

COLOCALIZATION OF LOCAL HOMOLOGY MODULES

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ABSTRACT. Let I be an ideal of Noetherian local ring (R, \mathfrak{m}) and M an artinian R -module. In this paper, we study colocalization of local homology modules. In fact we give Colocal-global Principle for the artinianness and minimaxness of local homology modules, which is a dual case of Local-global Principle for the finiteness of local cohomology modules. We define the representation dimension $r^I(M)$ of M and the artinianness dimension $a^I(M)$ of M relative to I by $r^I(M) = \inf\{i \in \mathbb{N}_0 : H_i^I(M) \text{ is not representable}\}$, and $a^I(M) = \inf\{i \in \mathbb{N}_0 : H_i^I(M) \text{ is not artinian}\}$ and we will prove that

- i) $a^I(M) = r^I(M) = \inf\{r^{IR_{\mathfrak{p}}}(M) : \mathfrak{p} \in \text{Spec}(R)\} \geq \inf\{a^{IR_{\mathfrak{p}}}(M) : \mathfrak{p} \in \text{Spec}(R)\}$,
- ii) $\inf\{i \in \mathbb{N}_0 : H_i^I(M) \text{ is not minimax}\} = \inf\{r^{IR_{\mathfrak{p}}}(M) : \mathfrak{p} \in \text{Spec}(R) \setminus \{\mathfrak{m}\}\}$.

Also, we define the upper representation dimension $R^I(M)$ of M relative to I by $R^I(M) = \sup\{i \in \mathbb{N}_0 : H_i^I(M) \text{ is not representable}\}$, and we will show that

- i) $\sup\{i \in \mathbb{N}_0 : H_i^I(M) \neq 0\} = \sup\{i \in \mathbb{N}_0 : H_i^I(M) \text{ is not artinian}\} = \sup\{R^{IR_{\mathfrak{p}}}(M) : \mathfrak{p} \in \text{Spec}(R)\}$,
- ii) $\sup\{i \in \mathbb{N}_0 : H_i^I(M) \text{ is not finitely generated}\} = \sup\{i \in \mathbb{N}_0 : H_i^I(M) \text{ is not minimax}\} = \sup\{R^{IR_{\mathfrak{p}}}(M) : \mathfrak{p} \in \text{Spec}(R) \setminus \{\mathfrak{m}\}\}$.

1. Introduction

Throughout this paper assume that (R, \mathfrak{m}) is a commutative Noetherian local ring, I is an ideal of R and M is an R -module. Cuong and Nam in [2] defined the local homology modules $H_i^I(M)$ with respect to I by

$$H_i^I(M) = \varprojlim_n \text{Tor}_i^R(R/I^n, M).$$

This definition is dual to Grothendieck's definition of local cohomology modules and coincides with the definition of Greenless and May in [5] for an artinian R -module M . For basic results about local homology we refer the reader to [2], [4] and [13]; for local cohomology refer to [1].

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An important problem in local cohomology is Faltings' Local-global Principle for finiteness of local cohomology modules which states that for a positive integer n and an ideal \mathfrak{a} of R the $R_{\mathfrak{p}}$ -module $H_{\mathfrak{a}R_{\mathfrak{p}}}^i(M_{\mathfrak{p}})$ is finitely generated for all $i < n$ and for all $\mathfrak{p} \in \text{Spec}(R)$ if and only if $H_{\mathfrak{a}}^i(M)$ is finitely generated R -module for all $i < n$. In the other word, in terms of the finiteness dimension $f_{\mathfrak{a}}(M)$ of M relative to \mathfrak{a} we have

$$f_{\mathfrak{a}}(M) = \inf\{f_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) : \mathfrak{p} \in \text{Spec}(R)\}.$$

where, $f_{\mathfrak{a}}(M) := \inf\{i \in \mathbb{N}_0 : H_{\mathfrak{a}}^i(M) \text{ is not finitely generated}\}$.

In this paper, we investigate the dual of Faltings' Local-global Principle and we call it Colocal-global Principle for artinianness of local homology modules. At first, for a representable R -module M , we define the representation dimension $r^I(M)$ of M relative to I by

$$r^I(M) = \inf\{i \in \mathbb{N}_0 : H_i^I(M) \text{ is not representable}\}$$

and artinianness dimension $a^I(M)$ of M relative to I by

$$a^I(M) = \inf\{i \in \mathbb{N}_0 : H_i^I(M) \text{ is not artinian}\}$$

and we prove the following main result:

Theorem 1.1. *Let (R, \mathfrak{m}) be a Noetherian local ring, $I \subseteq \mathfrak{m}$ be an ideal of R and M an artinian R -module. Let n be an integer. Then the following conditions are equivalent*

- i) $H_i^I(M)$ is an artinian R -module for all $i < n$,
- ii) $H_i^I(M)$ is a representable R -module for all $i < n$,
- iii) ${}_{\mathfrak{p}}(H_i^I(M))$ is a representable $R_{\mathfrak{p}}$ -module for all $i < n$ and all $\mathfrak{p} \in \text{Spec}(R)$,

and we conclude that

$$\begin{aligned} a^I(M) = r^I(M) &= \inf\{r^{IR_{\mathfrak{p}}}({}_{\mathfrak{p}}M) : \mathfrak{p} \in \text{Spec}(R)\} \\ &\geq \inf\{a^{IR_{\mathfrak{p}}}({}_{\mathfrak{p}}M) : \mathfrak{p} \in \text{Spec}(R)\}. \end{aligned}$$

In the second main result we study Colocal-global Principle for minimax local homology modules by proving the following result:

