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NODAL SOLUTIONS FOR AN ELLIPTIC EQUATION IN AN ANNULUS WITHOUT THE SIGNUM CONDITION

Tianlan Chen, Yanqiong Lu, and Ruyun Ma

Abstract. This paper is concerned with the global behavior of components of radial nodal solutions of semilinear elliptic problems

$$-\Delta v = \lambda h(x, v) \ \text{ in } \Omega, \quad v = 0 \ \text{ on } \partial\Omega,$$

where $\Omega = \{x \in \mathbb{R}^N : r_1 < |x| < r_2\}$ with $0 < r_1 < r_2,\ N \geq 2$. The nonlinear term is continuous and satisfies $h(x, 0) = h(x, s_1(x)) = h(x, s_2(x)) = 0$ for suitable positive, concave function s_1 and negative, convex function s_2, as well as $sh(x, s) > 0$ for $s \in \mathbb{R} \setminus \{0, s_1(x), s_2(x)\}$. Moreover, we give the intervals for the parameter λ which ensure the existence and multiplicity of radial nodal solutions for the above problem. For this, we use global bifurcation techniques to prove our main results.

1. Introduction

In this paper, by applying global bifurcation techniques, we study the existence and multiplicity of radial nodal solutions of elliptic equations in annular bounded domains, i.e.,

$$\begin{cases} -\Delta v = \lambda h(x, v) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega = \{x \in \mathbb{R}^N : r_1 < |x| < r_2\}$ with $0 < r_1 < r_2,\ N \geq 2$ and $\lambda > 0$ is a real parameter.

This model was studied by several authors (see e.g. [1, 2, 7, 8, 18]). In these papers, existence and multiplicity of positive radial solutions were studied under several behaviors of the nonlinearity, in particular, nonlinearities which are superlinear at infinity and with different behaviors at the origin are considered.

On the other hand, more details about the geometry of the nonlinearity become important when one is interested in the multiplicity of solutions, see

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for example the nice review in Lions [9], as was remarked in this paper, the presence of zeros in the nonlinearity usually provides multiple solutions.

This paper is motivated by the recent work of Iturriaga, Massa, Sánchez and Ubilla [8] who proved the existence and multiplicity of positive radial solutions of (1). Since the radial problem in the annulus may be reduced to an ODE's problem, it will be possible to take advantage of some techniques available in dimension one. Consequently, they restricted the attention to the problem

$$u'' + \lambda q(t)\tilde{f}(t,u) = 0,\ t \in (0,1), \quad u(0) = 0 = u(1). \tag{2}$$

In fact, the change of variables

$$t = -\frac{A}{r^{N-2}} + B \ \text{ and } \ u(t) = v(r) \ \text{ if } N \geq 3 \tag{3}$$

and

$$r = r_2\Big(\frac{r_1}{r_2}\Big)^t \ \text{ and } \ u(t) = v(r) \ \text{ if } N = 2 \tag{4}$$

transform (1) into (2), where $A = \frac{(r_1r_2)^{N-2}}{r_2^{N-2}-r_1^{N-2}}$, $B = \frac{r_2^{N-2}}{r_2^{N-2}-r_1^{N-2}}$. Note that, in both cases, the function $q(t)$ is well defined, continuous and bounded between positive constants in the interval $[0,1]$.

Moreover, Iturriaga, Massa, Sánchez and Ubilla [8] made the following assumptions for $\tilde{f}$:

(H1) $\tilde{f} : [0,1]\times[0,+\infty) \to [0,+\infty)$ is continuous and there exists a continuous function $s_1 : [0,1] \to (0,+\infty)$, which is concave, such that $\tilde{f}(t,0) = \tilde{f}(t,s_1(t)) = 0$ and $\tilde{f}(t,s) > 0$ for $0 < s < s_1(t)$;

(H2) there exists a continuous function $a : [0,1] \to (0,+\infty)$ such that

$$\lim_{s\to 0+} \frac{\tilde{f}(t,s)}{s} = a(t) \ \text{ uniformly on } [0,1];$$

(H3) there exist positive constants α, β ($\alpha < \beta < 1$) such that

$$\lim_{s\to +\infty} \frac{\tilde{f}(t,s)}{s} = +\infty \ \text{ uniformly on } [\alpha,\beta];$$

(M) (a) the function $\tilde{f}_s := \frac{\partial \tilde{f}}{\partial s}$ exists and is continuous for $t \in [0,1]$, $u \in [0, s_1(t)]$;

(b) $\tilde{f}_s < \frac{\tilde{f}(t,s)}{s}$ for $t \in (0,1)$, $u \in (0, s_1(t))$.

