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ON OPERATORS T COMMUTING WITH CTC WHERE C IS A CONJUGATION

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Abstract. In this paper, we study the properties of T satisfying $[CTC, T] = 0$ for some conjugation C where $[R, S] := RS - SR$. In particular, we show that if T is normal, then $[CTC, C] = 0$. Moreover, the class of operators T satisfy $[CTC, T] = 0$ is norm closed. Finally, we prove that if T is complex symmetric, then T is binormal if and only if $[C|T|C, |T|] = 0$.

1. Introduction

Let $\mathcal{H}$ be a separable complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *normal* if $T^*T = TT^*$ and *binormal* if T^*T and TT^* commute where T^* is the adjoint of T.

A *conjugation* on $\mathcal{H}$ is an antilinear operator $C : \mathcal{H} \to \mathcal{H}$ which satisfies $\langle Cx, Cy\rangle = \langle y, x\rangle$ for all $x, y \in \mathcal{H}$ and $C^2 = I$. Given $T \in \mathcal{L}(\mathcal{H})$ and a conjugation C on $\mathcal{H}$, let $\mathcal{C}_C(T) := \{S \in \mathcal{L}(\mathcal{H}) \mid [CTC, S] = 0\}$ where $[R, S] := RS-SR$.

In this paper, we study the case when $T \in \mathcal{C}_C(T)$, i.e., $[CTC, T] = 0$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *complex symmetric* and *skew complex symmetric* if there exists a conjugation C such that $CTC = T^*$ and $CTC = -T^*$, respectively. In this case, we say that T is (skew) complex symmetric with a conjugation C. It is clear that if $T \in \mathcal{C}_C(T)$ is complex symmetric (or skew complex symmetric) with a conjugation C, then T is normal. Throughout the paper, we denote the spectrum and the approximate point spectrum of $T \in \mathcal{L}(\mathcal{H})$ by $\sigma(T)$ and $\sigma_a(T)$, respectively. For a set $F \subset \mathbb{C}$, let $F^* = \{\overline{z} : z \in F\}$.

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The following examples show that $\mathcal{C}_C(T)$ need not contain complex symmetric operators.

Example 1.1. Let $\mathcal{H} = \ell^2$, let $\{e_n\}$ be an orthonormal basis of $\mathcal{H}$ and let $C : \mathcal{H} \to \mathcal{H}$ be the conjugation given by $C(\sum_{n=0}^{\infty} x_n\, e_n) = \sum_{n=0}^{\infty} \overline{x_n}\, e_n$ where $\{x_n\}$ is a sequence in $\mathbb{C}$ with $\sum_{n=0}^{\infty} |x_n|^2 < \infty$ and $Ce_n = e_n$ for all n. If $W \in \mathcal{L}(\mathcal{H})$ is the weighted shift given by $We_n = \alpha_n e_{n+1}$ for all $n \geq 1$, then it is easy to compute $WCWCe_n = CWCWe_n$ for all n. Hence $W \in \mathcal{C}_C(W)$. In particular, if $\alpha_n = 1$ for all n, then $W = S$ is the unilateral shift and so $S \in \mathcal{C}_C(S)$. However, S is not complex symmetric.

Example 1.2. Let C and J be conjugations on $\mathcal{H}$. Assume that $T = \left(\begin{smallmatrix} 0 & CJ \\ I & 0 \end{smallmatrix}\right)$ and $\mathcal{J} = \left(\begin{smallmatrix} 0 & J \\ J & 0 \end{smallmatrix}\right)$ on $\mathcal{H} \oplus \mathcal{H}$. Then $\mathcal{J}T\mathcal{J}T = T\mathcal{J}T\mathcal{J} = \left(\begin{smallmatrix} I & 0 \\ 0 & I \end{smallmatrix}\right)$. Hence $T \in \mathcal{C}_{\mathcal{J}}(T)$ is normal.