Theorem 1.2. *Let (R, \mathfrak{m}) be a Noetherian local ring, $I \subseteq \mathfrak{m}$ be an ideal of R and M an artinian R -module. Let n be an integer. Then the following conditions are equivalent:*

- i) $H_i^I(M)$ is minimax R -module for all $i < n$,
- ii) ${}_{\mathfrak{p}}(H_i^I(M))$ is a representable $R_{\mathfrak{p}}$ -module for all $i < n$ and all $\mathfrak{p} \in \text{Spec}(R) \setminus \{\mathfrak{m}\}$,

And so we obtain that $\inf\{i : H_i^I(M) \text{ is not minimax}\} = \inf\{r^{IR_{\mathfrak{p}}}({}_{\mathfrak{p}}M) : \mathfrak{p} \in \text{Spec}(R) \setminus \{\mathfrak{m}\}\}$. Also, we define the upper representation dimension $R^I(M)$ of M relative to I by

$$R^I(M) = \sup\{i \in \mathbb{N}_0 : H_i^I(M) \text{ is not representable}\},$$

and we will show that

- i) $\sup\{i \in \mathbb{N}_0 : H_i^I(M) \neq 0\} = \sup\{i \in \mathbb{N}_0 : H_i^I(M) \text{ is not artinian}\} = \sup\{R^{IR_{\mathfrak{p}}}(M) : \mathfrak{p} \in \text{Spec}(R)\},$
- ii) $\sup\{i \in \mathbb{N}_0 : H_i^I(M) \text{ is not finitely generated}\} = \sup\{i \in \mathbb{N}_0 : H_i^I(M) \text{ is not minimax}\} = \sup\{R^{IR_{\mathfrak{p}}}(M) : \mathfrak{p} \in \text{Spec}(R) \setminus \{\mathfrak{m}\}\}.$

2. The results

A Hausdorff linearly topologized R -module M is said to be linearly compact if M has the following property: if \mathcal{F} is a family of closed cosets (i.e., the cosets of closed submodules) in M which has the finite intersection property, then the cosets in \mathcal{F} have a non-empty intersection. It is clear that artinian R -modules are linearly compact with the discrete topology. If (R, \mathfrak{m}) is a complete local ring, then finitely generated R -modules are also linearly compact and discrete. The local homology modules $H_i^I(M)$ of a linearly compact R -module M are also linearly compact R -modules by [4, Proposition 3.3]. For more facts about linearly compact modules see [7] and [15].

We need the concept of Noetherian dimension of an R -module in some of our proofs. Let M be an artinian R -module. The Noetherian dimension of M , $\text{Ndim}_R(M)$, is defined by induction. If $M = 0$, we put $\text{Ndim}_R(M) = 1$. For any integer $t \geq 0$, if $\text{Ndim}_R(M) < t$ is false and whenever $M_1 \subseteq M_2 \subseteq \dots$ is an ascending chain of submodules of M then there exists an integer m_0 such that $\text{Ndim}_R(M_{m+1}/M_m) < t$ for all $m \geq m_0$, then we put $\text{Ndim}_R(M) = t$. In case M is an artinian module, $\text{Ndim}_R(M) < \infty$ (see [6]).

The R -module M is said to be a minimax module if there is a finitely generated submodule N of M , such that M/N is artinian. The class of minimax modules includes all finitely generated and all artinian modules. Moreover it is closed under taking submodules, quotients and extensions, i.e., it is a Serre subcategory of the category of R -modules.

A module is called cocyclic if it is a submodule of $E(R/\mathfrak{m})$ for some maximal ideal \mathfrak{m} of R . A prime ideal \mathfrak{p} is called coassociated to a non-zero R -module M if there is a cocyclic homomorphic image T of M with $\mathfrak{p} = \text{Ann}_R T$ [14]. The set of coassociated primes of M is denoted by $\text{Coass}_R(M)$.

Let $S \subseteq R$ be a multiplicative set. The R_S -module $\text{Hom}_R(R_S, M)$ is called the co-localization of M with respect to S and denoted by ${}_S M$. When M is an artinian R -module, it is known that ${}_S M$ is almost never an artinian R_S -module (see [9]), while by [9, Theorem 3.2] ${}_S M$ is a representable R_S -module. Thus the functor co-localization is not closed on the category artinian modules. In the case M is linearly compact R -module, ${}_S M$ is a linearly compact R -module.

The cosupport of M is defined by $\text{Cos}_R M = \{\mathfrak{p} \in \text{Spec } R : {}_{\mathfrak{p}} M \neq 0\}$ (see [9]). Next, Yassemi [14] defined the co-support of an R -module M , denoted by $\text{Cosupp}_R(M)$, to be the set of primes \mathfrak{p} such that there exists a cocyclic homomorphic image L of M with $\text{Ann}(L) \subseteq \mathfrak{p}$. It is well known that in case

M is an artinian R -module or M is a linearly compact R -module the equality $\text{Cos}_R(M) = \text{Cosupp}_R(M)$ is true.

Definition 2.1. Let M be a representable R -module. We define the representation dimension $r^I(M)$ of M relative to I by

$$r^I(M) = \inf\{i \in \mathbb{N}_0 : H_i^I(M) \text{ is not representable}\},$$

artinianness dimension $a^I(M)$ of M relative to I by

$$a^I(M) = \inf\{i \in \mathbb{N}_0 : H_i^I(M) \text{ is not artinian}\}$$

and the upper representation dimension $R^I(M)$ of M relative to I by

$$R^I(M) = \sup\{i \in \mathbb{N}_0 : H_i^I(M) \text{ is not representable}\}.$$

Note that, if M is an artinian R -module, then in [12, Theorem 2.3] we have proved that

$\inf\{i \in \mathbb{N}_0 : H_i^I(M) \text{ is not artinian}\} = \inf\{i \in \mathbb{N}_0 : H_i^I(M) \text{ is not representable}\}$
which implies that $r^I(M) = a^I(M)$.