Let $\lambda_{k,m}$ be the k-th eigenvalue of the eigenvalue problem

$$u'' + \lambda m(t)u = 0,\ t \in (0,1), \quad u(0) = 0 = u(1), \tag{5}$$

where $m : [0,1] \to \mathbb{R}$ is a continuous function. Using the the sub and super-solutions method, a priori estimates and degree theory, they established the following results.

Theorem 1.1. *Assume* (H1)-(H3). *Then there exists a positive solution of problem* (2) *for every* $0 < \lambda < \lambda_{1,qa}$.

Theorem 1.2. *Assume* (H1)-(H3) *and* (M). *Then there exists at least two ordered positive solutions of problem* (2) *for every* $\lambda > \lambda_{1,qa}$.

In this paper, we extend the function $\tilde{f}$ to continuous function f satisfying our assumptions

(A1) $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists a continuous and concave function $s_1 : [0,1] \to (0,+\infty)$ such that $f(t, s_1(t)) = 0 = f(t,0)$ and $sf(t,s) > 0$ for $s \in \mathbb{R} \setminus \{0, s_1(t)\}$;

(A2) $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists a continuous and convex function $s_2 : [0,1] \to (-\infty,0)$ such that $f(t, s_2(t)) = 0 = f(t,0)$ and $sf(t,s) > 0$ for $s \in \mathbb{R} \setminus \{0, s_2(t)\}$;

(A3) there exists a continuous function $a : [0,1] \to (0,+\infty)$ such that

$$\lim_{|s|\to 0} \frac{f(t,s)}{s} = a(t) \ \text{ uniformly on } [0,1];$$

(A4) there exist positive constants $0 < \alpha < \beta < 1$ such that

$$\lim_{|s|\to \infty} \frac{f(t,s)}{s} = +\infty \ \text{ uniformly on } [\alpha,\beta].$$

Applying global bifurcation techniques, we show the existence and multiplicity of nodal solutions to the problem of the similar form

$$u'' + \lambda q(t) f(t,u) = 0,\ t \in (0,1), \quad u(0) = 0 = u(1). \tag{6}$$

We shall obtain the similar results as in [10] or [4] (with $p = 2$) for problem (6), see Theorem 3.4 and Corollary 3.5 below. Moreover, as we said at the beginning, our results for problem (6) may be applied on the existence and multiplicity of radial nodal solutions for (1) by the change of variables (3) and (4).

On the other hand, assume that

(A5) there exists a continuous function $b : [0,1] \to (0,+\infty)$ such that

$$\lim_{|s|\to \infty} \frac{f(t,s)}{s} = b(t) \ \text{ uniformly on } [0,1].$$

According to the proof of [3, Theorems 4.1 and 4.2] with $p = 2$, hypotheses (A2) and (A5) imply that the component from the trivial solution at $(\lambda_{k,qa}, 0)_p$ and the component from infinity at $(\lambda_{k,qb}, \infty)_p$ are coincident (we shall denote by $(a,b)_p$ the 'point' in some product spaces, and (a,b) the usual open interval in this paper). Furthermore, if there exist positive constants $f_0, f_\infty \in (0,\infty)$ such that $a(t) \equiv f_0$, $b(t) \equiv f_\infty$, see [14] for details. However, we shall show in Section 3 that these two components are disjoint under assumptions (A1)-(A3) and (A5), see Theorem 3.1 and Corollaries 3.2-3.3 below. Hence the essential role is played by the fact of whether f possesses these variable zeros.

For other results on the global bifurcation structure of nodal solutions of nonlinear elliptic problems, see e.g. Ma, Chen and Lu [13] and the references therein.

The rest of this paper is arranged as follows. In Section 2, we state some known results that will be used in the paper. Finally in Section 3, we study the global behavior of the components of nodal solutions of problem (6).

2. Preliminaries

We state some properties of the superior limit of a certain infinity collection of connected sets. Let M be a metric space and $\{C_n \mid n = 1, 2, \ldots\}$ be a family of subsets of M. Then the superior limit $\mathbb{D}$ of $\{C_n\}$ is defined by

$$\mathbb{D} := \limsup_{n\to\infty} C_n = \{x \in M \mid \exists\ \{n_k\} \subset \mathbb{N},\ x_{n_k} \in C_{n_k} \text{ such that } x_{n_k} \to x\}. \tag{7}$$

A *component* of a set M means a maximal connected subset of M, see [19] for the detail.

