

## MODULAR INVARIANTS UNDER THE ACTIONS OF SOME REFLECTION GROUPS RELATED TO WEYL GROUPS

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ABSTRACT. Some modular representations of reflection groups related to Weyl groups are considered. The rational cohomology of the classifying space of a compact connected Lie group  $G$  with a maximal torus  $T$  is expressed as the ring of invariants,  $H^*(BG; \mathbb{Q}) \cong H^*(BT; \mathbb{Q})^{W(G)}$ , which is a polynomial ring. If such Lie groups are locally isomorphic, the rational representations of their Weyl groups are equivalent. However, the integral representations need not be equivalent. Under the mod  $p$  reductions, we consider the structure of the rings, particularly for the Weyl group of symplectic groups  $Sp(n)$  and for the alternating groups  $A_n$  as the subgroup of  $W(SU(n))$ . We will ask if such rings of invariants are polynomial rings, and if each of them can be realized as the mod  $p$  cohomology of a space. For  $n = 3, 4$ , the rings under a conjugate of  $W(Sp(n))$  are shown to be polynomial, and for  $n = 6, 8$ , they are non-polynomial. The structures of  $H^*(BT^{n-1}; \mathbb{F}_p)^{A_n}$  will be also discussed for  $n = 3, 4$ .

The invariant theory of some finite groups will be discussed, [20] and [19]. For any prime  $p$  we note that  $H^*(BT^n; \mathbb{F}_p) = \mathbb{F}_p[t_1, t_2, \dots, t_n]$ , a polynomial ring generated by  $n$  elements of degree 2. When  $p$  does not divide the order of a subgroup  $W$  of  $GL(n, \mathbb{F}_p)$ , it is well-known that the invariant ring  $\mathbb{F}_p[t_1, t_2, \dots, t_n]^W$  is a polynomial ring if and only if  $W$  is a pseudo-reflection group, [13, §20-2, §20-3]. This result can fail in a modular case. Namely, even if  $W$  is a pseudo-reflection group, the invariant ring need not be a polynomial ring for  $|W| \equiv 0 \pmod{p}$ . This paper concerns such uncertainty.

The Weyl group of  $SU(n)$  is isomorphic to the symmetric group  $\Sigma_n$ . The reflection group  $W(SU(n))$  is generated by the permutation matrices  $\Sigma_{n-1}$  and an  $(n-1) \times (n-1)$  reflection, [14, Ch. 3]. For instance,

$$W(SU(3)) = \left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} \right\rangle.$$

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In [15], Kudo considers the invariant ring  $H^*(BT^{n-1}; \mathbb{F}_p)^{W_{n,d}}$ , where  $W_{n,d} = \phi_d W(SU(n)) \phi_d^{-1} = W(SU(n)/\mathbb{Z}_d)$  for the following matrix:

$$\phi_d = \begin{pmatrix} 1 & & & 0 \\ & \ddots & & \vdots \\ & & 1 & 0 \\ \frac{1-d}{d} & \dots & \frac{1-d}{d} & \frac{1}{d} \end{pmatrix}.$$

In this paper, we consider some mod 2 invariant rings related to the symplectic groups, namely  $H^*(BT^n; \mathbb{F}_2)^{\overline{W}_n}$  where  $\overline{W}_n$  is the mod 2 reduction of  $W_n = \phi_2 W(Sp(n)) \phi_2^{-1}$ . We note that the reflection group  $W(Sp(n))$  is generated by the permutation matrices  $\Sigma_n$  and the  $n \times n$  diagonal matrix  $\begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$ . We also note that  $\phi_d W(Sp(n)) \phi_d^{-1} \subset GL(n, \mathbb{Z})$  if and only if  $d = 2$ .

For a subgroup  $W$  of  $GL(n, \mathbb{F}_2)$ , we see [18, §8.1] that the Dickson algebra  $\mathbb{F}_2[t_1, t_2, \dots, t_n]^{GL(n, \mathbb{F}_2)}$  is included in the invariant ring  $\mathbb{F}_2[t_1, t_2, \dots, t_n]^W$ , and  $\mathbb{F}_2[t_1, t_2, \dots, t_n]^{GL(n, \mathbb{F}_2)} = \mathbb{F}_2[c_{n,n-1}, c_{n,n-2}, \dots, c_{n,0}]$ . Let  $W^*$  denote the dual representation of a subgroup  $W$  of  $GL(n, \mathbb{F}_p)$ . Some comparisons between  $H^*(BT^n; \mathbb{F}_p)^W$  and  $H^*(BT^n; \mathbb{F}_p)^{W^*}$  are done in [9] and [10]. In the case of  $n = d$ , the representation  $W_{n,n} = W(PSU(n))$  is equivalent to the dual representation  $W(SU(n))^*$ . It is known, [7], that for  $p \geq 5$ , the invariant ring  $H^*(BT^{p-1}; \mathbb{F}_p)^{W(SU(p))^*}$  is not polynomial. At  $p = 2$  or 3, the invariant rings are polynomial. Kudo shows [15] that even  $H^*(BT^3; \mathbb{F}_2)^{W_{4,4}}$  is also polynomial. However, for  $n = 6, 8$ , the rings  $H^*(BT^{n-1}; \mathbb{F}_2)^{W_{n,n}}$  are not polynomial, and the rings  $H^*(BT^{n-1}; \mathbb{F}_3)^{W_{n,n}}$  are not polynomial for  $n = 6, 9$ . Duan announced some related work at The 2nd Pan-Pacific International Conference on Topology and Applications, [4].