We need the following lemma in the proof of some of our results.

Lemma 2.2. *Let R be a Noetherian ring, I an ideal of R and M a linearly compact R -module. Then*

- i) IM and M/IM are linearly compact R -modules,
- ii) ${}_p(IM) \simeq I_p({}_p M)$ for any prime ideal $\mathfrak{p} \in \text{Spec}(R)$,
- iii) $\text{Coass}_R(IM) \subseteq \text{Coass}_R M$.

Proof. i) See [7, Lemma 3.14] and [7, Lemma 3.5].

ii) Let $\mathfrak{p} \in \text{Spec}(R)$. By (i), $0 \rightarrow IM \rightarrow M \rightarrow M/IM \rightarrow 0$ is a short exact sequence of linearly compact R -modules. Now [10, Lemma 3.2] implies that

$$0 \rightarrow {}_p(IM) \rightarrow {}_p M \rightarrow {}_p(M/IM) \rightarrow 0$$

is also exact. But, by using [10, Lemma 3.4] we have

$${}_p(M/IM) \simeq {}_p(R/I \otimes_R M) \simeq R_p/IR_p \otimes_{R_p} {}_p M \simeq {}_p M/I_p({}_p M).$$

Now, by using the above short exact sequence and the exact sequence

$$0 \rightarrow I_p({}_p M) \rightarrow {}_p M \rightarrow {}_p M/I_p({}_p M) \rightarrow 0,$$

we obtain the result.

iii) Since IM is a homomorphic image of $I \otimes_R M$ we have $\text{Coass}_R(IM) \subseteq \text{Coass}_R(I \otimes_R M)$. But by [14, Theorem 1.21] $\text{Coass}_R(I \otimes_R M) \subseteq \text{Coass}_R M$. Thus we conclude that $\text{Coass}_R(IM) \subseteq \text{Coass}_R M$. \square

Theorem 2.3. *Let (R, \mathfrak{m}) be a Noetherian local ring, $I \subseteq \mathfrak{m}$ be an ideal of R and M an artinian R -module. Let n be an integer. Then the following conditions are equivalent:*

- i) $H_i^I(M)$ is an artinian R -module for all $i < n$,
- ii) $H_i^I(M)$ is a representable R -module for all $i < n$,

- iii) ${}_p(H_i^I(M))$ is a representable R_p -module for all $i < n$ and all $p \in \text{Spec}(R)$,

Proof. i) \Leftrightarrow ii) By [12, Theorem 2.3].

i) \Rightarrow iii) By [9, Theorem 3.2], colocalization of any artinian R -module is representable.

iii) \Rightarrow i) By induction on n . Let $n = 1$. There exists an epimorphism $M \rightarrow M/I^t M \rightarrow 0$ for all $t > 0$ and so we have an epimorphism $M \rightarrow H_0^I(M) \rightarrow 0$. Hence $H_0^I(M)$ is an artinian R -module. Suppose that $n > 1$. By the inductive hypothesis, $H_i^I(M)$ is an artinian R -module for all $i < n - 1$. We show that $H_{n-1}^I(M)$ is artinian. By [3, Theorem 3.6] ${}_p(H_{n-1}^I(M)) \simeq H_{n-1}^{IR_p}({}_p M)$. Thus, by assumption, $H_{n-1}^{IR_p}({}_p M)$ is a representable R_p -module for all $p \in \text{Spec}(R)$. By [12, Corollary 2.2] for each $p \in \text{Spec}(R)$, there exists an integer n_p such that $(IR_p)^{n_p} H_{n-1}^{IR_p}({}_p M) = 0$. On the other hand, by [3, Theorem 4.5] $\text{Coass}_R(H_{n-1}^I(M))$ is finite. Let $k = \max\{n_p : p \in \text{Coass}_R(H_{n-1}^I(M))\}$. Thus $(IR_p)^k H_{n-1}^{IR_p}({}_p M) = 0$ for all $p \in \text{Coass}_R(H_{n-1}^I(M))$. Since by [4, Proposition 3.3] $H_{n-1}^I(M)$ is linearly compact R -module, Lemma 2.2(ii) implies that ${}_p(I^k H_{n-1}^I(M)) = 0$ for all $p \in \text{Coass}_R(H_{n-1}^I(M))$. But, by Lemma 2.2(iii) we get that ${}_p(I^k H_{n-1}^I(M)) = 0$ for all $p \in \text{Coass}_R(I^k H_{n-1}^I(M))$. By Lemma 2.2(i) $I^k H_{n-1}^I(M)$ is linearly compact R -module and so by [3, Theorem 4.2], it follows that $I^k H_{n-1}^I(M) = 0$. Now by using [2, Proposition 4.7] we conclude that $H_{n-1}^I(M)$ is artinian R -module and the proof is complete. \square

Corollary 2.4. *Let (R, \mathfrak{m}) be a Noetherian local ring, $I \subseteq \mathfrak{m}$ be an ideal of R and M an artinian R -module. Then*

$$r^I(M) = \inf\{r^{IR_p}({}_p M) : p \in \text{Spec}(R)\}.$$

Proof. For any integer i and any $p \in \text{Spec}(R)$ we have ${}_p(H_i^I(M)) \simeq H_i^{IR_p}({}_p M)$. Thus, the result follows by Theorem 2.3. \square