Lemma 2.1 (see [12, Lemma 2.4] and [11, Lemma 2.2])**.** *Let X be a Banach space and let $\{C_n\}$ be a family of closed connected subsets of X. Assume that*
(i) *there exist $z_n \in C_n$, $n = 1, 2, \ldots$ and $z_* \in X$ such that $z_n \to z_*$;*
(ii) $\lim_{n\to\infty} r_n = \lim_{n\to\infty} \sup\{\|u\| \mid u \in C_n\} = \infty$;
(iii) *for every $R > 0$, $(\bigcup_{n=1}^{\infty} C_n) \cap B_R$ is a relatively compact of X.*
Then there exists an unbounded component $\mathbb{C}$ in $\mathbb{D}$ and $z_* \in \mathbb{C}$.

Finally, we need to introduce the following strict monotonicity property with respect to the weight m for the eigenvalue problem (5), see [6].

Lemma 2.2 (see [6])**.** *Consider the eigenvalue problem* (5)*. Let m and $\widehat{m}$ be two bounded weights with $m \leq\not\equiv \widehat{m}$ ($\leq\not\equiv$ means inequality a.e. strict inequality on a set of positive measure) and let $j \in \mathbb{Z}_0$. Then $\lambda_{j,m} > \lambda_{j,\widehat{m}}$ whenever they exist.*

3. Global behavior of the components of nodal solutions

Let

$$Y = C[0,1], \quad E = \{u \in C^1[0,1] \mid u(0) = 0 = u(1)\}$$

be the Banach spaces endowed with the norms

$$\|u\|_\infty = \max_{t\in[0,1]} |u(t)|, \quad \|u\| = \max\{\|u\|_\infty,\ \|u'\|_\infty\},$$

respectively.

Let S_k^+ denote the set of functions in E which have exactly $k-1$ simple (non-degenerate) zeros in $(0,1)$ and are positive near $t = 0$. Set $S_k^- = -S_k^+$ and $S_k = S_k^- \cup S_k^+$. It is clear that S_k^- and S_k^+ are disjoint and open in E. Finally, for $\nu \in \{+,-\}$, let $\Phi_k^\nu = \mathbb{R} \times S_k^\nu$ and $\Phi_k = \mathbb{R} \times S_k$.

Let $\xi, \zeta \in C([0,1] \times \mathbb{R})$ be such that

$$f(t,u) = a(t)u + \zeta(t,u), \quad f(t,u) = b(t)u + \xi(t,u).$$

Clearly,

$$\lim_{|u|\to 0}\frac{\zeta(t,u)}{u}=0 \text{ and } \lim_{|u|\to\infty}\frac{\xi(t,u)}{u}=0 \text{ uniformly on } [0,1].$$

Let us consider

$$\begin{cases} u''+\lambda q(t)a(t)u+\lambda q(t)\zeta(t,u)=0, & t\in(0,1),\\ u(0)=0=u(1) \end{cases} \tag{8}$$

as a bifurcation problem from the trivial solution $u\equiv 0$, and

$$\begin{cases} u''+\lambda q(t)b(t)u+\lambda q(t)\xi(t,u)=0, & t\in(0,1),\\ u(0)=0=u(1) \end{cases} \tag{9}$$

as a bifurcation problem from infinity.

Applying Theorem 2 of [5] to (8), we have that for each integer $k\geq 1$, $\nu\in\{+,-\}$, there exists a continuum $\mathcal{C}_{k,0}^{\nu}$ of solutions of (6) joining $(\lambda_{k,qa},0)_p$ to infinity, and $\mathcal{C}_{k,0}^{\nu}\setminus\{(\lambda_{k,qa},0)_p\}\subset\Phi_k^{\nu}$.

Applying Theorem 1.6 and Corollary 1.8 of [17] to (9), we can show that for each integer $k\geq 1$, $\nu\in\{+,-\}$, there exists a continuum $\mathcal{D}_{k,\infty}^{\nu}$ of solutions of (6) meeting $(\lambda_{k,qb},\infty)_p$ and $\mathcal{D}_{k,\infty}^{\nu}\setminus\{(\lambda_{k,qb},\infty)_p\}\subset\Phi_k^{\nu}$.

Next, we prove that these two components are disjoint under the assumptions (A1) and (A2). Hence the essential role is played by the fact of whether f possesses these variable zeros.

Theorem 3.1. *Assume* (A1)-(A3) *and* (A5).