Although  $|\overline{W}_n| = \frac{|W(Sp(n))|}{2}$ , in a way, our results are similar to the ones in [15] as long as  $n$  is small. Namely, for  $n = 3, 4$ , we will show that all of the invariant rings are polynomial rings, though  $H^*(BT^n; \mathbb{F}_2)^{\overline{W}_n} \not\cong H^*(BT^n; \mathbb{F}_2)^{\overline{W}_n^*}$ . And, except  $H^*(BT^3; \mathbb{F}_2)^{\overline{W}_3^*} \cong H^*(BT^3; \mathbb{F}_2)^{W(SU(4))}$ , the other three invariant rings are not isomorphic to the mod 2 cohomology of spaces. See [3], [17, §3], [18, Ch 10], and [1] for a detail of the realization problem.

**Theorem 1.** (a)  $H^*(BT^3; \mathbb{F}_2)^{\overline{W}_3} = \mathbb{F}_2[x_2, x_8, x_{12}]$ , where  $x_2 = t_3$ ,  $x_8 = (t_1 + t_2)^4 + (t_1 t_2 + t_1 t_3 + t_2 t_3)^2 + t_1 t_2 t_3 (t_1 + t_2 + t_3)$ , and  $x_{12} = \{t_1 t_2 (t_1 + t_2) + t_1 t_3 (t_1 + t_3) + t_2 t_3 (t_2 + t_3)\}^2 + t_1 t_2 t_3 (t_1^3 + t_2^3 + t_3^3) + t_1^2 t_2^2 t_3^2$ .

(b)  $H^*(BT^3; \mathbb{F}_2)^{\overline{W}_3^*} = \mathbb{F}_2[y_4, y_6, y_8]$ , where  $y_4 = t_1^2 + t_2^2 + t_1 t_2$ ,  $y_6 = t_1 t_2 (t_1 + t_2)$ , and  $y_8 = c_{3,2} = (t_1 + t_2 + t_3)^4 + (t_1 t_2 + t_1 t_3 + t_2 t_3)^2 + t_1 t_2 t_3 (t_1 + t_2 + t_3)$ .

(c)  $H^*(BT^3; \mathbb{F}_2)^{\overline{W}_3}$  is not realizable.

**Theorem 2.** (a)  $H^*(BT^4; \mathbb{F}_2)^{\overline{W}_4} = \mathbb{F}_2[x_2, x_4, c_{4,3}, c_{4,2}]$ , where  $x_2 = t_4$ ,  $x_4 = (t_1 + t_2 + t_3)(t_1 + t_2 + t_3 + t_4)$ .

- (b)  $H^*(BT^4; \mathbb{F}_2)^{\overline{W}_4^*} = \mathbb{F}_2[z_4, z_6, z_8, c_{4,3}]$ , where  $H^*(BT^3; \mathbb{F}_2)^{W(SU(4))} = \mathbb{F}_2[z_4, z_6, z_8]$ .
- (c) Both of these invariant rings are not realizable.

When  $n$  gets larger, our results suggest that the ring  $H^*(BT^n; \mathbb{F}_2)^{\overline{W}_n}$  would not be a polynomial ring. A direct application of a method of [15] implies that, for  $n = 6, 8$ , they are not polynomial rings, as shown in Proposition 3.1. While doing this work, the case of  $n = 5$  had been remained open due to a heavy calculation involved. Now it can be seen [12] that the invariant rings are polynomial.

The mod  $p$  reduction of an integral reflection group is also a reflection group. The converse need not be true. We will see a few examples using the alternating groups  $A_n$  as the subgroup of  $W(SU(n))$ . For instance, one can see that  $A_3$  is a pseudo-reflection group at  $p$  if and only if  $p = 3$ . The following shows the structure of invariant rings under  $A_n$  for small  $n$ .

**Theorem 3.** For  $A_n \subset W(SU(n))$ , the following hold:

- (a)  $H^*(BT^2; \mathbb{F}_2)^{A_3} = \mathbb{F}_2[x_4, x_6, y_6]/x_4^3 + x_6^2 + y_6^2 + x_6y_6 = 0$ , where  $x_4 = t_1^2 + t_1t_2 + t_2^2$ ,  $x_6 = t_1^2t_2 + t_1t_2^2$ , and  $y_6 = t_1^3 + t_1^2t_2 + t_2^3$ .
- (b)  $H^*(BT^2; \mathbb{F}_3)^{A_3} = \mathbb{F}_3[t_1 - t_2, t_1t_2(t_1 + t_2)]$  and  $H^*(BT^2; \mathbb{F}_3)^{A_3^*} = \mathbb{F}_3[t_1 + t_2, t_1t_2(t_1 - t_2)]$ . Both of these are not realizable.
- (c) For  $p \geq 5$ , we have  $H^*(BT^2; \mathbb{F}_p)^{A_3} = \mathbb{F}_p[x_4, x_6, z_6]/4x_4^3 = 27x_6^2 + z_6^2$ , where  $x_4 = t_1^2 + t_1t_2 + t_2^2$ ,  $x_6 = t_1^2t_2 + t_1t_2^2$ , and  $z_6 = (t_1 - t_2)(t_1 + 2t_2)(2t_1 + t_2)$ .
- (d)  $H^*(BT^3; \mathbb{F}_2)^{A_4} = \mathbb{F}_2[\alpha_4, \alpha_6, \beta_6, x_8]/\alpha_4^3 + \alpha_6^2 + \beta_6^2 + \alpha_6\beta_6 = 0$ , where  $\alpha_4 = (t_1 + t_2)^2 + (t_1 + t_2)(t_1 + t_3) + (t_1 + t_3)^2$ ,  $\alpha_6 = (t_1 + t_2)(t_1 + t_3)(t_2 + t_3)$ ,  $\beta_6 = (t_1 + t_2)^3 + (t_1 + t_2)^2(t_1 + t_3) + (t_1 + t_3)^3$ , and  $x_8 = t_1t_2t_3(t_1 + t_2 + t_3)$ .