Corollary 2.5. *Let (R, \mathfrak{m}) be a Noetherian local ring, $I \subseteq \mathfrak{m}$ be an ideal of R and M an artinian R -module. Let n be an integer. Then, if ${}_p(H_i^I(M))$ is artinian R_p -module for all $i < n$ and all $p \in \text{Spec}(R)$, then $H_i^I(M)$ is artinian R -module for all $i < n$. Thus*

$$a^I(M) \geq \inf\{a^{IR_p}({}_p M) : p \in \text{Spec}(R)\}.$$

Proof. Since any artinian R_p -module is a representable R_p -module, the result follows by Theorem 2.3 iii) \Rightarrow i). \square

Recall that, module M is called coatomic, if every proper submodule of M is contained in a maximal submodule of M . Over Noetherian rings, the coatomic modules are closed under taking quotients, submodules and extensions. It is clear that every finitely generated R -module is coatomic and that every coatomic, artinian module has finite length. Also, it is well known that M is

coatomic if and only if $\text{Coass}_R(M)$ consists only of maximal ideals. For more details of coatomic modules, we refer the reader to [16] and [17].

Theorem 2.6. *Let (R, \mathfrak{m}) be a Noetherian local ring, $I \subseteq \mathfrak{m}$ be an ideal of R and M an artinian R -module. Let n be an integer. Then the following conditions are equivalent:*

- i) $H_i^I(M)$ is of finite length R -module for all $i < n$,
- ii) ${}_{\mathfrak{p}}(H_i^I(M))$ is a coatomic and representable $R_{\mathfrak{p}}$ -module for all $i < n$ and all $\mathfrak{p} \in \text{Spec}(R)$.

Proof. i) \Rightarrow ii) By assumption $H_i^I(M)$ is an artinian R -module for all $i < n$ and so Theorem 2.3 implies that ${}_{\mathfrak{p}}(H_i^I(M))$ is a representable $R_{\mathfrak{p}}$ -module for all $\mathfrak{p} \in \text{Spec}(R)$. On the other hand, by [9, Proposition 7.4] we have $\text{Cos}_R(H_i^I(M)) \subseteq \{\mathfrak{m}\}$ for all $i < n$. Thus ${}_{\mathfrak{p}}(H_i^I(M)) = 0$ for all $\mathfrak{p} \in \text{Spec}(R) \setminus \{\mathfrak{m}\}$ and all $i < n$. It is sufficient to show that ${}_{\mathfrak{m}}(H_i^I(M))$ is coatomic $R_{\mathfrak{m}}$ -module for all $i < n$. By [9, Theorem 3.2] $\text{Att}_{R_{\mathfrak{m}}}({}_{\mathfrak{m}}(H_i^I(M))) = \{\mathfrak{q}R_{\mathfrak{m}} : \mathfrak{q} \in \text{Att}_R(H_i^I(M)) \text{ and } \mathfrak{q} \subseteq \mathfrak{m}\}$. But by assumption (i) and [9, Proposition 7.4] we have $\text{Att}_R(H_i^I(M)) \subseteq \{\mathfrak{m}\}$. Hence it follows that $\text{Att}_{R_{\mathfrak{m}}}({}_{\mathfrak{m}}(H_i^I(M))) \subseteq \{\mathfrak{m}R_{\mathfrak{m}}\}$. Since ${}_{\mathfrak{m}}(H_i^I(M))$ is representable by [14, Theorem 1.14] $\text{Att}_{R_{\mathfrak{m}}}({}_{\mathfrak{m}}(H_i^I(M))) = \text{Coass}_{R_{\mathfrak{m}}}({}_{\mathfrak{m}}(H_i^I(M)))$. Hence, $\text{Coass}_{R_{\mathfrak{m}}}({}_{\mathfrak{m}}(H_i^I(M))) \subseteq \{\mathfrak{m}R_{\mathfrak{m}}\}$ and so ${}_{\mathfrak{m}}(H_i^I(M))$ is coatomic $R_{\mathfrak{m}}$ -module for all $i < n$, as required.

ii) \Rightarrow i) By Theorem 2.3 $H_i^I(M)$ is an artinian R -module for all $i < n$. By [9, Proposition 7.4] it is sufficient to show that $\text{Cos}_R(H_i^I(M)) \subseteq \{\mathfrak{m}\}$ for all $i < n$. Let $i < n$ be an integer and $\mathfrak{p} \in \text{Cos}_R(H_i^I(M))$. Then ${}_{\mathfrak{p}}(H_i^I(M)) \neq 0$ and so $\mathfrak{p} \neq \text{Att}_{R_{\mathfrak{p}}}({}_{\mathfrak{p}}(H_i^I(M))) = \{\mathfrak{q}R_{\mathfrak{p}} : \mathfrak{q} \in \text{Att}_R(H_i^I(M)) \text{ and } \mathfrak{q} \subseteq \mathfrak{p}\}$. Thus there exists $\mathfrak{q} \subseteq \mathfrak{p}$ and $\mathfrak{q} \in \text{Att}_R(H_i^I(M))$. It follows that $\mathfrak{q}R_{\mathfrak{m}} \in \text{Att}_{R_{\mathfrak{m}}}({}_{\mathfrak{m}}(H_i^I(M)))$. But by assumption ${}_{\mathfrak{m}}(H_i^I(M))$ is a representable and coatomic $R_{\mathfrak{m}}$ -module and so $\text{Att}_{R_{\mathfrak{m}}}({}_{\mathfrak{m}}(H_i^I(M))) = \text{Coass}_{R_{\mathfrak{m}}}({}_{\mathfrak{m}}(H_i^I(M))) \subseteq \{\mathfrak{m}R_{\mathfrak{m}}\}$. Thus we conclude that $\mathfrak{q} = \mathfrak{m}$ and so $\mathfrak{p} = \mathfrak{m}$. Therefore $\text{Cos}_R(H_i^I(M)) \subseteq \{\mathfrak{m}\}$. \square