(i) *If* $(\lambda,u)\in(\mathcal{C}_{k,0}^{+}\cup\mathcal{C}_{k,0}^{-})$, *then*

$$s_2(t)<u(t)<s_1(t),\qquad t\in[0,1]; \tag{10}$$

(ii) *If* $(\lambda,u)\in(\mathcal{D}_{k,\infty}^{+}\cup\mathcal{D}_{k,\infty}^{-})$, *then either*

$$u(t_0)>s_1(t_0) \text{ or } u(t_0)<s_2(t_0) \tag{11}$$

for some $t_0\in(0,1)$.

Proof. Suppose, for sake of contradiction, that there exist $(\lambda,u)\in\mathcal{C}_{k,0}^{+}\cup\mathcal{C}_{k,0}^{-}\cup\mathcal{D}_{k,\infty}^{+}\cup\mathcal{D}_{k,\infty}^{-}$ and $t_1,t_2\in(0,1)$ such that

$$\max\{u(t)\mid t\in[0,1]\}=s_1(t_1),$$

or

$$\min\{u(t)\mid t\in[0,1]\}=s_2(t_2).$$

Let

$$0=\tau_0<\tau_1<\cdots<\tau_k=1$$

denote the zeros of u. In the following we will divide the proof into two cases.

Case 1. $\max\{u(t)\mid t\in[0,1]\}=s_1(t_1)$.

In this case, there exists $j\in\{0,\dots,k-1\}$ such that $t_1\in(\tau_j,\tau_{j+1})$ and

$$u(t)\leq s_1(t),\quad t\in[\tau_j,\tau_{j+1}].$$

We claim that there exists a constant $m > 0$ such that

$$f(t,u) \le m(s_1(t) - u) \quad \text{and} \quad 0 \le u \le s_1(t) \quad \text{for all } t \in [\tau_j, \tau_{j+1}]. \tag{12}$$

Obviously, the claim is true for the case $u = 0$ or $u = s_1(t)$ by (A1). Therefore, assume by contradiction that there exists $s_0 \in (0, s_1(t))$ such that $f(t, s_0) > m(s_1(t) - s_0)$ for any $m > 0$. It is apparent that $m < \frac{f(t,s_0)}{s_1(t)-s_0}$, which contradicts the arbitrariness of m.

Combining (12) and s_1 is concave, we have that

$$\begin{aligned} &-(s_1(t) - u(t))'' + \lambda m q(t)(s_1(t) - u(t)) \\ &\ge \lambda m q(t)(s_1(t) - u(t)) - \lambda q(t) f(t,u) \ge 0, \quad t \in (\tau_j, \tau_{j+1}). \end{aligned}$$

On the other hand, it can be easily seen from $s_1(t) > 0$ that

$$s_1(\tau_j) - u(\tau_j) > 0, \quad s_1(\tau_{j+1}) - u(\tau_{j+1}) > 0. \tag{13}$$

Consequently, the strong maximum principle of [16] implies that $s_1(t) > u(t)$ in $[\tau_j, \tau_{j+1}]$, which is a contradiction.

Case 2. $\min\{u(t) \mid t \in [0,1]\} = s_2(t_2)$.

In this case, there exists $j \in \{0, \dots, k-1\}$ such that $t_2 \in (\tau_j, \tau_{j+1})$ and

$$u(t) \ge s_2(t), \quad t \in [\tau_j, \tau_{j+1}].$$

By the similar argument to treat (12), owing to (A2), we can also show that there exists a constant $m > 0$ such that

$$f(t,u) \ge m(s_2(t) - u) \quad \text{and} \quad s_2(t) \le u \le 0 \quad \text{for all } t \in [\tau_j, \tau_{j+1}]. \tag{14}$$

Combining this with s_2 is convex, we get

$$\begin{aligned} &-(s_2(t) - u(t))'' + \lambda m q(t)(s_2(t) - u(t)) \\ &\le \lambda m q(t)(s_2(t) - u(t)) - \lambda q(t) f(t,u) \le 0, \quad t \in (\tau_j, \tau_{j+1}). \end{aligned}$$

On the other hand, it concludes from $s_2(t) < 0$ that

$$s_2(\tau_j) - u(\tau_j) < 0, \quad s_2(\tau_{j+1}) - u(\tau_{j+1}) < 0. \tag{15}$$

The strong maximum principle of [16] implies that $s_2(t) < u(t)$ in $[\tau_j, \tau_{j+1}]$, which is a contradiction and ends the proof. □

