

HYPERBOLICALLY CLOSE TO $Q_p^\#$ -SEQUENCES

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ABSTRACT. It is shown that each sequence lying sufficiently close in the hyperbolic sense to a $Q_p^\#$ -sequence for a meromorphic function f in the unit disc is also a $Q_p^\#$ -sequence for f .

1. Introduction and results

Let $\mathcal{M}(\mathbb{D})$ denote the class of meromorphic functions in the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ of the complex plane \mathbb{C} . Green's function in \mathbb{D} with logarithmic singularity at $a \in \mathbb{D}$ is $g(z, a) = -\log |\varphi_a(z)|$, where $\varphi_a(z) = (a-z)/(1-\bar{a}z)$ for all $z \in \mathbb{D}$. The function φ_a is the Möbius transformation of \mathbb{D} which interchanges the point $a \in \mathbb{D}$ and the origin, and it is its own inverse. For $0 < p < \infty$, the class $Q_p^\#$ consists of $f \in \mathcal{M}(\mathbb{D})$ such that

$$\|f\|_{Q_p^\#}^2 = \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} f^\#(z)^2 g^p(z, a) dA(z) < \infty,$$

where $f^\#(z) = |f'(z)|/(1+|f(z)|^2)$ is the spherical derivative of f at z and $dA(z) = r dr d\theta$ for $z = re^{i\theta}$ denotes the element of Lebesgue area measure on \mathbb{D} . It is known that $Q_1^\#$ coincides with the class UBC of functions in $\mathcal{M}(\mathbb{D})$ of uniformly bounded Nevanlinna characteristic in \mathbb{D} [5], and, for each $p > 1$, $Q_p^\#$ is the same as the class \mathcal{N} of meromorphic normal functions [2], defined by the condition

$$\|f\|_{\mathcal{N}} = \sup_{z \in \mathbb{D}} f^\#(z)(1-|z|^2) < \infty.$$

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Let $f \in \mathcal{M}(\mathbb{D})$. According to [1, Definition 1], a sequence $\{a_n\}_{n=1}^\infty$ in \mathbb{D} is a q_N -sequence for f if

$$\lim_{n \rightarrow \infty} f^\#(a_n)(1 - |a_n|^2) = \infty.$$

Further, we say that $\{a_n\}_{n=1}^\infty$ is an \mathcal{N} -sequence for f if

$$\limsup_{n \rightarrow \infty} f^\#(a_n)(1 - |a_n|^2) = \infty.$$

Then $\{a_n\}_{n=1}^\infty$ is an \mathcal{N} -sequence for f if and only if one of its subsequences is a q_N -sequence for f . Similarly, if $0 < p < \infty$, then $\{a_n\}_{n=1}^\infty$ is a q_p -sequence for $f \in \mathcal{M}(\mathbb{D})$ according to [1, Definition 2] if

$$\lim_{n \rightarrow \infty} \int_{\mathbb{D}} f^\#(z)^2 g^p(z, a_n) dA(z) = \infty,$$

and it is a $Q_p^\#$ -sequence for f if

$$\limsup_{n \rightarrow \infty} \int_{\mathbb{D}} f^\#(z)^2 g^p(z, a_n) dA(z) = \infty.$$

If $\{a_n\}_{n=1}^\infty$ is a q_N -sequence for $f \in \mathcal{M}(\mathbb{D})$, then each sequence $\{b_n\}_{n=1}^\infty$ in \mathbb{D} for which $\sigma(a_n, b_n) \rightarrow 0$, as $n \rightarrow \infty$, is a q_p -sequence for f for all $0 < p < \infty$ by [1, Theorem 1]. Here $\sigma(z, w) = |\varphi_z(w)|$ is the pseudohyperbolic distance between two points z and w in \mathbb{D} . Hence each $\{b_n\}_{n=1}^\infty$ satisfying $\sigma(a_n, b_n) \rightarrow 0$, as $n \rightarrow \infty$, is a $Q_p^\#$ -sequence for $f \in \mathcal{M}(\mathbb{D})$ if $\{a_n\}_{n=1}^\infty$ is an \mathcal{N} -sequence for $f \in \mathcal{M}(\mathbb{D})$.