Theorem 2.7. *Let (R, \mathfrak{m}) be a Noetherian local ring, $I \subseteq \mathfrak{m}$ be an ideal of R and M an artinian R -module. Let n and t be two arbitrary integers. If $H_i^I(M)$ is minimax for all $i < n$, then $H_n^I(M)/I^t H_n^I(M)$ is minimax.*

Proof. i) We use induction on n . When $n = 0$, for all positive integers k the canonical epi-morphisms $M \rightarrow M/I^k M$ induces an epi-morphism $M \rightarrow \Lambda_I(M)$ where $\Lambda_I(M) = \varprojlim_k M/I^k M$ denotes the I -adic completion of M . Thus

$\Lambda_I(M)$ is homomorphic image of an artinian module and so is minimax. Since $H_0^I(M) \cong \Lambda_I(M)$, the result follows in this case. Now suppose, inductively that $n > 0$ and the result is true for $n - 1$. By [2, Corollary 4.5], we can replace M by $\bigcap_{n>0} I^n M$. But $\bigcap_{n>0} I^n M = I^k M$ for some $k \in \mathbb{N}$ and so we may assume that $IM = M$. Thus there exists $x \in I$ such that $xM = M$ by [8, 2.8] and so

$x^t M = M$. From the exact sequence

$$0 \rightarrow (0 :_M x^t) \rightarrow M \xrightarrow{x^t} M \rightarrow 0$$

and using [2, Corollary 4.2], we obtain the following long exact sequence

$$\cdots \rightarrow H_n^I(M) \xrightarrow{x^t} H_n^I(M) \xrightarrow{f} H_{n-1}^I(0 :_M x^t) \xrightarrow{g} H_{n-1}^I(M) \rightarrow H_{n-1}^I(M) \rightarrow \cdots.$$

From the above exact sequence we get

$$0 \rightarrow \text{Im } f \rightarrow H_{n-1}^I(0 :_M x^t) \rightarrow \text{Im } g \rightarrow 0,$$

$$\cdots \rightarrow H_n^I(M) \xrightarrow{x^t} H_n^I(M) \rightarrow \text{Im } f \rightarrow 0.$$

Thus we have the following exact sequences:

$$\begin{aligned} \cdots \rightarrow \text{Tor}_1^R(R/I^t, \text{Im } g) \rightarrow \text{Im } f/I^t \text{Im } f \\ \rightarrow H_{n-1}^I(0 :_M x^t)/I^t H_{n-1}^I(0 :_M x^t) \rightarrow \cdots, \end{aligned}$$

$$\cdots \rightarrow H_n^I(M)/I^t H_n^I(M) \xrightarrow{x^t} H_n^I(M)/I^t H_n^I(M) \rightarrow \text{Im } f/I^t \text{Im } f \rightarrow 0.$$

Induction hypothesis implies that $H_{n-1}^I(0 :_M x^t)/I^t H_{n-1}^I(0 :_M x^t)$ is minimax. But the class of minimax modules is a Serre subcategory of the category of R -modules and so it is easy to see that since $\text{Im } g$ is minimax $\text{Tor}_1^R(R/I^t, \text{Im } g)$ is minimax. Hence $\text{Im } f/I^t \text{Im } f$ is minimax. Since $x^t \in I^t$, the first map in the above exact sequence is zero, and therefore $H_n^I(M)/I^t H_n^I(M) \simeq \text{Im } f/I^t \text{Im } f$. Hence, $H_n^I(M)/I^t H_n^I(M)$ is minimax, as desired. \square

Proposition 2.8. *Let (R, \mathfrak{m}) be a Noetherian local ring and L a linearly compact R -module. If L is a minimax R -module, then ${}_p L$ is a representable R_p -module for all $\mathfrak{p} \in \text{Spec}(R) \setminus \{\mathfrak{m}\}$,*

Proof. Let $\mathfrak{p} \in \text{Spec}(R) \setminus \{\mathfrak{m}\}$. By assumption there exists a finitely generated submodule N of L such that L/N is an artinian R -module. By [9, Theorem 3.2] ${}_p(L/N)$ is a representable R_p -module. On the other hand, since L/N is a linearly compact R -module with the discrete topology, N is an open submodule of L . Thus, N is open, hence N is closed in L . Thus by [4, Lemma 2.3] N is a linearly compact submodule of L . Now by [10, Lemma 3.2], the short exact sequence $0 \rightarrow N \rightarrow L \rightarrow L/N \rightarrow 0$ of linearly compact R -modules induces an exact sequence $0 \rightarrow {}_p N \rightarrow {}_p L \rightarrow {}_p(L/N) \rightarrow 0$. It follows that ${}_p L/{}_p N \simeq {}_p(L/N)$ and so ${}_p L/{}_p N$ is a representable R_p -module. Since N is finitely generated and $\mathfrak{p} \in \text{Spec}(R) \setminus \{\mathfrak{m}\}$ we have ${}_p N = 0$ by [14, Theorem 2.10]. Therefore ${}_p L$ is a representable R_p -module and the proof is complete. \square

Theorem 2.9. *Let (R, \mathfrak{m}) be a Noetherian local ring, $I \subseteq \mathfrak{m}$ be an ideal of R and M an artinian R -module. Let i be an integer. If $H_i^I(M)$ is a minimax R -module, then ${}_p(H_i^I(M))$ is a representable R_p -module for all $\mathfrak{p} \in \text{Spec}(R) \setminus \{\mathfrak{m}\}$,*

Proof. By [4, Proposition 3.3] $H_i^I(M)$ is a linearly compact R -module. Therefore, the result follows by Proposition 2.8. \square

Theorem 2.10. *Let (R, \mathfrak{m}) be a Noetherian local ring, $I \subseteq \mathfrak{m}$ be an ideal of R and M an artinian R -module. Let n be an integer. Then the following conditions are equivalent:*

- i) $H_i^I(M)$ is a minimax R -module for all $i < n$.
- ii) ${}_{\mathfrak{p}}(H_i^I(M))$ is a representable $R_{\mathfrak{p}}$ -module for all $i < n$ and all $\mathfrak{p} \in \text{Spec}(R) \setminus \{\mathfrak{m}\}$.