In §1, we will show some basic results. It includes the matrix presentations, the order of  $\overline{W}_n$ , systems of parameters and the Dickson algebras. In §2 both Theorem 1 and Theorem 2 will be proved. For the non-realizability, our proof uses the classification theorem of 2-compact groups, [2]. Finally in §3, using Poincaré series and others, non-polynomial cases are discussed.

Major results in this work were announced at a fall meeting of the Japan Math. Soc., [11] together with The 2nd Pan-Pacific International Conference on Topology and Applications, Busan Korea [21].

## 1. Basic results

The integral representation of the Weyl group  $W(Sp(n)) = (\mathbb{Z}/2)^n \rtimes \Sigma_n$  can be given by  $\left\langle \Sigma_n, \begin{pmatrix} -1 & & \\ & 1 & \\ & & \ddots \\ & & & 1 \end{pmatrix} \right\rangle$  as reflection groups, [14, Ch. 3]. Let  $\phi = \phi_2^{-1}$

$$\text{so that } \phi = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & 0 \\ 1 & \dots & 1 & 2 \end{pmatrix} = \left( \begin{array}{c|cc} E_{n-2} & & 0 \\ \hline 0 \dots 0 & 1 & 0 \\ 1 \dots 1 & 1 & 2 \end{array} \right).$$

We will show that  $\phi^{-1}W(Sp(n))\phi \subset GL(n, \mathbb{Z})$ .

**Proposition 1.1.** *The group  $W_n = \phi^{-1}W(Sp(n))\phi$  is included in  $GL(n, \mathbb{Z})$ .*

*Proof.* For each  $\sigma$  of the reflections which generate  $W(Sp(n))$ , it's enough to show that  $\phi^{-1}\sigma\phi \in GL(n, \mathbb{Z})$ . First recall that  $\Sigma_{n-1} \subset W(SU(n))$ , and that  $\phi^{-1}W(SU(n))\phi \subset GL(n-1, \mathbb{Z})$ . Since  $\Sigma_n$  is generated by  $\Sigma_{n-1}$  together with the transposition  $\sigma_{n-1}$  switching  $n-1$  and  $n$ , we see  $\phi^{-1}\Sigma_n\phi$  is included in  $GL(n, \mathbb{Z})$  by the following:

$$\begin{aligned} \phi^{-1}\sigma_{n-1}\phi &= \frac{1}{2} \left( \begin{array}{c|c} 2E_{n-2} & 0 \\ \hline 0 \dots 0 & 2 \ 0 \\ -1 \dots -1 & -1 \ 1 \end{array} \right) \left( \begin{array}{c|c} E_{n-2} & 0 \\ \hline 0 & 0 \ 1 \\ 1 & 0 \ 0 \end{array} \right) \left( \begin{array}{c|c} E_{n-2} & 0 \\ \hline 0 \dots 0 & 1 \ 0 \\ 1 \dots 1 & 1 \ 2 \end{array} \right) \\ &= \left( \begin{array}{c|c} E_{n-2} & 0 \\ \hline 1 \dots 1 & 1 \ 2 \\ -1 \dots -1 & 0 \ -1 \end{array} \right). \end{aligned}$$

Moreover we see that

$$\begin{aligned} &\phi^{-1} \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \phi \\ &= \frac{1}{2} \left( \begin{array}{c|c} 2E_{n-2} & 0 \\ \hline 0 \dots 0 & 2 \ 0 \\ -1 \dots -1 & -1 \ 1 \end{array} \right) \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \left( \begin{array}{c|c} E_{n-2} & 0 \\ \hline 0 \dots 0 & 1 \ 0 \\ 1 \dots 1 & 1 \ 2 \end{array} \right) \\ &= \left( \begin{array}{c|c} -1 & \\ & 1 & & \\ & & \ddots & \\ & & & 1 & 0 \\ \hline 0 & \dots & 0 & 1 & 0 \\ 1 & 0 & \dots & 0 & 1 \end{array} \right). \end{aligned}$$

This completes the proof.  $\square$

Next consider the mod 2 reduction  $\overline{W}_n$  of  $W_n$ . Since the scalar matrix  $-E_n \in W(Sp(n))$ , it is also contained in  $W_n$ . Hence the kernel of the projection  $W_n \rightarrow \overline{W}_n$  contains  $\mathbb{Z}/2$ . It turns out  $|\overline{W}_n| = \frac{|W_n|}{2} = 2^{n-1} \cdot n!$  for  $n \geq 3$ .

