(q)}}, \\ \sqrt[4]{\frac{(1+q)(1-q)^2(1-q^{\nu+1})^2(q^{\nu+2}-1)}{q^2Y_\nu(q)}} < r^c(g_\nu^{(3)}(z; q)) \\ < \sqrt{\frac{(1-q^{\nu+1})(1-q^{\nu+3})T(q)Y_\nu(q)}{3q(q+1)(\theta_\nu(q)+\varphi_\nu(q))}}, \end{aligned}$$

where

$$\theta_\nu(q) = 261q - 18q^{\nu+2} - 504q^{\nu+3} - 620q^{\nu+4} - 504q^{\nu+5} - 18q^{\nu+6} + 67q^{2\nu+5},$$

$$\varphi_\nu(q) = 261q^{2\nu+6} + 261q^{2\nu+7} + 243q^{2\nu+8} + 67q^3 + 261q^2 + 243$$

and

$$Y_\nu(q) = 81q^{\nu+3} + 31q^{\nu+2} - 31q - 81.$$

Note that by multiplying by $(1-q)^{-1}$ both sides of the above inequalities and taking the limit as $q \nearrow 1$ for $\nu > -1$ we obtain the following inequalities

$$(5) \quad \frac{\sqrt{\nu+1}}{3} < r^c(g_\nu) < 3\sqrt{\frac{(\nu+1)(\nu+2)}{56\nu+137}}$$

and

$$(6) \quad \sqrt[4]{\frac{(\nu+1)^2(\nu+2)}{56\nu+137}} < r^c(g_\nu) < \sqrt{\frac{(\nu+1)(\nu+3)(56\nu+137)}{2(208\nu^2+1172\nu+1693)}}.$$

Theorem 2.4. *Let $\nu > -1$. Then the radius of convexity $r^c(h_\nu^{(3)}(z; q))$ of the function*

$$z \mapsto h_\nu^{(3)}(z; q) = c_\nu(q)z^{1-\frac{\nu}{2}}J_\nu^{(3)}(\sqrt{z}; q)$$

is the smallest positive root of the equation

$$(2-\nu)^2 J_\nu^{(3)}(\sqrt{r}; q) + (5-2\nu)\sqrt{r}dJ_\nu^{(3)}(\sqrt{r}; q)/dr + rd^2 J_\nu^{(3)}(\sqrt{r}; q)/dr^2 = 0$$

and satisfies the following inequalities

$$\begin{aligned} \frac{(1-q)(1-q^{\nu+1})}{4q} < r^c(h_\nu^{(3)}(z; q)) < \frac{2(q^{\nu+1}-1)(q^{\nu+2}-1)(q^2-1)}{qE_\nu(q)}, \\ \sqrt{\frac{(1-q)^2(1+q)(1-q^{\nu+1})^2(q^{\nu+2}-1)}{2q^2E_\nu(q)}} < r^c(h_\nu^{(3)}(z; q)) \\ < \frac{(1-q^{\nu+1})(q^{\nu+3}-1)E_\nu(q)T(q)}{2q(1+q)(K_\nu(q)+L_\nu(q))}, \end{aligned}$$

where

$$E_\nu(q) = (8q^{\nu+3} - q^{\nu+2} + q - 8),$$

$$K_\nu(q) = 5q + 11q^{\nu+2} - 21q^{\nu+3} - 34q^{\nu+4} - 21q^{\nu+5} + 11q^{\nu+6}$$

and

$$L_\nu(q) = q^{2\nu+5} + 5q^{2\nu+6} + 5q^{2\nu+7} + 16q^{2\nu+8} + 5q^2 + q^3 + 16.$$

Here we would like to emphasize that by multiplying by $(1-q)^{-2}$ both sides of the above inequalities and taking the limit as $q \nearrow 1$ for $\nu > -1$, we obtain the next two inequalities

$$(7) \quad \frac{\nu+1}{4} < r^c(h_\nu) < \frac{4(\nu+1)(\nu+2)}{7\nu+23}$$

and

$$(8) \quad \sqrt{\frac{(\nu+1)^2(\nu+2)}{7\nu+23}} < r^c(h_\nu) < \frac{(\nu+1)(\nu+3)(7\nu+23)}{2(9\nu^2+60\nu+115)}.$$

It is important to mention that by making a comparison among of above obtained inequalities we have that the left-hand sides of (5) and (6) are weaker than the left hand sides of (1) and (2), respectively. However, the right-hand sides of (5) and (6) improve the right-hand sides of (1) and (2), respectively. On the other hand, the left-hand sides of (7) and (8) are weaker than the left-hand sides of (3) and (4), while the right-hand sides of (7) and (8) improve the right-hand sides of (3) and (4).

