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ON FOUR NEW MOCK THETA FUNCTIONS

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Abstract. In this paper, we first give some representations for four new mock theta functions defined by Andrews [1] and Bringmann, Hikami and Lovejoy [5] using divisor sums. Then, some transformation and summation formulae for these functions and corresponding bilateral series are derived as special cases of ${}_2\psi_2$ series

$$\sum_{n=-\infty}^{\infty} \frac{(a,c;q)_n}{(b,d;q)_n} z^n$$

and Ramanujan's sum

$$\sum_{n=-\infty}^{\infty} \frac{(a;q)_n}{(b;q)_n} z^n.$$

1. Introduction

In his letter to G. H. Hardy [12], S. Ramanujan defined seventeen functions $M(q), |q| < 1$, which he called mock θ-functions of order three, five and seven. But Ramanujan did not explain precisely what he meant by mock theta functions. The mock theta functions are interpreted by Andrews and Hickerson [2] to mean a function $f(q)$ defined by a q-series which converges for $|q| < 1$ and satisfies the following two conditions:

(0) For every root of unity ξ, there is a theta function $\theta_\xi(q)$ such that the difference $f(q) - \theta_\xi(q)$ is bounded as $q \to \xi$ radially.

(1) There is no single theta function which works for all ξ; i.e., for every theta function $\theta(q)$ there is some root of unity ξ for which $f(q) - \theta_\xi(q)$ is unbounded as $q \to \xi$ radially.

G. N. Watson [13] found three third-order mock theta functions. In his "lost" Notebook Ramanujan gave six sixth-order mock theta functions which were studied by G. E. Andrews and D. Hickerson [2] and four tenth-order mock theta functions which were studied by Choi [6]. B. Gordon and R. J. McIntosh [8] generated eight eighth-order mock theta functions, but four of them were

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later found of lower order. Hikami [9] found one more mock theta function of order two.

Andrews [1] generated new mock theta functions. Bringmann, Hikami and Lovejoy [5] developed two more mock theta functions.

Throughout this paper, we adopt the standard notations in [7]. For $|q| < 1$, the q-shifted factorial is defined by

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), \quad n = 1,2,3,\dots, \quad (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k).$$

It is easy to deduce from the definition of $(a;q)_n$ that, for a positive integer n,

$$(a;q)_{-n} = \frac{(-q/a)^n}{(q/a;q)_n} q^{n(n-1)/2}.$$

We also adopt the following compact notation for multiple q-shifted factorials:

$$(a_1, a_2, \dots, a_m; q)_n = (a_1;q)_n (a_2;q)_n \cdots (a_m;q)_n,$$

where n is an integer or ∞ and m is a positive integer.

If $|q_1| > 1$, then it holds

$$(a;q_1)_n = \frac{(\frac{1}{a}; \frac{1}{q_1})_\infty}{(\frac{1}{aq_1^n}; \frac{1}{q_1})_\infty} (-a)^n q_1^{\binom{n}{2}}.$$

The bilateral basic hypergeometric series is given by

$${}_r\psi_r \left[\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_r \end{matrix} ; q, z \right] = \sum_{n=-\infty}^{\infty} \frac{(a_1, \dots, a_s; q)_n}{(b_1, \dots, b_r; q)_n} z^n.$$

Two new mock theta functions of Andrews in [1] are defined by:

$$\overline{\psi}_0(q) = \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(-q;q)_{2n}}, \tag{1}$$

$$\overline{\psi}_1(q) = \sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(-q;q)_{2n+1}}, \tag{2}$$

and two new mock theta functions of Bringmann, Hikami and Lovejoy in [5] are defined by:

$$\overline{\phi}_0(q) = \sum_{n=0}^{\infty} q^n (-q;q)_{2n+1}, \tag{3}$$

$$\overline{\phi}_1(q) = \sum_{n=0}^{\infty} q^n (-q;q)_{2n}. \tag{4}$$

Lemma 1.1 ([11, Corollary 2.9])**.** *The following identities are true*:

$$\overline{\psi}_0(q) + 2q\overline{\phi}_0(q) = -\frac{\overline{J}_{3,8}}{J_2}(J_{1,2} - 2\overline{J}_{2,4}), \tag{5}$$

$$\overline{\psi}_1(q) + 2\overline{\phi}_1(q) = \frac{\overline{J}_{1,8}}{J_2}(J_{1,2} + 2\overline{J}_{2,4}). \tag{6}$$