For the convenience, we denote $u^+_{k,\infty}$, $u^-_{k,\infty}$, $u^+_{k,0}$ and $u^-_{k,0}$ by functions of $\mathcal{D}^+_{k,\infty}, \mathcal{D}^-_{k,\infty}$, $\mathcal{C}^+_{k,0}$ and $\mathcal{C}^-_{k,0}$ respectively, such that $u^+_{k,\infty}$ changes sign exactly $k-1$ times in $(0,1)$ and is positive near 0; $u^-_{k,\infty}$ changes sign exactly $k-1$ times in $(0,1)$ and is negative near 0; $u^+_{k,0}$ changes sign exactly $k-1$ times in $(0,1)$ and is positive near 0; $u^-_{k,0}$ changes sign exactly $k-1$ times in $(0,1)$ and is negative near 0.

According to Theorem 3.1 and Lemma 2.2, using the similar argument to prove [10, Corollaries 2.1 and 2.2] with obvious changes, we obtain the following results.

Corollary 3.2. *Assume* (A1)-(A3) *and* (A5). *Let* $a(t) \leq\not\equiv b(t)$. *Then for some* $k \in \mathbb{N}$,

(i) *if* $\lambda \in (\lambda_{k,qb}, \lambda_{k,qa}]$, *then problem* (6) *has two solutions* $u^+_{k,\infty}$ *and* $u^-_{k,\infty}$;

(ii) *if* $\lambda \in (\lambda_{k,qa}, +\infty)$, *then problem* (6) *has four solutions* $u^+_{k,\infty}, u^-_{k,\infty}, u^+_{k,0}$ *and* $u^-_{k,0}$.

Corollary 3.3. *Assume* (A1)-(A3) *and* (A5). *Let* $b(t) \leq\not\equiv a(t)$. *Then for some* $k \in \mathbb{N}$,

(i) *if* $\lambda \in (\lambda_{k,qa}, \lambda_{k,qb}]$, *then problem* (6) *has two solutions* $u^+_{k,0}$ *and* $u^-_{k,0}$;

(ii) *if* $\lambda \in (\lambda_{k,qb}, +\infty)$, *then problem* (6) *has four solutions* $u^+_{k,\infty}, u^-_{k,\infty}, u^+_{k,0}$ *and* $u^-_{k,0}$.

The next result establishes the global behavior of components of nodal solutions under the hypothesis (A4) (which means that the nonlinearity is locally superlinear at $+\infty$).

Theorem 3.4. *Assume* (A1), (A3) *and* (A4). *Then for some* $k \in \mathbb{N}$,

(i) *if* $\lambda \in (0, \lambda_{k,qa})$, *then problem* (6) *has two solutions* $u^+_{k,\infty}$ *and* u^-_k, *where* u^-_k *changes sign exactly* $k-1$ *times in* $(0,1)$ *and is negative near* 0;

(ii) *if* $\lambda = \lambda_{k,qa}$, *then problem* (6) *has one solution* $u^+_{k,\infty}$;

(iii) *if* $\lambda \in (\lambda_{k,qa}, +\infty)$, *then problem* (6) *has two solutions* $u^+_{k,\infty}$, $u^+_{k,0}$.

Proof. For any $n \in \mathbb{N}$ and $n > s_1(t)$. Let us define the function $f^{[n]} : [0,1]\times\mathbb{R} \to \mathbb{R}$,

$$f^{[n]}(t,s) := \begin{cases} f(t,s), & |s| \leq n, \quad t \in [0,1], \\ \frac{f(t,\ n)}{n}s, & |s| > n, \quad t \in [0,1]. \end{cases} \tag{16}$$

Then $f^{[n]} : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous and, for all $t \in [0,1]$,

$$f^{[n]}(t, s_1(t)) = f^{[n]}(t,0) = 0, \quad sf^{[n]}(t,s) > 0 \text{ on } s \in \mathbb{R} \setminus \{0, s_1(t)\},$$

and

$$(f^{[n]})_\infty = \frac{f(t,\ n)}{n}.$$

According to (A4), we have

$$\lim_{n\to\infty} (f^{[n]})_\infty = +\infty \text{ uniformly on } [\alpha, \beta]. \tag{17}$$

Consider the auxiliary problem

$$\begin{cases} u'' + \lambda q(t) f^{[n]}(t,u) = 0, \quad t \in (0,1), \\ u(0) = 0 = u(1). \end{cases} \tag{18}$$