Proof of Theorem 2.1. By using the Alexander duality theorem for starlike and convex functions we can say that the function $g_\nu^{(2)}(z; q)$ is convex if and only if $z \mapsto z \left(g_\nu^{(2)}(z; q) \right)'$ is starlike. But, the smallest positive zero of $z \mapsto \left(z \left(g_\nu^{(2)}(z; q) \right)' \right)'$ is actually the radius of starlikeness of $z \mapsto z \left(g_\nu^{(2)}(z; q) \right)'$, according to [9, 10]. Therefore, the radius of convexity $r^c(g_\nu^{(2)})$ is the smallest positive root of the equation $\left(r \left(g_\nu^{(2)}(r; q) \right)' \right)' = 0$. Thus, we get that the radius of convexity of the function $z \mapsto g_\nu^{(2)}(z; q)$ is the smallest positive root of the equation

$$(1 - \nu)^2 J_\nu^{(2)}(r; q) + (3 - 2\nu)rdJ_\nu^{(2)}(r; q)/dr + r^2 d^2 J_\nu^{(2)}(r; q)/dr^2 = 0.$$

Now, recall that the zeros $j_{\nu, n}(q), n \in \mathbb{N}$, of the Jackson's second q -Bessel function are all real and simple, according to [18, Theorem 4.2]. Then, the function $g_\nu^{(2)}(\cdot; q)$ belongs to the Laguerre-Pólya class \mathcal{LP} of real entire functions. Because of the properties of the class \mathcal{LP} the function $z \mapsto \left(z \left(g_\nu^{(2)}(z; q) \right)' \right)'$ belongs also to the class \mathcal{LP} . Hence the function $z \mapsto \left(z \left(g_\nu^{(2)}(z; q) \right)' \right)'$ has only real zeros. Also its growth order ρ is 0, that is

$$\rho = \lim_{n \rightarrow \infty} \frac{n \log n}{2n \log 2 + \log(q; q)_n + \log(q^{\nu+1}; q)_n - 2 \log(2n + 1) - n(n + \nu) \log q} = 0,$$

since as $n \rightarrow \infty$ we have $(q; q)_n \rightarrow (q; q)_\infty < \infty$ and $(q^{\nu+1}; q)_n \rightarrow (q^{\nu+1}; q)_\infty < \infty$. Now, by applying Hadamard's Theorem [24, p. 26] we obtain

$$G_\nu(z; q) = \left(z \left(g_\nu^{(2)}(z; q) \right)' \right)' = \prod_{n \geq 1} \left(1 - \frac{z^2}{(\alpha_{\nu, n}(q))^2} \right),$$

where $\alpha_{\nu, n}(q)$ is the n th zero of the function $G_\nu(\cdot; q)$. Now, via logarithmic derivation of $G_\nu(\cdot; q)$ we obtain

$$(9) \quad \frac{G'_\nu(z; q)}{G_\nu(z; q)} = -2 \sum_{k \geq 0} \epsilon_{k+1} z^{2k+1}, \quad |z| < \alpha_{\nu, 1}(q),$$

where $\epsilon_k = \sum_{n \geq 1} (\alpha_{\nu, n}(q))^{-2k}$. Also, by using the infinite sum representation of G_ν we get

$$(10) \quad \frac{G'_\nu(z; q)}{G_\nu(z; q)} = \sum_{n \geq 0} A_n z^{2n+1} \Big/ \sum_{n \geq 0} B_n z^{2n},$$

where

$$A_n = \frac{(-1)^{n+1}(2n+2)(2n+3)^2 q^{(n+1)(n+\nu+1)}}{2^{2n+2}(q; q)_{n+1}(q^{\nu+1}; q)_{n+1}}$$

and

$$B_n = \frac{(-1)^n(2n+1)^2 q^{n(n+\nu)}}{2^{2n}(q; q)_n(q^{\nu+1}; q)_n}.$$

By comparing (9) and (10) and matching all terms with the same degree we have the following Euler-Rayleigh sums $\epsilon_k = \sum_{n \geq 1} \alpha_{\nu, n}^{-2k}(q)$ in terms of ν and q . That is,