Here, let a and m be integers with m positive. Define

$$J_{a,m} := j(q^a; q^m), \quad J_m := J_{m,3m} = \prod_{i\geq 1}(1 - q^{mi}), \quad \overline{J}_{a,m} := j(-q^a; q^m).$$

In this paper, motivated by the work of Nikos Bagis [3] on divisor sums, we first give some representations of some new mock theta functions using divisor sums. Then, motivated by the work of James Mc Laughlin [10], some transformation and summation formulae for these functions and corresponding bilateral series are derived.

2. Divisor sums and new mock theta functions

Theorem 2.1. *The function $\overline{\psi}_0(q)$ is defined for all $q \in C - D$, where $D = z \in C : |z| = 1$. For $|q| < 1$, we have*

$$\begin{aligned} \overline{\psi}_0(q) &= \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(-q;q)_{2n}} \\ &= \frac{1}{\chi(q)} \sum_{n=0}^{\infty} q^{2n^2} \exp\left[-\sum_{s=1}^{\infty} \frac{q^s}{s} \sum_{0<d|s, d\geq 2n+1} (-1)^{s/d} d\right], \end{aligned} \tag{7}$$

$$\begin{aligned} \overline{\psi}_0(1/q) &= \sum_{n=0}^{\infty} \frac{q^{n}}{(-q;q)_{2n}} \\ &= \frac{1}{\chi(q)} \sum_{n=0}^{\infty} q^{n} \exp\left[-\sum_{s=1}^{\infty} \frac{q^s}{s} \sum_{0<d|s, d\geq 2n+1} (-1)^{s/d} d\right], \end{aligned} \tag{8}$$

$$\begin{aligned} \overline{\psi}_1(q) &= \sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(-q;q)_{2n+1}} \\ &= \frac{1}{\chi(q)} \sum_{n=0}^{\infty} q^{2n^2+2n} \exp\left[-\sum_{s=2}^{\infty} \frac{q^s}{s} \sum_{0<d|s, d\geq 2n+2} (-1)^{s/d} d\right], \end{aligned} \tag{9}$$

$$\begin{aligned} \overline{\psi}_1(1/q) &= \sum_{n=0}^{\infty} \frac{q^{n+1}}{(-q;q)_{2n+1}} \\ &= \frac{1}{\chi(q)} \sum_{n=0}^{\infty} q^{n+1} \exp\left[-\sum_{s=2}^{\infty} \frac{q^s}{s} \sum_{0<d|s, d\geq 2n+2} (-1)^{s/d} d\right], \end{aligned} \tag{10}$$

$$\overline{\phi}_0(q) = \sum_{n=0}^{\infty} q^n (-q;q)_{2n+1} \tag{11}$$

$$= \chi(q) \sum_{n=0}^{\infty} q^n \exp\left[\sum_{s=2}^{\infty} \frac{q^s}{s} \sum_{0<d|s, d\geq 2n+2} (-1)^{s/d} d\right],$$

(12) $$\overline{\phi}_1(q) = \sum_{n=0}^{\infty} q^n (-q;q)_{2n}$$
$$= \chi(q) \sum_{n=0}^{\infty} q^n \exp\left[\sum_{s=1}^{\infty} \frac{q^s}{s} \sum_{0<d|s, d\geq 2n+1} (-1)^{s/d} d\right],$$

where $\chi(q) = (-q;q)_\infty$.

Proof. We only give the proof of the identity (8) in details here.