**Proposition 1.2.** *For  $n \geq 3$ , the subgroup  $\overline{W}_n$  of  $GL(n, \mathbb{F}_2)$  is isomorphic to  $W_n/\mathbb{Z}/2$ .*

*Proof.* Note that  $W_n \cong (\mathbb{Z}/2)^n \rtimes \Sigma_n$ . A result of Minkowski tells us, [18, Lemma 10.7.1], that the kernel of the projection  $GL(n, \mathbb{Z}) \rightarrow GL(n, \mathbb{F}_2)$  is an

elementary 2-abelian. For  $n \geq 3$ , the homomorphism  $\Sigma_n \rightarrow GL(n, \mathbb{F}_2)$  should be injective. Thus the kernel of  $W_n \rightarrow \overline{W}_n$  has to come from diagonal matrices of  $W(Sp(n))$ . The desired result is obtained from the following observation:

$$\phi^{-1} \begin{pmatrix} \varepsilon_1 & & & \\ & \varepsilon_2 & & \\ & & \ddots & \\ & & & \varepsilon_n \end{pmatrix} \phi = \left( \begin{array}{ccc|c} \varepsilon_1 & & & 0 \\ & \ddots & & \\ & & \varepsilon_{n-1} & \\ \hline \frac{\varepsilon_n - \varepsilon_1}{2} & \dots & \frac{\varepsilon_n - \varepsilon_{n-1}}{2} & \varepsilon_n \end{array} \right). \quad \square$$

We recall how to see if a ring of invariants  $H^*(BT^n; \mathbb{F}_p)^W$  is polynomial for a prime  $p$  (see [8, 14, 16, 18]). An element of  $H^*(BT^n; \mathbb{F}_p)$  is considered as a function of  $n$  variables,  $t_1, t_2, \dots, t_n$ . A set of  $n$  elements  $x_1, x_2, \dots, x_n \in H^*(BT^n; \mathbb{F}_p)^W$  is said to be a system of parameters if the solution of the following system of equations

$$\begin{cases} x_1(t_1, t_2, \dots, t_n) = 0, \\ x_2(t_1, t_2, \dots, t_n) = 0, \\ \vdots \\ x_n(t_1, t_2, \dots, t_n) = 0 \end{cases}$$

is trivial. Namely  $t_1 = t_2 = \dots = t_n = 0$ . As before, we write  $H^*(BT^n; \mathbb{F}_p) = \mathbb{F}_p[t_1, t_2, \dots, t_n]$ . Let  $d(x)$  denote  $\frac{1}{2} \deg(x)$  so that  $d(t_i) = 1$  for  $1 \leq i \leq n$ . Usually  $d(x)$  is said to be the algebraic degree of  $x$ , while  $\deg(x)$  is the topological degree. According to [18, Proposition 5.5.5], for a finite group  $W$ , if we can find a system of parameters  $\{x_1, x_2, \dots, x_n\}$  with  $\prod_{i=1}^n d(x_i) = |W|$ , then  $H^*(BT^n; \mathbb{F}_p)^W = \mathbb{F}_p[x_1, x_2, \dots, x_n]$ .

Next we recall some basic things about generators of the Dickson algebra  $\mathbb{F}_p[t_1, t_2, \dots, t_n]^{GL(n, \mathbb{F}_p)}$ , [18, §8.1]. Let

$$V = \mathbb{F}_p\langle t_1 \rangle \oplus \mathbb{F}_p\langle t_2 \rangle \oplus \dots \oplus \mathbb{F}_p\langle t_n \rangle,$$

the vector space over  $\mathbb{F}_p$  with basis  $t_1, t_2, \dots, t_n$ . Consider the polynomial  $f(X) = \prod_{v \in V} (X - v)$ . Then  $f(X) = X^{p^n} + \sum_{i=0}^{n-1} (-1)^{n-i} c_{n,i} X^{p^i}$  and  $\mathbb{F}_p[t_1, t_2, \dots, t_n]^{GL(n, \mathbb{F}_p)} = \mathbb{F}_p[c_{n,n-1}, c_{n,n-2}, \dots, c_{n,0}]$  with  $d(c_{n,i}) = p^n - p^i$ . For instance, if  $p = 2$  and  $n = 2$ , then  $f(X) = X^4 + c_{2,1}X^2 + c_{2,0}X$  with  $c_{2,1} = t_1^2 + t_1t_2 + t_2^2$  and  $c_{2,0} = t_1^2t_2 + t_1t_2^2$ . The Dickson invariants  $c_{n,i}$  ( $0 \leq i \leq n-1$ ) can be expressed using determinants. Consider the following polynomial:

$$\Delta_n(X) = \begin{vmatrix} t_1 & \dots & t_n & X \\ t_1^p & \dots & t_n^p & X^p \\ \vdots & & \vdots & \vdots \\ t_1^{p^n} & \dots & t_n^{p^n} & X^{p^n} \end{vmatrix}.$$