$$\epsilon_1 = \frac{9q^{\nu+1}}{4(q^{\nu+1}-1)(q-1)},$$

$$\epsilon_2 = \frac{q^{2\nu+2} S_\nu(q)}{16(q^{\nu+1}-1)^2(q^{\nu+2}-1)(q-1)^2(q+1)}$$

and

$$\epsilon_3 = \frac{q^{3(\nu+1)}(P_\nu(q) + R_\nu(q))}{64(q^{\nu+1}-1)^3(1-q^{\nu+2})(1-q^{\nu+3})(q-1)^2 T(q)}.$$

Now, by considering these Euler-Rayleigh sums in the known Euler-Rayleigh inequalities

$$\epsilon_k^{-\frac{1}{k}} < (\alpha_{\nu, 1}(q))^2 < \frac{\epsilon_k}{\epsilon_{k+1}}$$

for $\nu > -1$ and $k \in \{1, 2\}$ we obtain the following inequalities

$$\sqrt{\frac{4(1-q^{\nu+1})(1-q)}{9q^{\nu+1}}} < r^c(g_\nu^{(2)}(z; q)) < \sqrt{\frac{36(q^2-1)(1-q^{\nu+1})(1-q^{\nu+2})}{q^{\nu+1} S_\nu(q)}}$$

and

$$2\sqrt[4]{\frac{(1+q)(1-q)^2(1-q^{\nu+1})^2(q^{\nu+2}-1)}{q^{2(\nu+1)} S_\nu(q)}} < r^c(g_\nu^{(2)}(z; q)) < \sqrt{\frac{4(1-q^{\nu+1})(1-q^{\nu+3})T(q)S_\nu(q)}{q^{\nu+1}(1+q)(P_\nu(q) + R_\nu(q))}}. \quad \square$$

Proof of Theorem 2.2. We proceed exactly as in the proof of Theorem 2.1. By using the Alexander duality theorem for starlike and convex functions we can say that the function $h_\nu^{(2)}(z; q)$ is convex if and only if $z \mapsto z \left(h_\nu^{(2)}(z; q) \right)'$ is starlike. But, the smallest positive zero of $z \mapsto \left(z \left(h_\nu^{(2)}(z; q) \right)' \right)'$ is actually the radius of starlikeness of $z \mapsto z \left(h_\nu^{(2)}(z; q) \right)'$, according to [9, 10]. Therefore, the radius of convexity $r^c(h_\nu^{(2)})$ is the smallest positive root of the equation

$\left(r \left(h_\nu^{(2)}(r; q)\right)'\right)' = 0$. Thus, we get that the radius of convexity of the function $z \mapsto h_\nu^{(2)}(z; q)$ is the smallest positive root of the equation

$$(2 - \nu)^2 J_\nu^{(2)}(\sqrt{r}; q) + (5 - 2\nu) \sqrt{r} dJ_\nu^{(2)}(\sqrt{r}; q)/dr + r d^2 J_\nu^{(2)}(\sqrt{r}; q)/dr^2 = 0.$$

Now, recall that the zeros $j_{\nu, n}(q)$, $n \in \mathbb{N}$, of the Jackson's second q -Bessel function are all real and simple, according to [18, Theorem 4.2]. Then, the function $h_\nu^{(2)}(\cdot; q)$ belongs to the Laguerre-Pólya class \mathcal{LP} of real entire functions.

Because of the properties of the class \mathcal{LP} the function $z \mapsto \left(z \left(h_\nu^{(2)}(z; q)\right)'\right)'$ belongs also to the class \mathcal{LP} . Hence the function $z \mapsto \left(z \left(h_\nu^{(2)}(z; q)\right)'\right)'$ has only real zeros. Also its growth order ρ is 0, that is

$$\begin{aligned} \rho &= \lim_{n \rightarrow \infty} \frac{n \log n}{2n \log 2 + \log(q; q)_n + \log(q^{\nu+1}; q)_n - 2 \log(n+1) - n(n+\nu) \log q} \\ &= 0, \end{aligned}$$

since as $n \rightarrow \infty$ we have $(q; q)_n \rightarrow (q; q)_\infty < \infty$ and $(q^{\nu+1}; q)_n \rightarrow (q^{\nu+1}; q)_\infty < \infty$. Now, by applying Hadamard's Theorem [24, p. 26] we obtain