Example 1.3. Let $\mathcal{H} = \mathbb{C}^n$ and $C(z_1, z_2, z_3, \dots, z_n) = (\overline{z_n}, \dots, \overline{z_3}, \overline{z_2}, \overline{z_1})$. If

$$T = \begin{pmatrix} 0 & \lambda_1 & 0 & & \dots & 0 \\ 0 & 0 & \lambda_2 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \dots & 0 \\ \vdots & \vdots & . & 0 & \ddots & 0 \\ . & . & . & . & 0 & \lambda_{n-1} \\ 0 & 0 & . & . & \dots & 0 \end{pmatrix} \quad \text{and} \quad e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \end{pmatrix}$$

for all nonzero $\lambda_j \in \mathbb{C}$, then $0 = (CTC)T\, e_1 \neq T(CTC)\, e_1 = \lambda_1 \cdot \overline{\lambda_{n-1}} \cdot e_1$. Hence $T \notin \mathcal{C}_C(T)$. But, it is clear that T is binormal.

From Example 1.3, we observe that there exists T such that $T \notin \mathcal{C}_C(T)$, in general.

The aim of this paper is to study some properties of an operator which satisfies $T \in \mathcal{C}_C(T)$ where C is a conjugation on $\mathcal{H}$. In particular, we prove that if T is normal, then $[CTC, C] = 0$. Moreover, the class of operators T satisfy $[CTC, T] = 0$ is norm closed. Finally, we show that if T is complex symmetric, then T is binormal if and only if $[C|T|C, |T|] = 0$.

2. Operators satisfying $T \in \mathcal{C}_C(T)$

In this section, we study several properties about operators which satisfy $T \in \mathcal{C}_C(T)$ where C is a conjugation on $\mathcal{H}$. Remark from [7] that if $T \in \mathcal{L}(\mathcal{H})$ is a complex symmetric operator with a conjugation C, then both $\operatorname{Re} T$ and $\operatorname{Im} T$ are complex symmetric operators with same conjugation C. In the following lemma, we consider the previous statement for operators which satisfy $T \in \mathcal{C}_C(T)$.

Lemma 2.1. *Let $T \in \mathcal{C}_C(T)$. Suppose that $R = \frac{T+CTC}{2}$ and $S = \frac{T-CTC}{2i}$. Then R and S belong to $\mathcal{C}_C(T)$ such that $T = R + iS$ and $[R, S] = 0$, $[R, C] = 0$, and $[S, C] = 0$ hold.*

Proof. Suppose that $T \in \mathcal{C}_C(T)$ for a conjugation C. Since $R = \frac{T+CTC}{2}$ and $S = \frac{T-CTC}{2i}$, we can easily see that $T = R + iS$ and $RS = SR$, $CRC = R$ and $CSC = S$ hold. □

Theorem 2.2. *If $T \in \mathcal{L}(\mathcal{H})$ is a normal operator, then T, T^*, $\mathrm{Re}\, T$, and $\mathrm{Im}\, T$ are in $\mathcal{C}_C(T)$ for some conjugation C.*

Proof. Assume that T is normal. Then T can be written in the form $U|T|$, where U may be taken to be unitary such that U and $|T|$ commute with each other by [6, Theorem 7, page 67]. Since U is a unitary operator, by Godič and Lucenko [10], there exist conjugations C and J such that $U = CJ$ and $(CJ)^* = JC$. On the other hand, since T is normal, it follows from [7] that T is complex symmetric. Thus $C|T| = |T|C$ and $J|T| = |T|J$ (see [8, Lemma 1 and Example 2] for more details). Therefore, it is easy to see $CTC\,T = T\,CTC$ by this conjugation C. Thus $T \in \mathcal{C}_C(T)$.