Then  $\Delta_n(X) = c_n f(X)$  where  $c_n = \Delta_{n-1}(t_n)$ . From this observation, for  $p = 2$ , we see the following:

$$c_{3,2} = \frac{\begin{vmatrix} t_1 & t_2 & t_3 \\ t_1^2 & t_2^2 & t_3^2 \\ t_1^8 & t_2^8 & t_3^8 \end{vmatrix}}{\begin{vmatrix} t_1 & t_2 & t_3 \\ t_1^2 & t_2^2 & t_3^2 \\ t_1^4 & t_2^4 & t_3^4 \end{vmatrix}}$$

and

$$c_{4,3} = \frac{\begin{vmatrix} t_1 & t_2 & t_3 & t_4 \\ t_1^2 & t_2^2 & t_3^2 & t_4^2 \\ t_1^4 & t_2^4 & t_3^4 & t_4^4 \\ t_1^{16} & t_2^{16} & t_3^{16} & t_4^{16} \end{vmatrix}}{\begin{vmatrix} t_1 & t_2 & t_3 & t_4 \\ t_1^2 & t_2^2 & t_3^2 & t_4^2 \\ t_1^4 & t_2^4 & t_3^4 & t_4^4 \\ t_1^8 & t_2^8 & t_3^8 & t_4^8 \end{vmatrix}}.$$

## 2. Polynomial rings and non-realizability

Some invariant elements can be found using orbit sums or orbit polynomials. A typical example is  $\prod_{i=1}^n (X + t_i)$  for the permutation representation of  $\Sigma_n$ . Another way of finding elements of unstable algebras  $H^*(BT^n; \mathbb{F}_p)^W$  can be given by use of cohomology operations, [18, Ch. 10]. In fact, in the part (a) of Theorem 1 we see  $x_{12} = Sq^4(x_8)$ .

*Proof of Theorem 1.* (a) Suppose  $x_2 = t_3$ ,  $x_8 = (t_1 + t_2)^4 + (t_1 t_2 + t_1 t_3 + t_2 t_3)^2 + t_1 t_2 t_3 (t_1 + t_2 + t_3)$ ,  $x_{12} = \{t_1 t_2 (t_1 + t_2) + t_1 t_3 (t_1 + t_3) + t_2 t_3 (t_2 + t_3)\}^2 + t_1 t_2 t_3 (t_1^3 + t_2^3 + t_3^3) + t_1^2 t_2^2 t_3^2$ . We notice that  $\overline{W}_3$  is generated by the 3 reflections:

$$\overline{W}_3 = \left\langle \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \right\rangle$$

It is easy to check that the elements  $x_2, x_8, x_{12}$  are  $\overline{W}_3$ -invariant. For instance, the middle matrix sends  $t_1$  to  $t_1 + t_2 + t_3$ , and fixes both  $t_2$  and  $t_3$ . Consequently  $x_8$  is sent to  $(t_1 + t_3)^4 + \{(t_1 + t_2 + t_3)(t_2 + t_3) + t_2 t_3\}^2 + (t_1 + t_2 + t_3)t_2 t_3 t_1 = x_8$ . The solution of the following system of equations

$$\begin{cases} t_3 = 0, \\ (t_1 + t_2)^4 + (t_1 t_2 + t_1 t_3 + t_2 t_3)^2 + t_1 t_2 t_3 (t_1 + t_2 + t_3) = 0, \\ \{t_1 t_2 (t_1 + t_2) + t_1 t_3 (t_1 + t_3) + t_2 t_3 (t_2 + t_3)\}^2 \\ + t_1 t_2 t_3 (t_1^3 + t_2^3 + t_3^3) + t_1^2 t_2^2 t_3^2 = 0 \end{cases}$$

is trivial. Hence  $\{x_2, x_8, x_{12}\}$  is a system of parameters. Consequently we see that  $H^*(BT^3; \mathbb{F}_2)^{\overline{W}_3}$  is the polynomial ring generated by these elements, since  $|\overline{W}_3| = 2^2 \cdot 3! = d(x_2) \cdot d(x_8) \cdot d(x_{12})$ .

(b) Notice that  $\{y_4, y_6, y_8\}$  is a system of parameters. Furthermore  $|\overline{W}_3^*| = 2^2 \cdot 3! = d(y_4) \cdot d(y_6) \cdot d(y_8)$ . A similar argument shows the desired result.

(c) If the unstable algebra  $H^*(BT^3; \mathbb{F}_2)^{\overline{W}_3}$  is realizable, there is a 2–compact group  $X$ , [6] such that  $H^*(BT^3; \mathbb{F}_2)^{\overline{W}_3} \cong H^*(BX; \mathbb{F}_2)$ . Since the polynomial algebra is generated by even-degree elements, the classifying space  $BX$  is 2–torsion free. So the 2–adic cohomology is also a polynomial algebra generated by elements of the same degree. We can find, [2], a compact connected Lie group  $G$  such that  $H^*(BX; \mathbb{Z}_2^\wedge) \cong H^*(BG; \mathbb{Z}_2^\wedge)$ . However, any Lie group  $G$  does not satisfy the condition that  $H^*(BG; \mathbb{F}_2) = \mathbb{F}_2[x_2, x_8, x_{12}]$ , since this cohomology does not contain a generator of degree 4. Thus,  $H^*(BT^3; \mathbb{F}_2)^{\overline{W}_3}$  is not realizable. This completes the proof.  $\square$

The representation of  $\Sigma_n = W(SU(n))$  is generated by the permutation matrices together with the following  $(n-1) \times (n-1)$  matrix:

$$\begin{pmatrix} 1 & & & -1 \\ & \ddots & & \vdots \\ & & 1 & \vdots \\ & & & -1 \end{pmatrix}.$$

We will prove Theorem 2.