First, we have

$$(-q^{-1}; q^{-1})_n = q^{-\binom{n+1}{2}} (-q;q)_n.$$

Then, by proper computations we get

$$\begin{aligned}\overline{\psi}_0(1/q) &= \sum_{n=0}^{\infty} \frac{q^n}{(-q;q)_{2n}} = \frac{1}{(-q;q)_\infty} \sum_{n=0}^{\infty} q^n (-q^{2n+1};q)_\infty \\ &= \frac{1}{(-q;q)_\infty} \sum_{n=0}^{\infty} q^n \exp\left[\log \prod_{k=1}^{\infty} (1+q^{2n+k})\right] \\ &= \frac{1}{(-q;q)_\infty} \sum_{n=0}^{\infty} q^n \exp\left[\sum_{k=1}^{\infty} \log(1+q^{2n+k})\right] \\ &= \frac{1}{(-q;q)_\infty} \sum_{n=0}^{\infty} q^n \exp\left[\sum_{k=1}^{\infty}\sum_{m=1}^{\infty} \frac{(-1)^{m+1} q^{(2n+k)m}}{m}\right].\end{aligned}$$

After substitutions $(2n+k)m = s$, $2n+k = d$, we get our desired result (8).

Similarly, other identities in Theorem 2.1 can be also proved. □

3. Some transformation and summation formulas associated to new mock theta functions

In this section, we will begin with transformation and summation formulas for basic bilateral hypergeometric series:

(13) $$\sum_{n=-\infty}^{\infty} \frac{(e,f;q)_n}{(aq/c, aq/d;q)_n} \left(\frac{aq}{ef}\right)^n$$
$$= \frac{(q/c, q/d, aq/e, aq/f;q)_\infty}{(aq, q/a, aq/cd, aq/ef;q)_\infty}$$
$$\times \sum_{n=-\infty}^{\infty} \frac{(1-aq^{2n})(c,d,e,f;q)_n}{(1-a)(aq/c, aq/d, aq/e, aq/f;q)_n} \left(\frac{a^3 q}{cdef}\right)^n q^{n^2},$$
$$|a^2q^2/cdef| < |aq/ef| < 1.$$

(14)
$$\sum_{n=-\infty}^{\infty}\frac{(a,c;q)_n}{(b,d;q)_n}z^n = \frac{(b/a,d/c,az,qb/acz;q)_\infty}{(b,q/c,z,bd/acz;q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(a,acz/b;q)_n}{(az,d;q)_n}(b/a)^n,$$
$$|b/a|<1,\ |d/c|<1,\ |bd/ac|<|z|<1.$$

(15)
$$\sum_{n=-\infty}^{\infty}\frac{(a,c;q)_n}{(b,d;q)_n}z^n = \frac{(az,cz,qb/acz,qd/acz;q)_\infty}{(b,d,q/a,q/c;q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(acz/b,acz/d;q)_n}{(az,cz;q)_n}\left(\frac{bd}{acz}\right)^n,$$
$$|bd/ac|<|z|<1.$$

(16)
$$\sum_{n=-\infty}^{\infty}\frac{(a;q)_n}{(b;q)_n}z^n = \frac{(q,b/a,az,q/az;q)_\infty}{(b,q/a,z,b/az;q)_\infty},\ |b/a|<|z|<1.$$

The bilateral transformations at (13), (14) and (15) are all due to Bailey [4]. The identity at (16) is Ramanujan's sum for ${}_1\psi_1\left[\begin{smallmatrix}a\\b\end{smallmatrix};q,z\right]$ (see [7, (5.2.1)]).

First, we consider a generalization, namely the series

$$G(c,d;z,q) = 1+\sum_{n=1}^{\infty}\frac{z^nq^{2n^2}}{(c,d;q^2)_n},$$

the bilateral series of which is defined by

$$G^*(c,d;z,q) = \sum_{n=-\infty}^{\infty}\frac{z^nq^{2n^2}}{(c,d;q^2)_n}.$$

Proposition 3.1. *For $|q|<1$, we have*

(17)
$$G^*(c,d;z,q) = \frac{(c/z,d/z;q^2)_\infty}{(zq^2,q^2/z,cd/zq^2;q^2)_\infty}\sum_{n=-\infty}^{\infty}\frac{1-zq^{4n}}{1-z}\frac{(zq^2/c,zq^2/d;q^2)_n}{(c,d;q^2)_n}(zcd)^nq^{4n^2-4n},$$