Put $\mathrm{Re}\, T := \frac{T+T^*}{2}$ and $\mathrm{Im}\, T := \frac{T-T^*}{2i}$. Since T is normal and $[CTC, T] = 0$, it follows from the Fuglede-Putnam Theorem that $T^*(CTC) = (CTC)T^*$, i.e., $[CTC, T^*] = 0$. Thus $T^* \in \mathcal{C}_C(T)$. Also, we get that

$$(\mathrm{Re}\ T)CTC = \frac{1}{2}(TCTC + T^*CTC) = \frac{1}{2}(CTCT + CTCT^*) = CTC(\mathrm{Re}T)$$

and

$$(\mathrm{Im}\ T)CTC = \frac{1}{2i}(TCTC - T^*CTC) = \frac{1}{2i}(CTCT - CTCT^*) = CTC(\mathrm{Im}T).$$

Hence $T, T^*, \mathrm{Re}\ T$, and $\mathrm{Im}\ T$ are in $\mathcal{C}_C(T)$ for the conjugation C. □

Remark 2.3. The converse of Theorem 2.2 does not hold.

Example 2.4. Let $\mathcal{H} = \mathbb{C}^2$ and let C be a conjugation on $\mathcal{H}$ given by $C(x, y) = (\overline{y}, \overline{x})$. Assume that $R = \begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix}$ on $\mathcal{H}$. Then $CRC = \begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix} = R$. Hence $R \in \mathcal{C}_C(R)$. However, R is not normal. We also note that $\mathrm{Re}\ T \notin \mathcal{C}_C(T)$ and $\mathrm{Im}\ T \notin \mathcal{C}_C(T)$.

Next, we state some basic properties of an operator $T \in \mathcal{C}_C(T)$.

Theorem 2.5. *Let C be a conjugation on $\mathcal{H}$. Then the following statements hold.*

(i) *If $T \in \mathcal{C}_C(T)$, then $f(T) \in \mathcal{C}_C(T)$ for every function f analytic on $\sigma(T)$.*
(ii) *If $T \in \mathcal{C}_C(T)$ is invertible, then $T^{-1} \in \mathcal{C}_C(T)$.*
(iii) *If $T_1, T_2 \in \mathcal{C}_C(T)$, then $T_1 + T_2, \alpha T_1, T_1T_2$, and T_2T_1 are in $\mathcal{C}_C(T)$ for any $\alpha \in \mathbb{C}$.*
(iv) *The class $\mathcal{C}_C(T)$ is closed in norm.*

Proof. (i) If $T \in \mathcal{C}_C(T)$, then $p(T) \in \mathcal{C}_C(T)$ for every polynomial p. If T is a function analytic on $\sigma(T)$, then there exists $\{p_n\}$, sequence of polynomials, such that $\{p_n\}$ converges uniformly to f on $\sigma(T)$. Since $p_n(T) \in \mathcal{C}_C(T)$, it follows that $f(T) \in \mathcal{C}_C(T)$.

(ii) Since $T \in \mathcal{C}_C(T)$ is invertible, it follows that

$$CTCT^{-1} = T^{-1}(TCTC)T^{-1} = T^{-1}(CTCT)T^{-1} = T^{-1}CTC.$$

Thus $T^{-1} \in \mathcal{C}_C(T)$.

(iii) Since $T_1, T_2 \in \mathcal{C}_C(T)$, we have $(T_1 + T_2)CTC = CTC(T_1 + T_2)$ and $T_1T_2(CTC) = T_1(CTC)T_2 = (CTC)T_1T_2$. Therefore $T_1 + T_2$ and T_1T_2 are in $\mathcal{C}_C(T)$. Similarly, T_2T_1 is in $\mathcal{C}_C(T)$.

(iv) If $\{S_n\}$ is a sequence of operators such that

$$S_n \in \mathcal{C}_C(T) \text{ and } \lim_{n\to\infty} \|S_n - T\| = 0,$$

then we obtain

$$\begin{aligned}\|TCTC - CTCT\| &\leq \|TCTC - S_nCTC\| + \|CTCS_n - CTCT\| \\ &\leq \|T - S_n\|\|CTC\| + 0 + \|CTC\|\|S_n - T\| \to 0 \text{ as } n \to \infty.\end{aligned}$$

Hence $T \in \mathcal{C}_C(T)$ and so the class $\mathcal{C}_C(T)$ is closed in norm. □

From Theorem 2.5, we observe that $\mathcal{C}_C(T)$ is a Banach space.