$$\Phi_\nu(z; q) = \left(z \left(h_\nu^{(2)}(z; q)\right)'\right)' = \prod_{n \geq 1} \left(1 - \frac{z}{\beta_{\nu, n}(q)}\right),$$

where $\beta_{\nu, n}(q)$ is the n th zero of the function $\Phi_\nu(\cdot; q)$. Now, via logarithmic derivation of $\Phi_\nu(\cdot; q)$ we obtain

$$(11) \quad \frac{\Phi_\nu'(z; q)}{\Phi_\nu(z; q)} = - \sum_{k \geq 0} \mu_{k+1} z^k, \quad |z| < \beta_{\nu, 1}(q),$$

where $\mu_k = \sum_{n \geq 1} (\beta_{\nu, n}(q))^{-k}$. Also, by using the infinite sum representation of Φ_ν we get

$$(12) \quad \frac{\Phi_\nu'(z; q)}{\Phi_\nu(z; q)} = \frac{\sum_{n \geq 0} C_n z^n}{\sum_{n \geq 0} D_n z^n},$$

where

$$C_n = \frac{(-1)^{n+1} (n+1)(n+2)^2 q^{(n+1)(n+\nu+1)}}{2^{2n+2} (q; q)_{n+1} (q^{\nu+1}; q)_{n+1}}$$

and

$$D_n = \frac{(-1)^n (2n+1)^2 q^{n(n+\nu)}}{2^{2n} (q; q)_n (q^{\nu+1}; q)_n}.$$

By comparing (11) and (12) and matching all terms with the same degree we have the following Euler-Rayleigh sums $\mu_k = \sum_{n \geq 1} \beta_{\nu, n}^{-k}(q)$ in terms of ν and

q . That is,

$$\mu_1 = \frac{q^{\nu+1}}{(q^{\nu+1} - 1)(q - 1)},$$

$$\mu_2 = \frac{q^{2\nu+2}U_\nu(q)}{8(q^{\nu+1} - 1)^2(1 - q^{\nu+2})(q - 1)^2(q + 1)}$$

and

$$\mu_3 = \frac{q^{3(\nu+1)}(M_\nu(q) + N_\nu(q))}{16(q^{\nu+1} - 1)^3(q^{\nu+2} - 1)(q^{\nu+3} - 1)(q - 1)^2T(q)}.$$

Now, by considering these Euler-Rayleigh sums in the known Euler-Rayleigh inequalities

$$\mu_k^{-\frac{1}{k}} < \beta_{\nu,1}(q) < \frac{\mu_k}{\mu_{k+1}}$$

for $\nu > -1$ and $k \in \{1, 2\}$ we obtain the next inequalities

$$\frac{(1 - q)(1 - q^{\nu+1})}{q^{\nu+1}} < r^c \left(h_\nu^{(2)}(z; q) \right) < \frac{8(q^{\nu+1} - 1)(q^{\nu+2} - 1)(1 - q^2)}{q^{\nu+1}U_\nu(q)},$$

$$\sqrt{\frac{8(1 - q)^2(1 + q)(1 - q^{\nu+1})^2(1 - q^{\nu+2})}{q^{2\nu+2}U_\nu(q)}} < r^c \left(h_\nu^{(2)}(z; q) \right)$$

$$< \frac{2(1 - q^{\nu+1})(q^{\nu+3} - 1)U_\nu(q)T(q)}{(1 + q)q^{\nu+1}(M_\nu(q) + N_\nu(q))}. \quad \square$$

Proof of Theorem 2.3. By using the Alexander duality theorem for starlike and convex functions we can say that the function $g_\nu^{(3)}(z; q)$ is convex if and only if $z \mapsto z \left(g_\nu^{(3)}(z; q) \right)'$ is starlike. But, the smallest positive zero of $z \mapsto \left(z \left(g_\nu^{(3)}(z; q) \right)' \right)'$ is actually the radius of starlikeness of $z \mapsto z \left(g_\nu^{(3)}(z; q) \right)'$, according to [9, 10]. Therefore, the radius of convexity $r^c(g_\nu^{(3)})$ is the smallest positive root of the equation $\left(r \left(g_\nu^{(3)}(r; q) \right)' \right)' = 0$. Thus, we get that the radius of convexity of the function $z \mapsto g_\nu^{(3)}(z; q)$ is the smallest positive root of the equation

$$(1 - \nu)^2 J_\nu^{(3)}(r; q) + (3 - 2\nu)rdJ_\nu^{(3)}(r; q)/dr + r^2 d^2 J_\nu^{(3)}(r; q)/dr^2 = 0.$$