If $\{a_n\}_{n=1}^\infty$ is a $Q_p^\#$ -sequence for $f \in \mathcal{M}(\mathbb{D})$, then each sequence $\{b_n\}_{n=1}^\infty$ in \mathbb{D} for which $\sigma(a_n, b_n) \rightarrow 0$, as $n \rightarrow \infty$, is also a $Q_p^\#$ -sequence for f by the proof of [1, Theorem 5]. The following theorem improves this result.

Theorem 1.1. *Let $0 < p < \infty$ and $f \in \mathcal{M}(\mathbb{D})$, and let $\{a_n\}_{n=1}^\infty$ be a $Q_p^\#$ -sequence for f . Then there exists $\delta = \delta(f, p) \in (0, 1)$ such that each sequence $\{b_n\}_{n=1}^\infty$ in \mathbb{D} satisfying $\sigma(a_n, b_n) \leq \delta$ for all $n \in \mathbb{N}$ is a $Q_p^\#$ -sequence for f .*

By using Theorem 1.1 we obtain the following improvement of [1, Theorem 6].

Corollary 1.2. *Let $0 < p < p' < \infty$ and $f \in \mathcal{M}(\mathbb{D})$, and let $\{a_n\}_{n=1}^\infty$ be a $Q_p^\#$ -sequence for f . If*

$$(1.1) \quad \limsup_{n \rightarrow \infty} \int_{\mathbb{D} \setminus \Delta(a_n, r)} f^\#(z)^2 (1 - |\varphi_{a_n}(z)|^2)^p dA(z) < \infty$$

for some $r \in (0, 1)$, then there exists $\delta = \delta(f, p') \in (0, 1)$ such that each sequence $\{b_n\}_{n=1}^\infty$ in \mathbb{D} satisfying $\sigma(a_n, b_n) \leq \delta$ for all $n \in \mathbb{N}$ is a $Q_{p'}^\#$ -sequence for f .

In the forthcoming sections we will prove our results in the order of appearance.

2. Auxiliary result

To proof Theorem 1.1 we need the following auxiliary result.

Lemma 2.1. *Let $0 < p < \infty$ and $f \in \mathcal{M}(\mathbb{D})$, and let $\{a_n\}_{n=1}^\infty$ be an \mathcal{N} -sequence for f . Then there exists $\delta = \delta(f, p) \in (0, 1)$ such that each sequence $\{b_n\}_{n=1}^\infty$ satisfying $\sigma(a_n, b_n) \leq \delta$ for all $n \in \mathbb{N}$ is a $Q_p^\#$ -sequence for f .*

Proof. Assume on the contrary to the assertion that for each $\delta \in (0, 1)$ there exists a sequence $\{b_n\}_{n=1}^\infty$ in \mathbb{D} such that $\sigma(a_n, b_n) \leq \delta$ for all $n \in \mathbb{N}$, but

$$(2.1) \quad K = K(f, p, \{b_n\}) = \sup_{n \in \mathbb{N}} \int_{\mathbb{D}} f^\#(z)^2 g^p(z, b_n) dA(z) < \infty.$$

Let $\Delta(z, r) = \{\zeta \in \mathbb{D} : \sigma(z, \zeta) < r\}$ denote the pseudohyperbolic disc with center $z \in \mathbb{D}$ and radius $r \in (0, 1)$, and let $D(z, R) = \{\zeta \in \mathbb{C} : |z - \zeta| < R\}$ be the Euclidean disc with center $z \in \mathbb{C}$ and radius $R > 0$. By the hypothesis, we can pick up a subsequence $\{a_n^1\}_{n=1}^\infty$ of $\{a_n\}_{n=1}^\infty$ which is a $q_{\mathcal{N}}$ -sequence for f . Let $\{b_n^1\}_{n=1}^\infty$ be the corresponding subsequence of $\{b_n\}_{n=1}^\infty$. Then (2.1) yields