Proof. i) \Rightarrow ii) By Theorem 2.9.

ii) \Rightarrow i) We use induction on n . Let $n = 1$. There exists an epimorphism $M \rightarrow M/I^t M \rightarrow 0$ for all $t > 0$ and so we have an epimorphism $M \rightarrow H_0^I(M) \rightarrow 0$. Hence $H_0^I(M)$ is an artinian R -module and so $H_0^I(M)$ is a minimax R -module. Suppose that $n > 1$. By the inductive hypothesis, $H_i^I(M)$ is a minimax R -module for all $i < n - 1$. It is sufficient to show that $H_{n-1}^I(M)$ is a minimax R -module. At first, we prove that there exists an integer t such that $\text{Cosupp}_R(I^t H_{n-1}^I(M)) \subseteq \{\mathfrak{m}\}$. By [11, Theorem 3.2] $\text{Coass}_R(H_{n-1}^I(M))$ is finite. Let $\text{Coass}_R(H_{n-1}^I(M)) \setminus \{\mathfrak{m}\} = \{\mathfrak{p}_1, \dots, \mathfrak{p}_k\}$. By assumption ${}_{\mathfrak{p}_i}(H_{n-1}^I(M))$ is a representable $R_{\mathfrak{p}_i}$ -module for all $1 \leq i \leq k$. Since ${}_{\mathfrak{p}_i}(H_{n-1}^I(M)) \simeq H_{n-1}^{IR_{\mathfrak{p}_i}}({}_{\mathfrak{p}_i}M)$ for all $1 \leq i \leq k$, it follows by [12, Theorem 2.3] that for any integer $1 \leq i \leq k$ there exists an integer t_i such that $(IR_{\mathfrak{p}_i})^{t_i} H_{n-1}^{IR_{\mathfrak{p}_i}}({}_{\mathfrak{p}_i}M) = 0$. Let $t = \text{Max}\{t_1, \dots, t_k\}$. Thus $(IR_{\mathfrak{p}_i})^t H_{n-1}^{IR_{\mathfrak{p}_i}}({}_{\mathfrak{p}_i}M) = 0$ for all $1 \leq i \leq k$. By Lemma 2.2(ii), $(IR_{\mathfrak{p}_i})^t H_{n-1}^{IR_{\mathfrak{p}_i}}({}_{\mathfrak{p}_i}M) \simeq {}_{\mathfrak{p}_i}(I^t H_{n-1}^I(M))$ for all $1 \leq i \leq k$ and so ${}_{\mathfrak{p}_i}(I^t H_{n-1}^I(M)) = 0$ for all $1 \leq i \leq k$. Hence $\mathfrak{p}_i \notin \text{Cosupp}_R(I^t H_{n-1}^I(M))$ for all $1 \leq i \leq k$ and so

$$\{\mathfrak{p}_1, \dots, \mathfrak{p}_k\} \cap \text{Coass}_R(I^t H_{n-1}^I(M)) = \emptyset.$$

On the other hand, by Lemma 2.2(iii)

$$\text{Coass}_R(I^t H_{n-1}^I(M)) \setminus \{\mathfrak{m}\} \subseteq \text{Coass}_R(H_{n-1}^I(M)) \setminus \{\mathfrak{m}\} = \{\mathfrak{p}_1, \dots, \mathfrak{p}_k\}.$$

It follows that $\text{Coass}_R(I^t H_{n-1}^I(M)) \subseteq \{\mathfrak{m}\}$. Now by [18, Satz 2.4], it follows that there exists an integer s such that $\mathfrak{m}^s I^t H_{n-1}^I(M)$ is finitely generated. Thus $I^{s+t} H_{n-1}^I(M)$ is finitely generated and so is minimax. On the other hand, by Theorem 2.7 $H_{n-1}^I(M)/I^{s+t} H_{n-1}^I(M)$ is minimax. Since the class of minimax modules is a Serre subcategory of the category of R -modules, it follows that $H_{n-1}^I(M)$ is minimax and the proof is complete. \square

Corollary 2.11. *Let (R, \mathfrak{m}) be a Noetherian local ring, $I \subseteq \mathfrak{m}$ be an ideal of R and M an artinian R -module. Then*

$$\inf\{i : H_i^I(M) \text{ is not minimax}\} = \inf\{r^{IR_{\mathfrak{p}}}({}_{\mathfrak{p}}M) \mid \mathfrak{p} \in \text{Spec}(R) \setminus \{\mathfrak{m}\}\}.$$

Proof. It follows by Theorem 2.10. \square

Theorem 2.12. *Let (R, \mathfrak{m}) be a Noetherian local ring, $I \subseteq \mathfrak{m}$ be an ideal of R and M an artinian R -module. Let n be an integer. Then for any $\mathfrak{p} \in \text{Spec}(R)$ the following conditions are equivalent:*

- i) ${}_p(H_i^I(M))$ is a representable R_p -module for all $i > n$.
- ii) $IR_p \subseteq \sqrt{(0 :_{R_p} {}_p(H_i^I(M)))}$ for all $i > n$.
- iii) ${}_p(H_i^I(M)) = 0$ for all $i > n$.