Corollary 2.6. *If $N \in \mathcal{L}(\mathcal{H})$ is normal, then $f(N) \in \mathcal{C}_C(N)$ for every function f analytic on $\sigma(N)$. In particular, if N is invertible, then $N^{-1} \in \mathcal{C}_C(N)$.*

Proof. The proof follows from Theorems 2.2 and 2.5. □

Proposition 2.7. *Let $T \in \mathcal{C}_C(T)$ for some conjugation C. Then the following statements hold.*

(i) *$T^* \in \mathcal{C}_C(T^*)$ and $T^{-1} \in \mathcal{C}_C(T^{-1})$ if T^{-1} exists.*
(ii) *If $X \in \mathcal{L}(\mathcal{H})$ is invertible with $[X, C] = 0$, then $X^{-1}TX \in \mathcal{C}_C(X^{-1}TX)$.*
(iii) *If $R \in \mathcal{L}(\mathcal{H})$ is unitarily equivalent to T, i.e., $R = UTU^*$ where U is unitary, then $R \in \mathcal{C}_D(R)$ for a conjugation $D = UCU^*$.*
(iv) *$[CT^nC, T^m] = 0$ for all $n, m \in \mathbb{N}$.*

Proof. (i) If $T \in \mathcal{C}_C(T)$, then it is clear that $T^* \in \mathcal{C}_C(T^*)$. If $T \in \mathcal{C}_C(T)$ is invertible, then $T(CTC) = (CTC)T$ implies

$$CT^{-1}CT^{-1} = [T(CTC)]^{-1} = [(CTC)T]^{-1} = T^{-1}CT^{-1}C.$$

(ii) If X is an invertible with $X = CXC$, then we obtain

$$\begin{aligned}C(X^{-1}TX)C(X^{-1}TX) &= CX^{-1}TXX^{-1}CTX \\ &= CX^{-1}TCTX = X^{-1}CTCTX \\ &= X^{-1}TCTCX = X^{-1}TXX^{-1}CTCX \\ &= X^{-1}TXCX^{-1}TCX = (X^{-1}TX)C(X^{-1}TX)C.\end{aligned}$$

Hence $X^{-1}TX \in \mathcal{C}_C(X^{-1}TX)$.

(iii) Since $[CTC, T] = 0$, $R = UTU^*$, and $D = UCU^*$, it follows that $[DRD, R] = U[CTC, T]U^* = 0$. Hence $R \in \mathcal{C}_D(R)$ for the conjugation D.

(iv) It is clear that $CTC\,T^2 = T^2\,CTC$ and $CT^2C\,T = T\,CT^2C$. Assume that $CT^kC\,T^j = T^j\,CT^kC$ for all $k \le n$ and $j \le m$. Then we have

$$CT^{n+1}C\,T^m = CTC\,CT^nC\,T^m = CTC\,T^m\,CT^nC = T^m\,CT^{n+1}C \tag{1}$$

and

$$CT^nC\,T^{m+1} = CT^nC\,T^m\,T = T^m\,CT^nC\,T = T^{m+1}\,CT^nC. \tag{2}$$

Since (1) and (2) hold for $n+1$ and $m+1$, it holds $CT^nC\,T^m = T^m\,CT^nC$ for every $n, m \in \mathbb{N}$. □

Let us recall that $\mathcal{H}_1 \otimes \mathcal{H}_2$ denotes the completion (endowed with a sensible uniform cross-norm) of the algebraic tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$ of $\mathcal{H}_1$ and $\mathcal{H}_2$ where $\mathcal{H}_1$ and $\mathcal{H}_2$ are separable complex Hilbert spaces. For operators $T \in \mathcal{L}(\mathcal{H}_1)$ and $S \in \mathcal{L}(\mathcal{H}_2)$, we define the *tensor product* operator $T \otimes S$ on $\mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ by

$$(T \otimes S)(\sum_{j=1}^{n} \alpha_j x_j \otimes y_j) = \sum_{j=1}^{n} \alpha_j T x_j \otimes S y_j.$$