$$(2.2) \quad \begin{aligned} K &\geq \int_{\mathbb{D}} f^\#(z)^2 g^p(z, b_n^1) dA(z) \geq \int_{\Delta(b_n^1, r)} f^\#(z)^2 g^p(z, b_n^1) dA(z) \\ &\geq \left(\log \frac{1}{r}\right)^p \int_{D(0, r)} f_n^\#(z)^2 dA(z), \quad n \in \mathbb{N}, \quad 0 < r < 1, \end{aligned}$$

where $f_n = f \circ \varphi_{b_n^1}$ for all $n \in \mathbb{N}$. Choose $r = r(K, p) \in (0, 1)$ sufficiently small such that $(-\log r)^p > K/\pi$. Then [3, Theorem 6] shows that $\{f_n : n \in \mathbb{N}\}$ is a normal family in $D(0, r)$. Therefore there exists a subsequence $\{f_{n_k}\}_{k=1}^\infty$ which converges uniformly on compact subsets of $D(0, r)$ to a meromorphic function h on $D(0, r)$ or to ∞ . In the latter case, (2.2) yields

$$\pi > \lim_{k \rightarrow \infty} \int_{D(0, r/2)} f_{n_k}^\#(z)^2 dA(z) = \infty,$$

which is a contradiction, and thus the assertion follows. If the limit h is meromorphic on $D(0, r)$, then $f_{n_k}^\#$ converges uniformly to $h^\#$ on $\overline{D(0, r/2)}$ as $k \rightarrow \infty$. Fix now $\delta = \delta(r) > 0$ sufficiently small such that $\Delta(\varphi_{b_n}(a_n), \delta) \subset D(0, r/2)$ for all $n \in \mathbb{N}$. This is possible because $\sigma(a_n, b_n) \leq \delta$ for all $n \in \mathbb{N}$. Then, by the uniform convergence on $\overline{D(0, r/2)}$,

$$\begin{aligned} &\lim_{k \rightarrow \infty} \int_{\Delta(a_{n_k}^1, \delta)} f^\#(z)^2 g^p(z, a_{n_k}^1) dA(z) \\ &= \lim_{k \rightarrow \infty} \int_{\Delta(\varphi_{b_{n_k}^1}(a_{n_k}^1), \delta)} f_{n_k}^\#(\zeta)^2 g^p(\zeta, \varphi_{b_{n_k}^1}(a_{n_k}^1)) dA(\zeta) \\ &\leq \lim_{k \rightarrow \infty} \int_{D(0, r/2)} f_{n_k}^\#(\zeta)^2 g^p(\zeta, \varphi_{b_{n_k}^1}(a_{n_k}^1)) dA(\zeta) \end{aligned}$$

$$\leq \sup_{c \in D(0, \delta)} \int_{D(0, r/2)} h^\#(\zeta)^2 g^p(\zeta, c) dA(\zeta) < \infty,$$

because h is meromorphic in $D(0, r)$. Since $\{a_n^1\}_{n=1}^\infty$ is a q_N -sequence for $f \in \mathcal{M}(\mathbb{D})$, so is its subsequence $\{a_{n_k}^1\}_{k=1}^\infty$. To complete the proof, it suffices to show that for each q_N -sequence $\{c_n\}_{n=1}^\infty$ for $f \in \mathcal{M}(\mathbb{D})$ there exists a subsequence $\{c_{n_j}\}_{j=1}^\infty$ such that

$$(2.3) \quad \lim_{j \rightarrow \infty} \int_{\Delta(c_{n_j}, \rho)} f^\#(z)^2 g^p(z, c_{n_j}) dA(z) = \infty$$

for each $\rho \in (0, 1)$ and $0 < p < \infty$. This for $\rho = \delta$ contradicts what just have been proved, and thus gives the assertion. To prove (2.3), we employ the method used in the proof of [1, Theorem 4]. Assume on the contrary that there exist $\rho \in (0, 1)$ and $0 < p < \infty$ such that

$$C = C(f, p, \{c_n\}) = \sup_{n \in \mathbb{N}} \int_{\Delta(c_n, \rho)} f^\#(z)^2 g^p(z, c_n) dA(z) < \infty.$$