Proof. i) \Rightarrow ii) Since ${}_p(H_i^I(M)) \simeq H_i^{IR_p}({}_p M)$, it follows by [12, Corollary 2.2].

ii) \Rightarrow iii) We use induction on $u := \text{Ndim}_R M$. Let $u = 0$. Since $H_i^I(M) = 0$ for all $i > 0$, by [2, Proposition 4.8], the result follows in this case. Now suppose, inductively that $u > 0$ and the result is true for $u - 1$. By Lemma [2, Corollary 4.5], we can replace M by $\bigcap_{n>0} I^n M$. But $\bigcap_{n>0} I^n M = I^k M$ for some $k \in \mathbb{N}$ and so we may assume that $IM = M$. Since M is artinian, $xM = M$ for some $x \in I$ by [8, 2.8]. Now for all $i > n$, from the exact sequence

$$0 \rightarrow (0 :_M x) \rightarrow M \xrightarrow{x} M \rightarrow 0$$

we obtain the following long exact sequence of linearly compact R -modules

$$\cdots \rightarrow H_{i+1}^I(M) \rightarrow H_i^I(0 :_M x) \rightarrow H_i^I(M) \xrightarrow{x} H_i^I(M) \rightarrow \cdots$$

By [10, Lemma 3.2] we have the following long exact sequence

$$\cdots \rightarrow {}_p(H_{i+1}^I(M)) \rightarrow {}_p(H_i^I(0 :_M x)) \rightarrow {}_p(H_i^I(M)) \xrightarrow{x/1} {}_p(H_i^I(M)) \rightarrow \cdots$$

By [4, Lemma 4.7] $\text{Ndim}_R(0 :_M x) \leq u - 1$ and so the induction hypothesis implies that ${}_p(H_i^I(0 :_M x)) = 0$ for all $i > n$. Thus for any $i > n$ we have an injection $0 \rightarrow {}_p(H_i^I(M)) \xrightarrow{x/1} {}_p(H_i^I(M))$. Suppose that ${}_p(H_i^I(M)) \neq 0$ for some $i > n$. Since $\frac{x}{1} \in IR_p$, by assumption (ii) there exists a positive integer k such that $(x/1)^k {}_p(H_i^I(M)) = 0$. Now from the above injection we get a contradiction. Hence ${}_p(H_i^I(M)) = 0$ for all $i > n$ and the proof is complete.

iii) \Rightarrow i) It is clear. \square

Theorem 2.13. *Let (R, \mathfrak{m}) be a Noetherian local ring, $I \subseteq \mathfrak{m}$ be an ideal of R and M an artinian R -module. Let n be an integer. Then the following conditions are equivalent:*

- i) $H_i^I(M)$ is an artinian R -module for all $i > n$.
- ii) ${}_p(H_i^I(M))$ is a representable R_p -module for all $i > n$ and all $p \in \text{Spec}(R)$.
- iii) ${}_p(H_i^I(M)) = 0$ for all $i > n$ and all $p \in \text{Spec}(R)$.
- iv) $H_i^I(M) = 0$ for all $i > n$.

Proof. i) \Rightarrow ii) By [9, Theorem 3.2], colocalization of any artinian R -module is representable.

ii) \Rightarrow iii) By Theorem 2.12 i) \Rightarrow iii).

iii) \Rightarrow iv) Assumption implies that $\text{Cos}_R(H_i^I(M)) = \phi$ for all $i > n$. Thus $H_i^I(M) = 0$ for all $i > n$ by [3, 4.3].

iv) \Rightarrow i) It is clear. \square

Theorem 2.14. *Let (R, \mathfrak{m}) be a Noetherian local ring, $I \subseteq \mathfrak{m}$ be an ideal of R and M an artinian R -module. Let n be an integer. Then the following conditions are equivalent:*

- i) $H_i^I(M)$ is a finitely generated R -module for all $i > n$.
- ii) $H_i^I(M)$ is a minimax R -module for all $i > n$.
- iii) ${}_{\mathfrak{p}}(H_i^I(M))$ is a representable $R_{\mathfrak{p}}$ -module for all $i > n$ and all $\mathfrak{p} \in \text{Spec}(R) \setminus \{\mathfrak{m}\}$.

Proof. i) \Rightarrow ii): Any finitely generated R -module is minimax.

ii) \Rightarrow iii): By Theorem 2.9.

iii) \Rightarrow i): By Theorem 2.12, ${}_{\mathfrak{p}}(H_i^I(M)) = 0$ for all $i > n$ and all $\mathfrak{p} \in \text{Spec}(R) \setminus \{\mathfrak{m}\}$. Thus $\text{Cos}_R((H_i^I(M))) \subseteq \{\mathfrak{m}\}$ for all $i > n$ and so

$$\text{Coass}_R((H_i^I(M))) \subseteq \{\mathfrak{m}\}$$

for all $i > n$. Now by [18, satz 2.4], for any $i > n$ there exists a positive integer t_i such that $I^{t_i} H_i^I(M)$ is finitely generated. Since $H_i^I(M) = 0$ for all $i > \text{Ndim}_R M$ by [2, Proposition 4.8], we can find an integer t such that $I^t H_i^I(M)$ is finitely generated for all $i > n$. Now we use induction on $u := \text{Ndim}_R M$. Let $u = 0$. Since $H_i^I(M) = 0$ for all $i > 0$, by [2, Proposition 4.8], the result follows in this case. Now suppose, inductively that $u > 0$ and the result is true for $u - 1$. By an argument analogue to that used in the proof of Theorem 2.12, we may assume that $xM = M$ for some $x \in I$ and so $x^t M = M$. Thus for all $i > n$, the exact sequence

$$0 \rightarrow (0 :_M x^t) \rightarrow M \xrightarrow{x^t} M \rightarrow 0$$

implies that

$$\cdots \rightarrow H_i^I(0 :_M x^t) \xrightarrow{\varphi_i} H_i^I(M) \xrightarrow{x^t} H_i^I(M) \rightarrow \cdots$$