*Proof of Theorem 2.* (a) We notice that  $\overline{W}_4$  is generated by the 4 reflections:

$$\overline{W}_4 = \left\langle \underbrace{\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \right\rangle.$$

$\Sigma_3$

The argument is analogous to the previous ones. Notice that  $\{x_2, x_4, c_{4,3}, c_{4,2}\}$  is a system of parameters. Furthermore  $|\overline{W}_4| = 2^6 \cdot 3 = d(x_2) \cdot d(x_4) \cdot d(c_{4,3}) \cdot d(c_{4,2})$ . Thus  $H^*(BT^4; \mathbb{F}_2)^{\overline{W}_4} = \mathbb{F}_2[x_2, x_4, c_{4,3}, c_{4,2}]$ .

(b) Recall that  $H^*(BSU(n); \mathbb{F}_2) = H^*(BT^{n-1}; \mathbb{F}_2)^{W(SU(n))}$  for  $n \geq 3$  and  $H^*(BSU(n); \mathbb{F}_2) \cong H^*(BU(n); \mathbb{F}_2)/(c_1)$ . Here notice that

$$\overline{W}_4^* = \left\{ \left( \left( \begin{array}{c|c} & \begin{matrix} b_1 \\ b_2 \\ b_3 \end{matrix} \\ \hline 0 & 0 & 0 & 1 \end{array} \right) \mid A \in W(SU(4)), \quad b_i \in \mathbb{F}_2 \text{ for } i = 1, 2, 3 \right\}.$$

Consequently we can see that, for  $H^*(BT^3; \mathbb{F}_2)^{W(SU(4))} = \mathbb{F}_2[z_4, z_6, z_8]$ , the set  $\{z_4, z_6, z_8, c_{4,3}\}$  is a system of parameters. And the product of their algebraic degree is equal to the order of  $\overline{W}_4^*$ . So the desired result follows.

(c) Again the argument is analogous to the part (c) of Theorem 1. Using the classification of Lie groups, we can show that  $H^*(BG; \mathbb{Z}_2^\wedge)$  is isomorphic to neither  $H^*(BT^4; \mathbb{F}_2)^{\overline{W}_4}$  nor  $H^*(BT^4; \mathbb{F}_2)^{\overline{W}_4^*}$  for any compact connected Lie group  $G$ . This completes the proof.  $\square$

*Remark 1.* The following is the case of  $n = 2$ . The group  $W_2 = \phi^{-1}W(Sp(2))\phi \cong D_8$  is generated by the two reflections  $\begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$  and  $\begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}$ . The mod 2 reduction is  $\overline{W}_2 = \mathbb{Z}/2 \langle \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \rangle$  and  $H^*(BT^2; \mathbb{F}_2)^{\overline{W}_2} = \mathbb{F}_2[t_2, t_1(t_1 + t_2)]$ .

*Remark 2.* Another set of generators for the polynomial ring  $H^*(BT^4; \mathbb{F}_2)^{\overline{W}_4}$  is obtained as follows. The higher dimensional elements are less computational. Let  $V = \mathbb{F}_2\langle t_1 \rangle \oplus \mathbb{F}_2\langle t_2 \rangle \oplus \mathbb{F}_2\langle t_3 \rangle \oplus \mathbb{F}_4\langle t_4 \rangle$ . The  $\overline{W}_4$ -action divides  $V$  into four invariant subsets. They are  $\{0\}$ ,  $\{t_4\}$ ,  $\{t_1 + t_2 + t_3, t_1 + t_2 + t_3 + t_4\}$  and the rest of 12 vectors. For  $A = t_1(t_1 + t_4)(t_2 + t_3)(t_2 + t_3 + t_4)$ ,  $B = t_2(t_1 + t_3)(t_2 + t_4)(t_1 + t_3 + t_4)$  and  $C = t_3(t_1 + t_2)(t_3 + t_4)(t_1 + t_2 + t_4)$ , we can see that  $\overline{W}_4$  permutes these three elements and  $A + B + C = 0$ . The invariant ring contains  $y_{16} = A^2 + AB + B^2$  and  $y_{24} = AB(A + B)$ . Since  $\{x_2, x_4, y_{16}, y_{24}\}$  is a system of parameters, it follows that  $H^*(BT^4; \mathbb{F}_2)^{\overline{W}_4} = \mathbb{F}_2[x_2, x_4, y_{16}, y_{24}]$ . The following is the orbit polynomial for the set  $U$  of the 12 vectors.

$$\begin{aligned} f(X) &= \prod_{u \in U} (X + u) \\ &= X^{12} + (x_2^2 + x_4)X^{10} + x_2x_4X^9 + (x_2^4 + x_4^2)X^8 + (x_2^6 + x_2^4x_4 + x_4^3)X^6 \\ &\quad + x_2x_4(x_2^4 + x_4^2)X^5 + \{x_2^2x_4^2(x_2^2 + x_4) + y_{16}\}X^4 + x_2^3x_4^3X^3 \\ &\quad + (x_2^2 + x_4)y_{16}X^2 + x_2x_4y_{16}X + y_{24}. \end{aligned}$$

### 3. Structure of invariant rings

We will see examples of invariant rings that are not polynomial in this section. We use a result of Dwyer–Wilkerson [7, Theorem 1.4]. Suppose that  $V$  is a finite dimensional vector space over the field  $\mathbb{F}_p$ , and that  $W$  is a subgroup of  $\text{Aut}(V)$ . Note that the symmetric algebra  $S(V)$  is isomorphic to  $H^*(BT^n; \mathbb{F}_p)$  if  $\dim V = n$ . Let  $U$  be a subset of  $V$ , and  $W_U$  the subgroup of  $W$  consisting of elements which fix  $U$  pointwise. Then if  $S(V)^{W^*}$  is a polynomial ring over  $\mathbb{F}_p$ , then  $W_U$  must be a pseudo-reflection group and  $S(V)^{W_U}$  is also a polynomial ring.