Choose $r = r(f, p, \{c_n\}) \in (0, \rho)$ sufficiently small such that $2(-\log r)^{-p} C \leq \pi$. Then a change of variable gives

$$C \geq \int_{\Delta(c_n, r)} f^\#(z)^2 g^p(z, c_n) dA(z) \geq \left(\log \frac{1}{r}\right)^p \int_{D(0, r)} (f \circ \varphi_{c_n})^\#(z)^2 dA(z),$$

and hence $g_n = f \circ \varphi_{c_n}$ satisfies $g_n^\#(0) = f^\#(c_n)(1 - |c_n|^2) \leq 1/r$ by Dufresnoy's theorem [4, p. 83]. This is a contradiction, and hence the claimed subsequence exists. \square

Note that the argument used in the end of the proof gives an easy way to see that each \mathcal{N} -sequence for $f \in \mathcal{M}(\mathbb{D})$ is a $Q_p^\#$ -sequence for f .

3. Proof of Theorem 1.1

Let $\{a_n\}_{n=1}^\infty$ be a $Q_p^\#$ -sequence for f , and let $\delta \in (0, 1)$ be that of Lemma 2.1. Then either

$$(3.1) \quad \limsup_{n \rightarrow \infty} \int_{\mathbb{D} \setminus \Delta(a_n, \delta)} f^\#(z)^2 g^p(z, a_n) dA(z) = \infty$$

or

$$(3.2) \quad \limsup_{n \rightarrow \infty} \int_{\Delta(a_n, \delta)} f^\#(z)^2 g^p(z, a_n) dA(z) = \infty.$$

Assume first (3.1). The inequalities $1 - t \leq -\log t \leq \frac{1}{t}(1 - t)$, valid for all $0 < t \leq 1$, imply

$$g(z, a_n) \leq \frac{1}{\delta} (1 - |\varphi_{a_n}(z)|) \leq \frac{C_1}{\delta} (1 - |\varphi_{b_n}(z)|) \leq \frac{C_1}{\delta} g(z, b_n),$$

where $C_1 = C_1(\delta) > 0$ is a constant, for all $z \in \mathbb{D} \setminus \Delta(a_n, \delta)$ and $b_n \in \overline{\Delta(a_n, \delta)}$. It follows that

$$\begin{aligned} & \int_{\mathbb{D} \setminus \Delta(a_n, \delta)} f^\#(z)^2 g^p(z, a_n) dA(z) \\ & \leq \left(\frac{C_1}{\delta}\right)^p \int_{\mathbb{D} \setminus \Delta(a_n, \delta)} f^\#(z)^2 g^p(z, b_n) dA(z) \\ & \leq \left(\frac{C_1}{\delta}\right)^p \int_{\mathbb{D}} f^\#(z)^2 g^p(z, b_n) dA(z), \quad b_n \in \overline{\Delta(a_n, \delta)}, \end{aligned}$$

and hence every $\{b_n\}_{n=1}^\infty$ satisfying $\sigma(a_n, b_n) \leq \delta$ for all $n \in \mathbb{N}$ is a $Q_p^\#$ -sequence for f .