(18)
$$G^*(c,d;z,q) = \frac{(c/z;q^2)_\infty}{(c,cd/zq^2;q^2)_\infty}\sum_{n=-\infty}^{\infty}\frac{(zq^2/c;q^2)_n}{(d;q^2)_n}(-c)^nq^{n^2-n},$$

(19)
$$G^*(c,d;z,q) = \frac{(c/z,d/z;q^2)_\infty}{(c,d;q^2)_\infty}\sum_{n=-\infty}^{\infty}(zq^2/c,zq^2/d;q^2)_n(cd/zq^2)^n.$$

Proof. First, replace c, d by $aq/c, aq/d$, respectively, and then let $q \to q^2, e \to \infty, f \to \infty$ in (13), to get that

$$\begin{aligned}
&\sum_{n=-\infty}^{\infty} \frac{a^n q^{2n^2}}{(c,d;q^2)_n} \\
&= \frac{(c/a, d/a; q^2)_\infty}{(aq^2, q^2/a, cd/aq^2; q^2)_\infty} \sum_{n=-\infty}^{\infty} \frac{1-aq^{4n}}{1-a} \frac{(aq^2/c, aq^2/d; q^2)_n}{(c,d;q^2)_n} (acd)^n q^{4n^2-4n}.
\end{aligned}$$

The identity (17) follows after replacing a by z.

Transformations (18) and (19) will follow as special cases of two more general identities. First by letting $q \to q^2, z \to zq^2/ac$, and then $a \to \infty, c \to \infty$ in (14) and (15), respectively, to get

$$\sum_{n=-\infty}^{\infty} \frac{z^n q^{2n^2}}{(b,d;q^2)_n} = \frac{(b/z;q^2)_\infty}{(b, bd/zq^2; q^2)_\infty} \sum_{n=-\infty}^{\infty} \frac{(zq^2/b;q^2)_n}{(d;q^2)_n} (-b)^n q^{n^2-n}$$

and

$$\sum_{n=-\infty}^{\infty} \frac{z^n q^{2n^2}}{(b,d;q^2)_n} = \frac{(b/z, d/z; q^2)_\infty}{(b,d;q^2)_\infty} \sum_{n=-\infty}^{\infty} (zq^2/b, zq^2/d; q^2)_n (bd/zq^2)^n.$$

After replacing b by c, we obtain our desired results (18) and (19). This completes the proof. □

Theorem 3.2. *For $|q| < 1$, we have*

$$\overline{\psi}_0(q) + 2q\overline{\phi}_0(q) = \frac{1}{J_1^2} \sum_{n=-\infty}^{\infty} (8n+1) q^{4n^2+n}, \tag{20}$$

$$\overline{\psi}_1(q) + 2\overline{\phi}_1(q) = \frac{1}{J_1^2} \sum_{n=-\infty}^{\infty} (8n+3) q^{4n^2+3n}. \tag{21}$$

Proof. Replacing z, c, d by $z^2, -zq, -zq^2$ in (17), respectively, we get

$$\begin{aligned}
&\sum_{n=-\infty}^{\infty} \frac{z^{2n} q^{2n^2}}{(-zq;q)_{2n}} \\
&= \frac{(-q/z, -q^2/z; q^2)_\infty}{(z^2q^2, q^2/z^2, q; q^2)_\infty} \sum_{n=-\infty}^{\infty} \frac{1-zq^{2n}}{1-z} z^{4n} q^{4n^2-n} \\
&= \frac{(-q/z;q)_\infty}{(z^2q^2, q^2/z^2, q; q^2)_\infty} \left[1 + \sum_{n=1}^{\infty} \frac{z^{-4n} q^{4n^2-n} \left\{ z^{8n}(1-zq^{2n}) + q^{2n}(1-zq^{-2n}) \right\}}{1-z} \right] \\
&= \frac{(-q/z;q)_\infty}{(z^2q^2, q^2/z^2, q; q^2)_\infty} \left[1 + \sum_{n=1}^{\infty} \frac{z^{-4n} q^{4n^2-n} \left\{ -z(1-z^{8n-1}) + q^{2n}(1-z^{8n+1}) \right\}}{1-z} \right].
\end{aligned}$$