Let $\xi^{[n]} \in C([0,1] \times \mathbb{R})$ be such that

$$f^{[n]}(t,u) = (f^{[n]})_\infty u + \xi^{[n]}(t,u).$$

Then

$$\lim_{|u|\to\infty}\frac{\xi^{[n]}(t,u)}{u}=0 \quad \text{uniformly on } [0,1]. \tag{19}$$

Let us consider

$$\begin{cases} u''+\lambda q(t)(f^{[n]})_\infty u+\lambda q(t)\xi^{[n]}(t,u)=0, & t\in[0,1],\\ u(0)=0=u(1) \end{cases} \tag{20}$$

as a bifurcation problem from infinity.

Applying Theorem 1.6 and Corollary 1.8 of [17] to (20), we have that for each integer $k\geq 1$ and $n\in\mathbb{N}$ with $n>s_1(t)$, there exists a continuum $\mathcal{D}_{k,\infty}^{[n],+}$ of solutions of (18) meeting $(\frac{\lambda_{k,q}}{(f^{[n]})_\infty},\infty)_p$ and $\mathcal{D}_{k,\infty}^{[n],+}\setminus\{(\frac{\lambda_{k,q}}{(f^{[n]})_\infty},\infty)_p\}\subset\Phi_k^+$.

By the similar argument to prove Theorem 3.1, for any $(\lambda,u)\in\mathcal{D}_{k,\infty}^{[n],+}$, we can show that $u(t_0)>s_1(t_0)$ for some $t_0\in(0,1)$. Moreover,

$$\sup\{\lambda\mid(\lambda,u)\in\mathcal{D}_{k,\infty}^{[n],+}\}=\infty. \tag{21}$$

Next we show that, for each $n\in\mathbb{N}$ and $n>s_1(t)$, there exists a positive constant A such that

$$\sup\{\|u\|_\infty\mid(\lambda,u)\in\mathcal{D}_{k,\infty}^{[n],+} \text{ and } \lambda\in I\}\leq A, \tag{22}$$

if $I\subset(\frac{\lambda_{k,q}}{(f^{[n]})_\infty},\infty)$ is a closed and bounded interval.

Assume by contradiction that there exists a sequence $\{(\eta_m,u_m)\}\subset\mathcal{D}_{k,\infty}^{[n],+}\cap(I\times E)$ satisfying

$$\|u_m\|\to\infty \quad \text{as } m\to\infty. \tag{23}$$

We claim that

$$\|u_m\|_\infty\to\infty \quad \text{as } m\to\infty. \tag{24}$$

Indeed, it is easy to see that (η_m,u_m) satisfies

$$u_m''+\eta_m q(t)f^{[n]}(t,u_m)=0,\ t\in(0,1), \qquad u_m(0)=0=u_m(1), \tag{25}$$

therefore, there exists $t_m\in(0,1)$ with $u_m'(t_m)=0$ and

$$u_m'(t)=-\int_{t_m}^{t}\eta_m q(s)f^{[n]}(s,u_m(s))ds. \tag{26}$$

If there exists a positive constant M such that $\|u_m\|_\infty\leq M$ for each m, then, according to (24) and the definition of $f^{[n]}$, we obtain that

$$\|u_m'\|_\infty\leq M_1 \quad \text{for some } M_1>0 \text{ and all } m,$$

which contradicts (23) and ends the proof of (24).

Let

$$0=\tau(0,m)<\tau(1,m)<\cdots<\tau(k,m)=1 \tag{27}$$

denote the zeros of u_m. Taking a subsequence and relabeling if necessary, we may assume that for each $l \in \{0, \dots, k\}$,

$$\lim_{m\to\infty} \tau(l, m) = \tau(l, \infty). \tag{28}$$

On the other hand, we can easily obtain that for all $l \in \{0, \cdots, k-1\}$,

$$\begin{aligned} &\min\{(-1)^l u_m(t) \ : \ t \in J(l, m)\} \\ &\geq \gamma \max\{|u_m(t)| \ : \ t \in [\tau(l, m), \tau(l+1, m)]\}, \end{aligned} \tag{29}$$

where

$$J(l, m) := \big[\tau(l, m) + \frac{\tau(l+1, m) - \tau(l, m)}{4},\ \tau(l+1, m) - \frac{\tau(l+1, m) - \tau(l, m)}{4}\big].$$