Assume now (3.2). Then there either exists a q_N -sequence $\{c_{n_k}\}_{k=1}^\infty$ satisfying $\sigma(a_{n_k}, c_{n_k}) \leq \delta/2$ for all $k \in \mathbb{N}$ or a constant $C_2 > 0$ such that $f^\#(z)(1 - |z|^2) \leq C_2$ for all $z \in \Delta(a_n, \delta/2)$ and all $n \in \mathbb{N}$. In the first case, $\{b_{n_k}\}_{k=1}^\infty$ is a $Q_p^\#$ -sequence for f by Theorem 2.1 if $\sigma(a_{n_k}, b_{n_k}) \leq \delta/2$ for all $k \in \mathbb{N}$ because then $\sigma(c_{n_k}, b_{n_k}) \leq \sigma(c_{n_k}, a_{n_k}) + \sigma(a_{n_k}, b_{n_k}) \leq \delta/2 + \delta/2 = \delta$. In the latter case, by a change of variable,

$$(3.3) \quad \int_{\Delta(a_n, \delta/2)} f^\#(z)^2 g^p(z, a_n) dA(z) \leq C_2^2 \int_{D(0, \delta/2)} \frac{\left(\log \frac{1}{|\zeta|}\right)^p}{(1 - |\zeta|^2)} dA(\zeta) < \infty, \quad n \in \mathbb{N}.$$

If now $\sigma(a_n, b_n) \leq \frac{1-\delta}{2}$ for all $n \in \mathbb{N}$, then $\Delta(a_n, \delta) \subset \Delta(b_n, \frac{1+\delta}{2})$ because $\sigma(z, b_n) \leq \sigma(z, a_n) + \sigma(a_n, b_n) \leq \delta + \frac{1-\delta}{2} = \frac{1+\delta}{2}$ for all $z \in \Delta(a_n, \delta)$. This observation together with (3.2) and (3.3) shows that

$$\begin{aligned} & \int_{\Delta(b_n, \frac{1+\delta}{2}) \setminus \Delta(a_n, \delta/2)} f^\#(z)^2 g^p(z, b_n) dA(z) \\ & \geq \left(\log \frac{2}{1+\delta}\right)^p \int_{\Delta(a_n, \delta) \setminus \Delta(a_n, \delta/2)} f^\#(z)^2 dA(z) \\ & \geq \left(\frac{\log \frac{2}{1+\delta}}{\log \frac{2}{\delta}}\right)^p \int_{\Delta(a_n, \delta) \setminus \Delta(a_n, \delta/2)} f^\#(z)^2 g^p(z, a_n) dA(z). \end{aligned}$$

Therefore

$$\limsup_{n \rightarrow \infty} \int_{\Delta(b_n, \frac{1+\delta}{2}) \setminus \Delta(a_n, \delta/2)} f^\#(z)^2 g^p(z, b_n) dA(z) = \infty,$$

and thus $\{b_n\}_{n=1}^\infty$ is a $Q_p^\#$ -sequence for f . \square

4. Proof of Corollary 1.2

The hypothesis (1.1) implies

$$\limsup_{n \rightarrow \infty} \int_{\Delta(a_n, r)} f^\#(z)^2 g^p(z, a_n) dA(z) = \infty.$$

If $r \leq \frac{1}{e}$, then $g^p(z, a_n) \leq g^{p'}(z, a_n)$ for all $z \in \Delta(a_n, r)$, and hence $\{a_n\}_{n=1}^{\infty}$ is a $Q_p^\#$ -sequence for f . The assertion now follows by Theorem 1.1. If $r > \frac{1}{e}$, then either

$$\limsup_{n \rightarrow \infty} \int_{\Delta(a_n, \frac{1}{e})} f^\#(z)^2 g^p(z, a_n) dA(z) = \infty$$

or

$$\limsup_{n \rightarrow \infty} \int_{\Delta(a_n, r) \setminus \Delta(a_n, \frac{1}{e})} f^\#(z)^2 g^p(z, a_n) dA(z) = \infty.$$

In the former case we may proceed as before, meanwhile in the latter case we have

$$g^p(z, a_n) \leq \frac{1}{|\varphi_{a_n}(z)|^p} (1 - |\varphi_{a_n}(z)|)^p \leq \frac{e^p}{(1-r)^{p'-p}} g^{p'}(z, a_n)$$

for all $z \in \Delta(a_n, r) \setminus \Delta(a_n, \frac{1}{e})$. Again the assertion follows by Theorem 1.1. \square

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