Letting $z \to 1$, we obtain

$$\begin{aligned}
&\sum_{n=-\infty}^{\infty} \frac{q^{2n^2}}{(-q;q)_{2n}} \\
&= \frac{(-q;q)_\infty}{(q;q)_\infty (q^2;q^2)_\infty} \left[1 + \sum_{n=1}^{\infty} q^{4n^2-n} \left(-(8n-1) + q^{2n}(8n+1) \right) \right] \\
&= \frac{(-q;q)_\infty}{(q;q)_\infty (q^2;q^2)_\infty} \sum_{n=-\infty}^{\infty} (1+8n) q^{4n^2+n} \\
&= \frac{1}{J_1^2} \sum_{n=-\infty}^{\infty} (1+8n) q^{4n^2+n}.
\end{aligned} \tag{22}$$

In fact, the left side of (22) can be directly written as

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{q^{2n^2}}{(-q;q)_{2n}} + \sum_{n=-1}^{-\infty} \frac{q^{2n^2}}{(-q;q)_{2n}} &= \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(-q;q)_{2n}} + 2q \sum_{n=0}^{\infty} q^n (-q;q)_{2n+1} \\
&= \overline{\psi}_0(q) + 2q\overline{\phi}_0(q).
\end{aligned}$$

Thus, the identity (20) is obtained.

For (21), first letting $z \to z^2q^2$, and then replacing c, d by $-zq^2, -zq^3$ in (17), respectively, we get

$$\begin{aligned}
&\sum_{n=-\infty}^{\infty} \frac{z^{2n} q^{2n^2+2n}}{(-zq^2, -zq^3; q^2)_n} \\
&= \frac{-z^3 q(-1/z, -1/zq; q^2)_\infty}{(1+z)(z^2q^2, q^2/z^2, q; q^2)_\infty} \sum_{n=-\infty}^{\infty} \frac{1 - zq^{2n+1}}{1-z} z^{4n} q^{4n^2+3n} \\
&= \frac{-z^3 q(-1/z, -1/zq; q^2)_\infty}{(1+z)(z^2q^2, q^2/z^2, q; q^2)_\infty} \\
&\quad \times \sum_{n=0}^{\infty} z^{4n} q^{4n^2+3n} \left(\frac{1 - z^{-8n-3}}{1-z} - zq^{2n+1} \frac{1 - z^{-8n-5}}{1-z} \right).
\end{aligned}$$

Letting $z \to 1$, we obtain

$$\begin{aligned}
&\sum_{n=-\infty}^{\infty} \frac{q^{2n^2+2n}}{(-q;q)_{2n+1}} \\
&= \frac{(-q, -q^2; q^2)_\infty}{(q^2, q^2, q; q^2)_\infty} \left(\sum_{n=0}^{\infty} q^{4n^2+3n}(8n+3) - \sum_{n=0}^{\infty} q^{4n^2+5n+1}(8n+5) \right) \\
&= \frac{1}{J_1^2} \sum_{n=-\infty}^{\infty} q^{4n^2+3n}(8n+3).
\end{aligned}$$

Since

$$\sum_{n=-\infty}^{\infty}\frac{q^{2n^2+2n}}{(-q;q)_{2n+1}}=\sum_{n=0}^{\infty}\frac{q^{2n^2+2n}}{(-q;q)_{2n+1}}+2\sum_{n=0}^{\infty}q^n(-q;q)_{2n}=\overline{\psi}_1(q)+2\overline{\phi}_1(q),$$

then, we have

$$\overline{\psi}_1(q)+2\overline{\phi}_1(q)=\frac{1}{J_1^2}\sum_{n=-\infty}^{\infty}q^{4n^2+3n}(8n+3).$$