**Proposition 3.1.** *Let  $n = 6, 8$ . Then  $H^*(BT^n; \mathbb{F}_2)^{\overline{W}_n}$  is not a polynomial ring.*

*Proof.* According to the result of Dwyer–Wilkerson, we need to find a subset  $U$  such that the subgroup  $W_U$  is not generated by pseudo-reflections. Our method is an immediate consequence of that of Kudo, [15].



The dual representation  $\overline{W}_n^*$  is expressed as follows:

$$\overline{W}_n^* = \left\{ \left( \left( \begin{array}{c|c} A & \begin{matrix} b_1 \\ \vdots \\ b_{n-1} \end{matrix} \\ \hline 0 \dots 0 & 1 \end{array} \right) \mid \begin{array}{l} A \in W(SU(n)), \\ b_i \in \mathbb{F}_2 \text{ for } 1 \leq i \leq n-1 \end{array} \right) \right\}.$$

First we consider the case of  $n = 6$ . Let  $U = \{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$  for

$$\mathbf{x} = {}^t(1, 1, 1, 0, 0, 0), \quad \mathbf{y} = {}^t(1, 1, 0, 1, 1, 0), \quad \mathbf{z} = {}^t(0, 0, 0, 0, 0, 1).$$

Recall that any element of  $W(SU(6))$  is a  $5 \times 5$  matrix such that each column is one of the set of the standard basis  $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5\}$  and the vector  $\mathbf{b} = {}^t(1, 1, 1, 1, 1)$  at  $p = 2$ . As in [15, proof of Theorem 3], it follows that

$$W_U = \left\{ e, (1, 2), (4, 5), (1, 2)(4, 5), (1, 4)(2, 5)(3, 6), \right. \\ \left. (1, 5, 2, 4)(3, 6), (1, 4, 2, 5)(3, 6), (1, 5)(2, 4)(3, 6) \right\}.$$

Since  $W_U$  is not a pseudo-reflection group, we see that  $H^*(BT^6; \mathbb{F}_2)^{\overline{W}_6}$  is not a polynomial ring by [7].

The case of  $n = 8$  is analogous. Let  $U = \{\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w}\}$  for

$$\mathbf{x} = {}^t(1, 1, 1, 1, 0, 0, 0, 0), \quad \mathbf{y} = {}^t(1, 1, 0, 0, 1, 1, 0, 0), \\ \mathbf{z} = {}^t(1, 0, 1, 0, 1, 0, 1, 0), \quad \mathbf{w} = {}^t(0, 0, 0, 0, 0, 0, 0, 1).$$

Again, the group  $W_U$  is not a pseudo-reflection group, hence  $H^*(BT^8; \mathbb{F}_2)^{\overline{W}_8}$  is not a polynomial ring.  $\square$

The concept of the Poincaré series can be useful to find the structure of invariant rings, [18] and [14]. For a graded vector space  $M = \bigoplus_{i=0}^{\infty} M_{2i}$  over a field  $\mathbb{F}$ , we define the Poincaré series by  $P_{\mathbb{F}}(M, t) = \sum_{i=0}^{\infty} (\dim_{\mathbb{F}} M_{2i}) \cdot t^i$ . If  $M = \mathbb{F}[f_1, \dots, f_m]/(h_1, \dots, h_k)$ , where  $\{f_1, \dots, f_m\}$  are generators and  $\{h_1, \dots, h_k\}$  are relations, then the following holds:

$$P_{\mathbb{F}}(M, t) = \frac{\prod_{i=1}^k (1 - t^{\text{d}(h_i)})}{\prod_{j=1}^m (1 - t^{\text{d}(f_j)})}.$$

*Proof of Theorem 3.* (a) The alternating group  $A_3$  as a subgroup of  $W(SU(3))$  is generated by  $\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$ . In a non-modular case, the Poincaré series can be calculated by Molien's theorem:

$$P_{\mathbb{F}_2}(H^*(BT^2; \mathbb{F}_2)^{A_3}, t) = \frac{1}{|A_3|} \sum_{w \in A_3} \frac{1}{\det(E_2 - tw)} = \frac{(1 - t^6)}{(1 - t^2)(1 - t^3)^2}.$$

The three elements  $\{x_4, x_6, y_6\}$  are  $A_3$ -invariant with  $x_4^3 + x_6^2 + y_6^2 + x_6 y_6 = 0$ . So we obtain the desired result.

(b) It is easy to show that both  $\{t_1 - t_2, t_1 t_2 (t_1 + t_2)\}$  and  $\{t_1 + t_2, t_1 t_2 (t_1 - t_2)\}$  are systems of parameters. Clearly the product of their algebraic degrees is equal to the order of  $A_3$ . Thus the two invariant rings have to be polynomial

rings. The nonrealizability of each invariant ring is based on a result of [5]. If a polynomial ring  $H^*(BT^n; \mathbb{F}_p)^W$  is realizable for an odd prime  $p$ , the modular representation  $W \rightarrow GL(n, \mathbb{F}_p)$  should lift to a  $p$ -adic representation as a pseudo-reflection group. This is impossible in each case.