Then it is well known that $T \otimes S \in \mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_2)$. The definition of $T \otimes S$ is extended from these finite linear combinations of simple tensors to the whole space. It was known from [3] that if C_1 and C_2 are conjugations on $\mathcal{H}$, we define $C_1 \otimes C_2$ on $\mathcal{H} \otimes \mathcal{H}$ by

$$(C_1 \otimes C_2)(\sum_{j=1}^{n} \alpha_j x_j \otimes y_j) = \sum_{j=1}^{n} \overline{\alpha_j} C_1 x_j \otimes C_2 y_j.$$

Then $C_1 \otimes C_2$ is a conjugation on $\mathcal{H} \otimes \mathcal{H}$.

We also observe the following easy consequences. So we omit its proof.

Proposition 2.8. *Let C, C_1, C_2 be conjugations on $\mathcal{H}$. Then the following statements hold.*

(i) *If $T_i \in \mathcal{C}_{C_i}(T_i)$ for conjugations C_i with $i = 1, 2$, respectively, then $T_1 \oplus T_2 \in \mathcal{C}_{C_1 \oplus C_2}(T_1 \oplus T_2)$ for a conjugation $C_1 \oplus C_2$.*

(ii) *Let $T \in \mathcal{C}_C(T)$ and $S \in \mathcal{C}_C(S)$. If $[T, S] = 0$ and $[CTC, S] = 0$, then $T + S \in \mathcal{C}_C(T + S)$ and $TS \in \mathcal{C}_C(TS)$ for a conjugation C.*

(iii) *If $T \in \mathcal{C}_{C_1}(T)$ and $S \in \mathcal{C}_{C_2}(S)$ for conjugations C_1 and C_2, respectively, then $T \otimes S \in \mathcal{C}_{C_1 \otimes C_2}(T \otimes S)$ for a conjugation $C_1 \otimes C_2$.*

For the next result, we need the following lemma.

Lemma 2.9 ([11, Lemma 3.21])**.** *Let $T \in \mathcal{L}(\mathcal{H})$ and let C be a conjugation on $\mathcal{H}$. Then $\sigma(CTC) = \sigma(T)^*$ and $\sigma_a(CTC) = \sigma_a(T)^*$.*

If T satisfies $CTC = T$, then $\sigma(T) = \sigma(T)^*$ from Lemma 2.9, that is, $\sigma(T)$ is a symmetric set with the real line. For a commuting pair $\mathbf{T} = (T_1, T_2) \in \mathcal{L}(\mathcal{H})^2$, $\sigma_T(T_1, T_2)$ (or $\sigma_T(\mathbf{T})$) and $\sigma_{ja}(T_1, T_2)$ (or $\sigma_{ja}(\mathbf{T})$) denote the *Taylor spectrum* and the *joint approximate point spectrum* of (T_1, T_2), respectively. We explain the Taylor spectrum for a commuting 2-tuple $\mathbf{T} = (T_1, T_2)$ case. Consider the following chain complex $E(\mathbf{T})$ as follows;

$$E(\mathbf{T}): \quad 0 \longrightarrow \mathcal{H} \xrightarrow{\delta^1_{\mathbf{T}}} \mathcal{H} \oplus \mathcal{H} \xrightarrow{\delta^2_{\mathbf{T}}} \mathcal{H} \longrightarrow 0,$$

where $\delta^1_{\mathbf{T}}(x) := (-T_2 x) \oplus (T_1 x)$ and $\delta^2_{\mathbf{T}}(x_1 \oplus x_2) := T_1 x_1 + T_2 x_2$. Then it is easy to see that $\delta^2_{\mathbf{T}} \circ \delta^1_{\mathbf{T}} = 0$. The commuting 2-tuple $\mathbf{T} = (T_1, T_2)$ is said to be *non-singular* if the chain complex $E(\mathbf{T})$ is exact, i.e., $\ker \delta^1_{\mathbf{T}} = \{0\}$, $\text{image}\, \delta^1_{\mathbf{T}} = \ker \delta^2_{\mathbf{T}}$ and $\text{image}\, \delta^2_{\mathbf{T}} = \mathcal{H}$. It is well known that $\mathbf{T}$ is non-singular if and only if