This completes the proofs of (20) and (21). □

Corollary 3.3. *We have*

$$\begin{aligned}\sum_{n=-\infty}^{\infty}(1+8n)q^{4n^2+n}&=J_1J_2\sum_{n=-\infty}^{\infty}\frac{(-q;q^2)_n}{(-q^2;q^2)_n}q^{n^2}\\&=-\frac{J_1^2\overline{J}_{3,8}}{J_2}(J_{1,2}-2\overline{J}_{2,4}),\\\sum_{n=-\infty}^{\infty}(8n+3)q^{4n^2+3n}&=2J_1J_2\sum_{n=-\infty}^{\infty}\frac{(-q^2;q^2)_n}{(-q;q^2)_{n+1}}q^{n^2+n}\\&=\frac{\overline{J}_{1,8}J_1^2}{J_2}(J_{1,2}+2\overline{J}_{2,4}).\end{aligned}$$

Proof. From (18), letting $c=-q, d=-q^2, z=1$, we get

$$\overline{\psi}_0(q)+2q\overline{\phi}_0(q)=\frac{J_2}{J_1}\sum_{n=-\infty}^{\infty}\frac{(-q;q^2)_n}{(-q^2;q^2)_n}q^{n^2}.$$

Again from (18), letting $c=-q^2, d=-q^3, z=q^2$, we obtain

$$\overline{\psi}_1(q)+2\overline{\phi}_1(q)=\frac{2J_2}{J_1}\sum_{n=-\infty}^{\infty}\frac{(-q^2;q^2)_n}{(-q;q^2)_{n+1}}q^{n^2+n}.$$

Together with the previous results, Corollary 3.3 can be easily proved. □

Theorem 3.4. *For $|q|<1$, we have*

$$\left(\overline{\psi}_0(q^2)+2q^2\overline{\phi}_0(q^2)\right)+q\left(\overline{\psi}_1(q^2)+2\overline{\phi}_1(q^2)\right)=\frac{\overline{J}_{1,2}J_2^2}{J_4J_1}. \tag{23}$$

Proof. Rewrite the identity (16) as

$$\sum_{n=-\infty}^{\infty}\frac{(a;q)_{2n}}{(b;q)_{2n}}z^{2n}+\sum_{n=-\infty}^{\infty}\frac{(a;q)_{2n+1}}{(b;q)_{2n+1}}z^{2n+1}=\frac{(q,b/a,az,q/az;q)_\infty}{(b,q/a,z,b/az;q)_\infty}.$$

Making the substitutions $a=-q$, $b=0$ and $z=\sqrt{q}$ in (16), we obtain

$$\sum_{n=-\infty}^{\infty}(-q;q)_{2n}q^n+\sqrt{q}\sum_{n=-\infty}^{\infty}(-q;q)_{2n+1}q^n=\frac{(q,-q^{3/2},-q^{-1/2};q)_\infty}{(-1,q^{1/2};q)_\infty}, \tag{24}$$

i.e.,

$$\begin{aligned}&\left(\sum_{n=0}^{\infty}(-q;q)_{2n}q^n+\frac{1}{2}\sum_{n=0}^{\infty}\frac{q^{2n^2+2n}}{(-q;q)_{2n+1}}\right)\\&+\sqrt{q}\left(\sum_{n=0}^{\infty}(-q;q)_{2n+1}q^n+\frac{1}{2q}\sum_{n=0}^{\infty}\frac{q^{2n^2}}{(-q;q)_{2n}}\right)\\=&\frac{(q,-q^{3/2},-q^{-1/2};q)_\infty}{2(-q,q^{1/2};q)_\infty}.\end{aligned}$$

Finally, replacing q by q^2 and after some simplifications, we get our desired result. This completes the proof. □

Combining (5), (6) with (23), we obtain the following result:

Corollary 3.5. *For $|q|<1$, we have*

$$q\overline{J}_{2,16}(J_{2,4}+2\overline{J}_{4,8})-\overline{J}_{6,16}(J_{2,4}-2\overline{J}_{4,8})=\frac{\overline{J}_{1,2}J_2^2}{J_1}. \tag{25}$$

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