Now, recall that the zeros $l_{\nu,n}(q)$, $n \in \mathbb{N}$, of the Jackson's third q -Bessel function are all real and simple, according to [18, Theorem 4.2]. Then, the function $g_\nu^{(3)}(\cdot; q)$ belongs to the Laguerre-Pólya class \mathcal{LP} of real entire functions. Because of the properties of the class \mathcal{LP} the function $z \mapsto \left(z \left(g_\nu^{(3)}(z; q) \right)' \right)'$

belongs also to the class \mathcal{LP} . Hence the function $z \mapsto \left(z \left(g_\nu^{(3)}(z; q) \right)' \right)'$ has only real zeros. Also its growth order ρ is 0, that is

$$\rho = \lim_{n \rightarrow \infty} \frac{n \log n}{\log(q; q)_n + \log(q^{\nu+1}; q)_n - 2 \log(2n+1) - \frac{n(n+1)}{2} \log q} = 0,$$

since as $n \rightarrow \infty$ we have $(q; q)_n \rightarrow (q; q)_\infty < \infty$ and $(q^{\nu+1}; q)_n \rightarrow (q^{\nu+1}; q)_\infty < \infty$. Now, by applying Hadamard's Theorem [24, p. 26] we obtain

$$\mathcal{H}_\nu(z; q) = \left(z \left(g_\nu^{(3)}(z; q) \right)' \right)' = \prod_{n \geq 1} \left(1 - \frac{z^2}{(h_{\nu, n}(q))^2} \right),$$

where $h_{\nu, n}(q)$ is the n th zero of the function $\mathcal{H}_\nu(\cdot; q)$. Now, via logarithmic derivation of $\mathcal{H}_\nu(\cdot; q)$ we obtain

$$(13) \quad \frac{\mathcal{H}'_\nu(z; q)}{\mathcal{H}_\nu(z; q)} = -2 \sum_{k \geq 0} \eta_{k+1} z^{2k+1}, \quad |z| < h_{\nu, 1}(q),$$

where $\eta_k = \sum_{n \geq 1} (h_{\nu, n}(q))^{-2k}$. Also, by using the infinite sum representation of \mathcal{H}_ν we have

$$(14) \quad \frac{\mathcal{H}'_\nu(z; q)}{\mathcal{H}_\nu(z; q)} = \sum_{n \geq 0} E_n z^{2n+1} \Big/ \sum_{n \geq 0} F_n z^{2n},$$

where

$$E_n = \frac{(-1)^{n+1} (2n+2)(2n+3)^2 q^{\frac{(n+1)(n+2)}{2}}}{(q; q)_{n+1} (q^{\nu+1}; q)_{n+1}}$$

and

$$F_n = \frac{(-1)^n (2n+1)^2 q^{\frac{n(n+1)}{2}}}{(q; q)_n (q^{\nu+1}; q)_n}.$$

By comparing (13) and (14) and matching all terms with the same degree we have the following Euler-Rayleigh sums $\eta_k = \sum_{n \geq 1} h_{\nu, n}^{-2k}(q)$ in terms of ν and q . That is,

$$\eta_1 = \frac{9q}{(q^{\nu+1} - 1)(q - 1)},$$

$$\eta_2 = \frac{q^2 Y_\nu(q)}{(q^{\nu+1} - 1)^2 (q^{\nu+2} - 1)(q - 1)^2 (q + 1)}$$

and

$$\eta_3 = \frac{3q^3 (\theta_\nu(q) + \varphi_\nu(q))}{64(q^{\nu+1} - 1)^3 (1 - q^{\nu+2})(1 - q^{\nu+3})(q - 1)^2 T(q)}.$$