$$\alpha(\mathbf{T}) = \begin{pmatrix} T_1 & T_2 \\ -T_2^* & T_1^* \end{pmatrix}$$

is invertible on $\mathcal{H} \oplus \mathcal{H}$ (see [14]). For $\mathbf{z} = (z_1, z_2) \in \mathbb{C}^2$, let $\mathbf{T} - \mathbf{z} = (T_1 - z_1, T_2 - z_2)$. Then we define the Taylor spectrum $\sigma_T(\mathbf{T})$ of $\mathbf{T} = (T_1, T_2)$ as $\mathbf{z} = (z_1, z_2) \in \sigma_T(\mathbf{T})$ if the chain complex $E(\mathbf{T} - \mathbf{z})$ is not exact.

For a commuting 2-tuple $(T, S) \in \mathcal{L}(\mathcal{H})^{(2)}$, a number $\lambda = (\lambda_1, \lambda_2) \in \mathbb{C}^2$ is in the *joint approximate point spectrum* $\sigma_{ja}(T, S)$ if and only if there exists a sequence $\{x_n\}_n \subset \mathcal{H}$ such that $\|x_n\| = 1$ and

$$(T - \lambda_1)x_n \longrightarrow 0 \text{ and } (S - \lambda_2)x_n \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

It is well known $\sigma_{ja}(T, S) \subset \sigma_T(T, S)$ (see [1] and [13]).

Proposition 2.10. *Let $T \in \mathcal{C}_C(T)$. Then there exist commuting operators R and S such that the following statements hold*:

(i) *$T = R + iS$ and (T, R, S) is a commuting 3-tuple.*
(ii) *$\sigma(R)$ and $\sigma(S)$ are symmetric sets with the real line.*
(iii) *If $\lambda \in \sigma(T)$, then there exist $\alpha \in \sigma(R)$ and $\beta \in \sigma(S)$ such that $\lambda = \alpha + i\beta$.*
(iv) *If $\alpha \in \sigma(R)$, then there exist $\lambda \in \sigma(T)$ and $\beta \in \sigma(S)$ such that $\lambda = \alpha + i\beta$.*
(v) *If $\beta \in \sigma(S)$, then there exist $\lambda \in \sigma(T)$ and $\alpha \in \sigma(R)$ such that $\lambda = \alpha + i\beta$.*

Proof. The proofs of (i) and (ii) follow from Theorem 2.2 and Lemma 2.9.

(iii) Since (R, S) is a commuting pair and $T = R + iS$, the proof follows from the spectral mapping theorem for $f(a, b) = a + ib$ of the Taylor spectrum.

(iv) Since (T, S) is a commuting pair and $R = -T + iS$, the proof follows from the spectral mapping theorem for $g(a, b) = -a + ib$ of the Taylor spectrum.

(v) The proof follows from a similar method of (iv). □

Remark 2.11. The statements (iii), (iv) and (v) hold for the approximate point spectra $\sigma_a(T), \sigma_a(R)$ and $\sigma_a(S)$. Please see [1] for the spectral mapping theorem for the joint approximate point spectrum.

For an operator $T \in \mathcal{L}(\mathcal{H})$ and a conjugation C, we define the operator $\alpha_m(T;C)$ by

$$\alpha_m(T;C) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} CT^{m-j}C \cdot T^j.$$

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be an $[m,C]$-symmetric operator if $\alpha_m(T;C) = 0$. See [4] for properties of $[m,C]$-symmetric operators.

Proposition 2.12. *If $T \in \mathcal{C}_C(T)$ is an $[m,C]$-symmetric operator, then the following statements hold.*

(i) *$CTC - T$ is m-nilpotent, i.e., $(CTC - T)^m = 0$.*
(ii) *$\sigma_T(CTC, T) = \{(\lambda, \lambda) : \lambda \in \sigma(T)\}$. In this case, it holds $\sigma(CTC) = \sigma(T) = \sigma(T)^*$. Moreover, it holds $\sigma_{ja}(CTC, T) = \{(\lambda, \lambda) : \lambda \in \sigma_a(T)\}$.*

Proof. (i) Since T commutes with CTC, the proof follows that $0 = \alpha_m(T;C) = (CTC - T)^m$.