Since $\text{Ndim}_R(0 :_M x^t) \leq u - 1$, induction hypothesis implies that $H_i^I(0 :_M x^t)$ is finitely generated for all $i > n$ and so we have the exact sequence $0 \rightarrow \text{Im } \varphi_i \rightarrow H_i^I(M) \rightarrow x^t H_i^I(M) \rightarrow 0$ for all $i > n$. Since $\text{Im } \varphi_i$ and $x^t H_i^I(M)$ are finitely generated we conclude that $H_i^I(M)$ is finitely generated for all $i > n$. \square

Corollary 2.15. *Let (R, \mathfrak{m}) be a Noetherian local ring, $I \subseteq \mathfrak{m}$ be an ideal of R and M an artinian R -module. Then we have*

- i) $\sup\{i \in \mathbb{N}_0 : H_i^I(M) \neq 0\} = \sup\{i \in \mathbb{N}_0 : H_i^I(M) \text{ is not artinian}\} = \sup\{R^{IR_{\mathfrak{p}}}(M) : \mathfrak{p} \in \text{Spec}(R)\}$,
- ii) $\sup\{i \in \mathbb{N}_0 : H_i^I(M) \text{ is not finitely generated}\} = \sup\{i \in \mathbb{N}_0 : H_i^I(M) \text{ is not minimax}\} = \sup\{R^{IR_{\mathfrak{p}}}(M) : \mathfrak{p} \in \text{Spec}(R) \setminus \{\mathfrak{m}\}\}$.

Proof. It follows by Theorems 2.13 and 2.14. \square

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References

- [1] M. P. Brodmann and R. Y. Sharp, *Local cohomology: an algebraic introduction with geometric applications*, Cambridge Studies in Advanced Mathematics, **60**, Cambridge University Press, Cambridge, 1998. <https://doi.org/10.1017/CB09780511629204>
- [2] N. T. Cuong and T. T. Nam, *The I -adic completion and local homology for Artinian modules*, Math. Proc. Cambridge Philos. Soc. **131** (2001), no. 1, 61–75. <https://doi.org/10.1017/S0305004100004771>
- [3] ———, *On the co-localization, co-support and co-associated primes of local homology modules*, Vietnam J. Math. **29** (2001), no. 4, 359–368.
- [4] ———, *A local homology theory for linearly compact modules*, J. Algebra **319** (2008), no. 11, 4712–4737. <https://doi.org/10.1016/j.jalgebra.2007.11.030>
- [5] J. P. C. Greenlees and J. P. May, *Derived functors of I -adic completion and local homology*, J. Algebra **149** (1992), no. 2, 438–453. [https://doi.org/10.1016/0021-8693\(92\)90026-1](https://doi.org/10.1016/0021-8693(92)90026-1)
- [6] D. Kirby, *Dimension and length for Artinian modules*, Quart. J. Math. Oxford Ser. (2) **41** (1990), no. 164, 419–429. <https://doi.org/10.1093/qmath/41.4.419>
- [7] I. G. Macdonald, *Duality over complete local rings*, Topology **1** (1962), 213–235. [https://doi.org/10.1016/0040-9383\(62\)90104-0](https://doi.org/10.1016/0040-9383(62)90104-0)
- [8] ———, *Secondary representation of modules over a commutative ring*, in Symposia Mathematica, Vol. XI (Convegno di Algebra Commutativa, INDAM, Rome, 1971), 23–43, Academic Press, London, 1973.
- [9] L. Melkersson and P. Schenzel, *The co-localization of an Artinian module*, Proc. Edinburgh Math. Soc. (2) **38** (1995), no. 1, 121–131. <https://doi.org/10.1017/S0013091500006258>
- [10] T. T. Nam, *Co-support and coartinian modules*, Algebra Colloq. **15** (2008), no. 1, 83–96. <https://doi.org/10.1142/S1005386708000084>
- [11] ———, *Minimax modules, local homology and local cohomology*, Internat. J. Math. **26** (2015), no. 12, 1550102, 16 pp. <https://doi.org/10.1142/S0129167X15501025>
- [12] S. Rezaei, *Some results on local homology and local cohomology modules*, Illinois J. Math. **57** (2013), no. 1, 17–23. <http://projecteuclid.org/euclid.ijm/1403534483>
- [13] Z. M. Tang, *Local homology theory for Artinian modules*, Comm. Algebra **22** (1994), no. 5, 1675–1684. <https://doi.org/10.1080/00927879408824928>
- [14] S. Yassemi, *Coassociated primes*, Comm. Algebra **23** (1995), no. 4, 1473–1498. <https://doi.org/10.1080/00927879508825288>
- [15] D. Zelinsky, *Linearly compact modules and rings*, Amer. J. Math. **75** (1953), 79–90. <https://doi.org/10.2307/2372616>
- [16] H. Zöschinger, *Koatomare Moduln*, Math. Z. **170** (1980), no. 3, 221–232.
- [17] ———, *Linear-kompakte Moduln über noetherschen Ringen*, Arch. Math. (Basel) **41** (1983), no. 2, 121–130.
- [18] ———, *Über koassozierte Primideale*, Math. Scand. **63** (1988), no. 2, 196–211.

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