Now, by considering these Euler-Rayleigh sums in the known Euler-Rayleigh inequalities

$$\eta_k^{-\frac{1}{k}} < (h_{\nu, 1}(q))^2 < \frac{\eta_k}{\eta_{k+1}}$$

for $\nu > -1$ and $k \in \{1, 2\}$ we obtain the following inequalities

$$\begin{aligned} \sqrt{\frac{(1-q^{\nu+1})(1-q)}{9q}} < r^c(g_\nu^{(3)}(z; q)) < \sqrt{\frac{9(q^2-1)(1-q^{\nu+1})(1-q^{\nu+2})}{qY_\nu(q)}}, \\ \sqrt[4]{\frac{(1+q)(1-q)^2(1-q^{\nu+1})^2(q^{\nu+2}-1)}{q^2Y_\nu(q)}} < r^c(g_\nu^{(3)}(z; q)) \\ < \sqrt{\frac{(1-q^{\nu+1})(1-q^{\nu+3})T(q)Y_\nu(q)}{3q(q+1)(\theta_\nu(q)+\varphi_\nu(q))}}. \end{aligned}$$

□

Proof of Theorem 2.4. By using the Alexander duality theorem for starlike and convex functions we can say that the function $h_\nu^{(3)}(z; q)$ is convex if and only if $z \mapsto z \left(h_\nu^{(3)}(z; q) \right)'$ is starlike. But, the smallest positive zero of $z \mapsto \left(z \left(h_\nu^{(3)}(z; q) \right)' \right)'$ is actually the radius of starlikeness of $z \mapsto z \left(h_\nu^{(3)}(z; q) \right)'$, according to [9, 10]. Therefore, the radius of convexity $r^c(h_\nu^{(3)})$ is the smallest positive root of the equation $\left(r \left(h_\nu^{(3)}(r; q) \right)' \right)' = 0$. Thus, we get that the radius of convexity of the function $z \mapsto h_\nu^{(3)}(z; q)$ is the smallest positive root of the equation

$$(2-\nu)^2 J_\nu^{(3)}(\sqrt{r}; q) + (5-2\nu) \sqrt{r} dJ_\nu^{(3)}(\sqrt{r}; q)/dr + rd^2 J_\nu^{(3)}(\sqrt{r}; q)/dr^2 = 0.$$

Now, recall that the zeros $l_{\nu, n}(q)$, $n \in \mathbb{N}$, of the Jackson's third q -Bessel function are all real and simple, according to [18, Theorem 4.2]. Then, the function $h_\nu^{(3)}(\cdot; q)$ belongs to the Laguerre-Pólya class \mathcal{LP} of real entire functions. Because of the properties of the class \mathcal{LP} the function $z \mapsto \left(z \left(h_\nu^{(3)}(z; q) \right)' \right)'$ belongs also to the class \mathcal{LP} . Hence, the function $z \mapsto \left(z \left(h_\nu^{(3)}(z; q) \right)' \right)'$ has only real zeros. Also its growth order ρ is 0, that is

$$\rho = \lim_{n \rightarrow \infty} \frac{n \log n}{\log(q; q)_n + \log(q^{\nu+1}; q)_n - 2 \log(n+1) - \frac{n(n+1)}{2} \log q} = 0,$$

since as $n \rightarrow \infty$ we have $(q; q)_n \rightarrow (q; q)_\infty < \infty$ and $(q^{\nu+1}; q)_n \rightarrow (q^{\nu+1}; q)_\infty < \infty$. Now, by applying Hadamard's Theorem [24, p. 26] we obtain

$$\psi_\nu(z; q) = \left(z \left(h_\nu^{(3)}(z; q) \right)' \right)' = \prod_{n \geq 1} \left(1 - \frac{z}{\gamma_{\nu, n}(q)} \right),$$

where $\gamma_{\nu,n}(q)$ is the n th zero of the function $\psi_{\nu}(\cdot; q)$. Now, via logarithmic derivation of $\psi_{\nu}(\cdot; q)$ we obtain

$$(15) \quad \frac{\psi'_{\nu}(z; q)}{\psi_{\nu}(z; q)} = - \sum_{k \geq 0} \sigma_{k+1} z^k, \quad |z| < \gamma_{\nu,1}(q),$$

where $\sigma_k = \sum_{n \geq 1} (\gamma_{\nu,n}(q))^{-k}$. Also, by using the infinite sum representation of Φ_{ν} we get

$$(16) \quad \frac{\psi'_{\nu}(z; q)}{\psi_{\nu}(z; q)} = \sum_{n \geq 0} G_n z^n \bigg/ \sum_{n \geq 0} H_n z^n,$$

where

$$G_n = \frac{(-1)^{n+1} (n+1)(n+2)^2 q^{\frac{(n+1)(n+2)}{2}}}{(q; q)_{n+1} (q^{\nu+1}; q)_{n+1}} \quad \text{and} \quad H_n = \frac{(-1)^n (n+1)^2 q^{\frac{n(n+1)}{2}}}{(q; q)_n (q^{\nu+1}; q)_n}.$$