According to (24) and (29), we must have that there exist $l^* \in \{0, \dots, k-1\}$ and a closed interval $J_1 \subset (\tau(l^*, \infty), \tau(l^*+1, \infty))$ with positive length such that

$$(-1)^{l^*} u_m(t) \to \infty \quad \text{as } m \to \infty \tag{30}$$

uniformly for $t \in J_1$. Owing to $\{\eta_m\} \subset I$, we can easily see that there exists η_* ($\eta_* > \frac{\lambda_{k,q}}{(f^{[n]})_\infty}$) such that

$$\lim_{m\to\infty} \eta_m = \eta_*. \tag{31}$$

According to (30) and (31), we are driven to

$$\lim_{m\to\infty} \eta_m \frac{f^{[n]}(t, u_m)}{u_m} = \eta_*(f^{[n]})_\infty \quad \text{uniformly on } t \in J_1. \tag{32}$$

This together with $\eta_*(f^{[n]})_\infty > \lambda_{k,q}$ and

$$u_m'' + \eta_m q(t) \frac{f^{[n]}(t, u_m)}{u_m} u_m = 0, \quad t \in J_1, \tag{33}$$

we deduce that u_m must change its sign on J_1 if m is large enough. This is a contradiction and ends the proof of (22).

Now let us verify that $\{\mathcal{D}_{k,\infty}^{[n],+}\}$ satisfy all of conditions of Lemma 2.1. Since

$$\lim_{n\to\infty} \frac{\lambda_{k,q}}{(f^{[n]})_\infty} = \lim_{n\to\infty} \frac{\lambda_{k,q}}{\frac{f(t,n)}{n}} = 0 \quad \text{uniformly on } t \in [\alpha, \beta],$$

it follows from (22) that we can find a closed interval $J \subset (0, \infty)$ and a positive constant r, set $\Gamma = \{u \in E \mid s_1(t_0) < \|u\|_\infty < r\}$, then there exists $z_{n_j} \in \mathcal{D}_{k,\infty}^{[n],+} \cap (J \times \Gamma)$ such that $z_{n_j} \to z_*$, so condition (a) in lemma 2.1 is satisfied, and obviously, (b) holds. (c) can be deduced directly from Arzela-Ascoli Theorem and the definition of $f^{[n]}$. Therefore, the superior limit of $\{\mathcal{D}_{k,\infty}^{[n],+}\}$ contain an unbounded connected component $\bar{\mathcal{D}}_{k,\infty}^{+}$. Moreover, it follows from (21) and the definition of $\limsup_{n\to\infty} \mathcal{D}_{k,\infty}^{[n],+}$ that

$$\sup\{\lambda \mid (\lambda, u) \in \bar{\mathcal{D}}_{k,\infty}^{+}\} = \infty. \tag{34}$$

By the similar method to prove Theorem 3.1, for $(\lambda,u)\in\bar{\mathcal{D}}^+_{k,\infty}$, it becomes apparent that $u(t_0)>s_1(t_0)$ for some $t_0\in(0,1)$.

We will show that

$$\lim_{(\lambda,u)\in\bar{\mathcal{D}}^+_{k,\infty},\|u\|\to\infty}\lambda=0. \tag{35}$$

Indeed, assume by contradiction that there exists $\{(\lambda_n,u_n)\}\subset\bar{\mathcal{D}}^+_{k,\infty}$ such that $\|u_n\|\to\infty,\lambda_n\geq a_0$ for some positive constant a_0. Then (24), (29) and (30) hold. According to (A4) and (30), one has that

$$\lim_{n\to\infty}\frac{f(t,u_n)}{u_n}=\infty \text{ uniformly on } t\in J_1,$$

which implies that, for all n sufficiently large, the solution u_n of

$$u_n''+\lambda_n q(t)\frac{f(t,u_n)}{u_n}u_n=0$$

must change its sign on J_1. This contradicts (30) and so (35) holds. Owing to (34) and (35), it becomes apparent that

$$\mathrm{Proj}_{\mathbb{R}}\bar{\mathcal{D}}^+_{k,\infty}=(0,\infty). \tag{36}$$

On the other hand, by Theorem 3.1(i), for any $(\lambda,u)\in\mathcal{C}^+_{k,0}$,

$$\|u\|_\infty<\|s_1\|_\infty. \tag{37}$$

This together with (8) imply that

$$\|u\|<\max\{\|s_1\|_\infty,\ \lambda\|q\|_\infty\max_{t\in[0,1],|s|\leq\|s_1\|_\infty}|f(t,s)|\}, \tag{38}$$

which means that the set $\{(\mu,z)\in\mathcal{C}^+_{k,0}\mid\mu\in[0,d]\}$ is bounded for any fixed $d\in(0,\infty)$. This together with the fact that $\mathcal{C}^+_{k,0}$ joins $(\lambda_{k,qa},0)_p$ to infinity yields that

$$\mathrm{Proj}_{\mathbb{R}}\mathcal{C}^+_{k,0}\supset(\lambda_{k,qa},+\infty). \tag{39}$$