(c) For  $p \geq 5$ , we see the following:

$$P_{\mathbb{F}_p}(H^*(BT^2; \mathbb{F}_p)^{A_3}, t) = \frac{(1-t^6)}{(1-t^2)(1-t^3)^2}.$$

The three elements  $\{x_4, x_6, z_6\}$  are  $A_3$ -invariant with  $4x_4^3 = 27x_6^2 + z_6^2$ , and the desired result follows.

(d) Recall that  $A_4 = (\mathbb{Z}/2\langle a \rangle \times \mathbb{Z}/2\langle b \rangle) \rtimes \mathbb{Z}/3\langle c \rangle$ , where  $a = (12)(34)$ ,  $b = (13)(24)$ ,  $c = (123)$  with  $c^{-1}ac = ab$  and  $c^{-1}bc = a$ . Each of the integral matrix presentations is as follows:

$$a = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 0 & 0 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & -1 & 1 \\ 0 & -1 & 0 \\ 1 & -1 & 0 \end{pmatrix}, \quad c = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Under the mod 2 reduction, we can show that  $\mathbb{Z}/2\langle a \rangle \times \mathbb{Z}/2\langle b \rangle$  is a reflection group and  $H^*(BT^3; \mathbb{F}_2)^{\mathbb{Z}/2\langle a \rangle \times \mathbb{Z}/2\langle b \rangle} = \mathbb{F}_2[x_2, y_2, x_8]$ , where  $x_2 = t_1 + t_2$ ,  $y_2 = t_1 + t_3$ ,  $x_8 = t_1 t_2 t_3 (t_1 + t_2 + t_3)$ . The group  $\mathbb{Z}/3\langle c \rangle$  acts on  $\mathbb{Z}/2\langle a \rangle \times \mathbb{Z}/2\langle b \rangle$  as  $A_3$ , and fixes  $x_4$ . Thus the Poincaré series of  $\mathbb{F}_2[x_1, y_1]^{\mathbb{Z}/3\langle c \rangle}$  is given by the following:

$$P_{\mathbb{F}_2}(\mathbb{F}_2[x_1, y_1]^{\mathbb{Z}/3\langle c \rangle}, t) = \frac{(1-t^6)}{(1-t^2)(1-t^3)^2}.$$

Since  $H^*(BT^3; \mathbb{F}_2)^{A_4} = (H^*(BT^3; \mathbb{F}_2)^{\mathbb{Z}/2\langle a \rangle \times \mathbb{Z}/2\langle b \rangle})^{\mathbb{Z}/3\langle c \rangle}$ , we obtain the desired result.  $\square$

*Remark 3.* As mentioned before, we will see that  $A_3$  is a pseudo-reflection group at  $p$  if and only if  $p = 3$ . For  $p \neq 3$  (non-modular case), it follows from the invariant ring not being polynomial. For  $p = 3$ , the rank of the matrix  $\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$  is 1 and the desired result follows.

*Remark 4.* We consider the rational invariant rings for the groups in Theorem 3 whose mod  $p$  reductions are pseudo-reflection groups. First it is straightforward to see the following:

$$H^*(BT^2; \mathbb{Q})^{A_3} = \mathbb{Q}[x_4, x_6, z_6]/4x_4^3 = 27x_6^2 + z_6^2,$$

where  $x_4 = t_1^2 + t_1 t_2 + t_2^2$ ,  $x_6 = t_1^2 t_2 + t_1 t_2^2$ , and  $z_6 = (t_1 - t_2)(t_1 + 2t_2)(2t_1 + t_2)$ .

Next we consider  $H^*(BT^3; \mathbb{Q})^{\mathbb{Z}/2\langle a \rangle \times \mathbb{Z}/2\langle b \rangle}$ . The Poincaré series is the following:

$$P_{\mathbb{Q}}(H^*(BT^3; \mathbb{Q})^{\mathbb{Z}/2\langle a \rangle \times \mathbb{Z}/2\langle b \rangle}, t) = \frac{(1-t^6)}{(1-t^2)^3(1-t^3)}.$$

Let  $x_4 = t_1^2 + t_2^2 + t_3^2 + t_1t_2 + t_1t_3 + t_2t_3$ ,  $y_4 = (t_1 - t_2)(t_1 + t_2 + 2t_3)$ ,  $z_4 = (t_1 - t_3)(t_1 + 2t_2 + t_3)$  and  $x_6 = t_1t_2t_3 - (t_1t_2 + t_1t_3 + t_2t_3)(t_1 + t_2 + t_3)$ . Then

$$H^*(BT^3; \mathbb{Q})^{\mathbb{Z}/2\langle a \rangle \times \mathbb{Z}/2\langle b \rangle} = \mathbb{Q}[x_4, y_4, z_4, x_6] / \sim,$$

where  $27x_6^2 = 8x_4^3 + 2y_4^3 + 2z_4^3 - 6x_4y_4^2 - 6x_4z_4^2 + 6x_4y_4z_4 - 3y_4^2z_4 - 3y_4z_4^2$  must be satisfied.

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