(ii) Since (CTC, T) is a commuting pair, by the spectral mapping theorem of the Taylor spectrum, it holds

$$f(\sigma_T(CTC, T)) = \sigma(CTC - T),$$

where $f(\mu, \lambda) = \mu - \lambda$. By Proposition 2.12, we have $\sigma(CTC - T) = \{0\}$ and hence $\mu = \lambda$. Thus $\sigma_T(CTC, T) = \{(\lambda, \lambda) : \lambda \in \sigma(T)\}$ and we have $\sigma(CTC) = \sigma(T) = \sigma(T)^*$ from [11]. Since $\sigma_{ja}(CTC, T) = \{(\lambda, \lambda) : \lambda \in \sigma_a(T)\}$, the proof follows from the spectral mapping theorem of the joint approximate point spectrum. □

For an operator $T \in \mathcal{L}(\mathcal{H})$, T is said to be *normaloid* if $r(T) = \|T\|$, where $r(T)$ is the spectral radius of T. Then we have the following corollary.

Corollary 2.13. *Let $T \in \mathcal{C}_C(T)$ be an $[m,C]$-symmetric operator. If $CTC - T$ is normaloid, then $CTC - T = 0$.*

Proof. By Proposition 2.12, we have $\sigma(CTC - T) = \{0\}$. Since $CTC - T$ is normaloid, it holds $CTC - T = 0$. □

For an operator $T \in \mathcal{L}(\mathcal{H})$ and a conjugation C, we define the operator $\lambda_m(T;C)$ by

$$\lambda_m(T;C) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} CT^{m-j}C \cdot T^{m-j}.$$

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be an $[m,C]$-isometric operator if $\lambda_m(T;C) = 0$. See [3] for properties of $[m,C]$-isometric operators.

Proposition 2.14. *If $T \in \mathcal{C}_C(T)$ is an $[m,C]$-isometric operator, then the following statements hold.*

(i) *$CTCT - I$ is m-nilpotent, i.e., $(CTCT - I)^m = 0$.*

(ii) $\sigma_T(CTC,T) = \{(\frac{1}{\lambda},\lambda) : \lambda \in \sigma(T)\}$. *In this case, it holds* $\sigma(CTC) = \{\frac{1}{\lambda} : \lambda \in \sigma(T)\}$. *Moreover, it holds* $\sigma_{ja}(CTC,T) = \{(\frac{1}{\lambda},\lambda) : \lambda \in \sigma_a(T)\}$.

Proof. (i) Since T commutes with CTC, we have $\lambda_m(T;C) = (CTCT - I)^m$. Hence, we have $(CTCT - I)^m = 0$.

(ii) Since (CTC,T) is a commuting pair, by the spectral mapping theorem of the Taylor spectrum, it holds

$$f(\sigma_T(CTC,T)) = \sigma(CTCT - I),$$

where $f(\mu,\lambda) = \mu\cdot\lambda - 1$. By Proposition 2.14(i), we have $\sigma(CTCT - I) = \{0\}$ and hence $\mu\cdot\lambda = 1$. Therefore, $\sigma_T(CTC,T) = \{(\frac{1}{\lambda},\lambda) : \lambda \in \sigma(T)\}$ and we have $\sigma(CTC) = \{\frac{1}{\lambda} : \lambda \in \sigma(T)\}$. By the same way, we get

$$\sigma_{ja}(CTC,T) = \{(\frac{1}{\lambda},\lambda) : \lambda \in \sigma_a(T)\}.$$ □

Finally, we focus on the binormalilty of T when $T \in \mathcal{C}_C(T)$ for a conjugation C on $\mathcal{H}$.