Finally, by applying Theorem 2 of [5] to (8), for each integer $k\geq1$, we have that there exists unbounded continuum $\mathcal{C}^-_k$ joining $(\lambda_{k,qa},0)_p$ to infinity such that $\mathcal{C}^-_k\setminus\{(\lambda_{k,qa},0)_p\}\subset\Phi^-_k$.

In the following, we shall use some idea from the proof of [15, Theorem 1.1] to prove $\mathcal{C}^-_k$ joins $(\lambda_{k,qa},0)_p$ to $(0,\infty)_p$.

Let $\{(\mu_m,u_m)\}\subset\mathcal{C}^-_k$ be such that $|\mu_m|+\|u_m\|\to\infty$ as $m\to\infty$. If $\{\|u_m\|\}$ is bounded, then we may assume that

$$\lim_{m\to\infty}\mu_m=\infty. \tag{40}$$

Using (27), (28) and noting that $\sum_{l=0}^{k-1}[\tau(l+1,\infty)-\tau(l,\infty)]=1$, it follows that there exists $l_0\in\{0,\dots,k-1\}$ such that

$$\tau(l_0,\infty)<\tau(l_0+1,\infty). \tag{41}$$

Thus there exist $m_0 \in \mathbb{N}$ and a closed interval $I_0 \subset (\tau(l_0, \infty), \tau(l_0+1, \infty))$ with positive length such that $I_0 \subset (\tau(l_0, m), \tau(l_0+1, m))$ for all $m \geq m_0$, and hence

$$(-1)^{l_0+1} u_m > 0 \quad \text{for all } m \geq m_0,\ t \in I_0. \tag{42}$$

Then it follows from (40) and

$$-u_m'' = \mu_m q(t) f(t, u_m), \quad t \in I_0, \tag{43}$$

that u_m must change its sign on I_0 if m is large enough, which contradicts (42). Hence $\{\|u_m\|\}$ is unbounded. Thus, by using the similar argument to prove (35), we can also show $\lim_{m\to\infty} \mu_m = 0$, and so

$$\mathrm{Proj}_{\mathbb{R}} \mathcal{C}_k^- \supset (0, \lambda_{k,qa}). \tag{44}$$

Consequently, according to (36), (39) and (44), we get the desired results. □

As an immediate consequence of Theorem 3.4, we get the following:

Corollary 3.5. *Assume* (A2), (A3) *and* (A4). *Then for some* $k \in \mathbb{N}$,

(i) *if* $\lambda \in (0, \lambda_{k,qa})$, *then problem* (6) *has at least two solutions* $u_{k,\infty}^-$ *and* u_k^+, *where* u_k^+ *has exactly* $k-1$ *zeros in* $(0,1)$ *and is positive near* 0;

(i) *if* $\lambda = \lambda_{k,qa}$, *then problem* (6) *has at least one solution* $u_{k,\infty}^-$;

(ii) *if* $\lambda \in (\lambda_{k,qa}, +\infty)$, *then problem* (6) *has at least two solutions* $u_{k,\infty}^-$, $u_{k,0}^-$.

Remark 3.6. If (A1) holds, then for all $t \in [0,1]$,

$$f(t,s) > 0, \quad s \in (0, s_1(t)) \cup (s_1(t), \infty),$$

which is a stronger condition imposed on f than (H1). However, under (A1), we can get the same interval in which (6) has one positive solution and one negative solution without the assumption (M). Compared with Theorems 1.1 and 1.2, by using the global bifurcation theory, Theorem 3.4 implies that there exists at least one solution for (6) if $\lambda = \lambda_{k,qa}$.

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Tianlan Chen
Department of Mathematics
Northwest Normal University
Lanzhou 730070, P. R. China
Email address: `chentianlan511@126.com`

Yanqiong Lu
Department of Mathematics
Northwest Normal University
Lanzhou 730070, P. R. China
Email address: linmu8610@163.com

Ruyun Ma
Department of Mathematics
Northwest Normal University
Lanzhou 730070, P. R. China
Email address: mary@nwnu.edu.cn