By comparing (15) and (16) and matching all terms with the same degree we have the following Euler-Rayleigh sums $\sigma_k = \sum_{n \geq 1} \gamma_{\nu,n}^{-k}(q)$ in terms of ν and q . That is,

$$\sigma_1 = \frac{4q}{(q^{\nu+1} - 1)(q - 1)},$$

$$\sigma_2 = \frac{2q^2 E_{\nu}(q)}{(q^{\nu+1} - 1)^2 (1 - q^{\nu+2})(1 - q)^2 (1 - q)}$$

and

$$\sigma_3 = \frac{4q^3 (K_{\nu}(q) + L_{\nu}(q))}{(1 - q^{\nu+1})^3 (1 - q^{\nu+2})(1 - q^{\nu+3})(1 - q)^2 T(q)}.$$

Now, by considering these Euler-Rayleigh sums in the known Euler-Rayleigh inequalities

$$\sigma_k^{-\frac{1}{k}} < \gamma_{\nu,1}(q) < \frac{\sigma_k}{\sigma_{k+1}}$$

for $\nu > -1$ and $k \in \{1, 2\}$ we obtain the next two inequalities

$$\frac{(1 - q)(1 - q^{\nu+1})}{4q} < r^c \left(h_{\nu}^{(3)}(z; q) \right) < \frac{2(q^{\nu+1} - 1)(q^{\nu+2} - 1)(q^2 - 1)}{q E_{\nu}(q)},$$

$$\begin{aligned} \sqrt{\frac{(1 - q)^2 (1 + q)(1 - q^{\nu+1})^2 (q^{\nu+2} - 1)}{2q^2 E_{\nu}(q)}} &< r^c \left(h_{\nu}^{(3)}(z; q) \right) \\ &< \frac{(1 - q^{\nu+1})(q^{\nu+3} - 1) E_{\nu}(q) T(q)}{2q(1 + q) (K_{\nu}(q) + L_{\nu}(q))}. \quad \square \end{aligned}$$

References

- [1] İ. Aktaş and Á. Baricz, *Bounds for radii of starlikeness of some q -Bessel functions*, Results Math. **72** (2017), no. 1-2, 947–963. <https://doi.org/10.1007/s00025-017-0668-6>
- [2] İ. Aktaş, Á. Baricz, and H. Orhan, *Bounds for radii of starlikeness and convexity of some special functions*, Turkish J. Math. **42** (2018), no. 1, 211–226. <https://doi.org/10.3906/mat-1610-41>
- [3] İ. Aktaş, Á. Baricz, and N. Yağmur, *Bounds for the radii of univalence of some special functions*, Math. Inequal. Appl. **20** (2017), no. 3, 825–843. <https://doi.org/10.7153/mia-20-52>
- [4] Á. Baricz, *Geometric properties of generalized Bessel functions of complex order*, Mathematica **48(71)** (2006), no. 1, 13–18.
- [5] ———, *Geometric properties of generalized Bessel functions*, Publ. Math. Debrecen **73** (2008), no. 1-2, 155–178.
- [6] ———, *Generalized Bessel functions of the first kind*, Lecture Notes in Mathematics, 1994, Springer-Verlag, Berlin, 2010. <https://doi.org/10.1007/978-3-642-12230-9>
- [7] Á. Baricz, D. K. Dimitrov, and I. Mező, *Radii of starlikeness and convexity of some q -Bessel functions*, J. Math. Anal. Appl. **435** (2016), no. 1, 968–985. <https://doi.org/10.1016/j.jmaa.2015.10.065>
- [8] Á. Baricz, D. K. Dimitrov, H. Orhan, and N. Yağmur, *Radii of starlikeness of some special functions*, Proc. Amer. Math. Soc. **144** (2016), no. 8, 3355–3367. <https://doi.org/10.1090/proc/13120>
- [9] Á. Baricz, P. A. Kupán, and R. Szász, *The radius of starlikeness of normalized Bessel functions of the first kind*, Proc. Amer. Math. Soc. **142** (2014), no. 6, 2019–2025. <https://doi.org/10.1090/S0002-9939-2014-11902-2>
- [10] Á. Baricz, H. Orhan, and R. Szász, *The radius of α -convexity of normalized Bessel functions of the first kind*, Comput. Methods Funct. Theory **16** (2016), no. 1, 93–103. <https://doi.org/10.1007/s40315-015-0123-1>
- [11] Á. Baricz and S. Ponnusamy, *Starlikeness and convexity of generalized Bessel functions*, Integral Transforms Spec. Funct. **21** (2010), no. 9-10, 641–653. <https://doi.org/10.1080/10652460903516736>
- [12] Á. Baricz and R. Szász, *The radius of convexity of normalized Bessel functions of the first kind*, Anal. Appl. (Singap.) **12** (2014), no. 5, 485–509. <https://doi.org/10.1142/S0219530514500316>
- [13] ———, *Close-to-convexity of some special functions and their derivatives*, Bull. Malays. Math. Sci. Soc. **39** (2016), no. 1, 427–437. <https://doi.org/10.1007/s40840-015-0180-7>
- [14] Á. Baricz, E. Toklu, and E. Kadioğlu, *Radii of starlikeness and convexity of Wright functions*, Math. Commun. **23** (2018), no. 1, 97–117.
- [15] Á. Baricz and N. Yağmur, *Geometric properties of some Lommel and Struve functions*, Ramanujan J. **42** (2017), no. 2, 325–346. <https://doi.org/10.1007/s11139-015-9724-6>
- [16] R. K. Brown, *Univalence of Bessel functions*, Proc. Amer. Math. Soc. **11** (1960), 278–283. <https://doi.org/10.2307/2032969>
- [17] P. L. Duren, *Univalent Functions*, Grundlehren der Mathematischen Wissenschaften, **259**, Springer-Verlag, New York, 1983.
- [18] M. E. H. Ismail, *The zeros of basic Bessel functions, the functions $J_{\nu+ax}(x)$, and associated orthogonal polynomials*, J. Math. Anal. Appl. **86** (1982), no. 1, 1–19. [https://doi.org/10.1016/0022-247X\(82\)90248-7](https://doi.org/10.1016/0022-247X(82)90248-7)
- [19] M. E. H. Ismail and M. E. Muldoon, *On the variation with respect to a parameter of zeros of Bessel and q -Bessel functions*, J. Math. Anal. Appl. **135** (1988), no. 1, 187–207. [https://doi.org/10.1016/0022-247X\(88\)90148-5](https://doi.org/10.1016/0022-247X(88)90148-5)

- [20] ———, *Bounds for the small real and purely imaginary zeros of Bessel and related functions*, *Methods Appl. Anal.* **2** (1995), no. 1, 1–21. <https://doi.org/10.4310/MAA.1995.v2.n1.a1>
- [21] H. T. Koelink and R. F. Swarttouw, *On the zeros of the Hahn-Exton q -Bessel function and associated q -Lommel polynomials*, *J. Math. Anal. Appl.* **186** (1994), no. 3, 690–710. <https://doi.org/10.1006/jmaa.1994.1327>
- [22] T. H. Koornwinder and R. F. Swarttouw, *On q -analogues of the Fourier and Hankel transforms*, *Trans. Amer. Math. Soc.* **333** (1992), no. 1, 445–461. <https://doi.org/10.2307/2154118>
- [23] E. Kreyszig and J. Todd, *The radius of univalence of Bessel functions. I*, *Illinois J. Math.* **4** (1960), 143–149. <http://projecteuclid.org/euclid.ijm/1255455740>
- [24] B. Ya. Levin, *Lectures on entire functions*, translated from the Russian manuscript by Tkachenko, *Translations of Mathematical Monographs*, **150**, American Mathematical Society, Providence, RI, 1996.
- [25] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, Cambridge, England, 1944.
- [26] H. S. Wilf, *The radius of univalence of certain entire functions*, *Illinois J. Math.* **6** (1962), 242–244. <http://projecteuclid.org/euclid.ijm/1255632321>

İBRAHİM AKTAŞ
DEPARTMENT OF MATHEMATICS
KAMIL ÖZDAĞ SCIENCE FACULTY
KARAMANOĞLU MEHMETBEY UNIVERSITY
KARAMAN, TURKEY
Email address: aktasibrahim38@gmail.com

HALİT ORHAN
DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
ATATÜRK UNIVERSITY
ERZURUM, TURKEY
Email address: orhanhalit607@gmail.com