Lemma 2.15. *Let* $T \in \mathcal{L}(\mathcal{H})$ *and let* C *be a conjugation on* $\mathcal{H}$*. If* $(T^*T)C = C(TT^*)$*, then* T *is binormal if and only if* $|T| \in \mathcal{C}_C(|T|)$*.*

Proof. Let T be binormal. Then $|T^*||T| = |T||T^*|$. Since $(T^*T)C = C(TT^*)$, it follows that $|T^*| = C|T|C$. Therefore $|T|C|T|C = C|T|C|T|$. Thus $|T| \in \mathcal{C}(|T|)$.

Conversely, if $|T| \in \mathcal{C}(|T|)$, then $|T|C|T|C = C|T|C|T|$ implies $|T^*||T| = |T||T^*|$. Thus T is binormal. □

It is well known that normal operators are binormal. The *Duggal transform* $\widetilde{T}^D$ of T is given by $\widetilde{T}^D := |T|U$ where U is the appropriate partial isometry satisfying $\ker(U) = \ker(T)$ and $\ker(U^*) = \ker(T^*)$ (see [5]).

Theorem 2.16. *Let* $T \in \mathcal{L}(\mathcal{H})$ *be complex symmetric with a conjugation* C*. Suppose that* $T = U|T|$ *is the polar decomposition of* T *where* $U = CJ$ *and* J *is a partial conjugation supported on* $\overline{ran(|T|)}$*, which commutes with* $|T|$*. Then the following statements are equivalent.*

(i) T *is binormal.*

(ii) $|T| \in \mathcal{C}_C(|T|)$.

(iii) $[|\widetilde{T}^D|,|T|] = 0$ *where* $\widetilde{T}^D := |T|U$ *is the Duggal transform of* T*.*

Proof. (i) ⇔ (ii) Let $T = U|T|$ be the polar decomposition of T. By [8], $U = CJ$ where C and J are conjugations and J commutes with $|T|$. Since T is complex symmetric with the conjugation C, it follows that $(T^*T)C = |T|^2C = C(TT^*)$. Hence the proof follows from Lemma 2.15.

(i) ⇔ (iii) Let $\widetilde{T}^D := |T|U$ be the Duggal transform of T. If T is binormal, then $\widetilde{T}^D$ is binormal by [12] and so $[|\widetilde{T}^D|,|(\widetilde{T}^D)^*|] = 0$. Since T is complex symmetric with the conjugation C, it follows that $(T^*T)C = CTCTC =$

$C(TCTC) = C(TT^*)$ and so $[C, |T|] = 0$. In this case, since

$$|\widetilde{T}^D| = U^*|T|U = JC|T|CJ = J|T|J \text{ and } |(\widetilde{T}^D)^*| = (U^*|T|UU^*|T|)^{\frac{1}{2}} = |T|,$$

it follows that $[|\widetilde{T}^D|, |T|] = [|\widetilde{T}^D|, |(\widetilde{T}^D)^*|] = 0$. The converse statement follows by a similar way. □

As some applications of Theorem 2.16, we get the following corollary.

Corollary 2.17. *Let $T \in \mathcal{L}(\mathcal{H})$ be such that T^2 is normal. Then $|T| \in \mathcal{C}_C(|T|)$.*

Proof. By [9, Corollary 3], T is complex symmetric. Hence by [8, Theorem 2], there exist a conjugation C on $\mathcal{H}$ and a partial conjugation J supported on $\overline{ran|T|}$ such that $T = CJ|T|$ and $J|T| = |T|J$. On the other hand, since T^2 is normal, it follows from the Fuglede-Putnam Theorem that $(T^2)T^* = T^*(T^2)$. Hence

$$[T^*T, TT^*] = T^*TTT^* - TT^*T^*T = TTT^*T^* - TTT^*T^* = 0$$

and so T is binormal (also see [2]). Therefore, $[C|T|C, |T|] = 0$ for this conjugation C. □

Applying Theorem 2.16, we provide examples of complex symmetric operators which are binormal or non-binormal.

Example 2.18. Let $T = \left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)$ on $\mathbb{C}^2$. Then T is complex symmetric with the conjugation C defined by $C(z_1, z_2) = (\overline{z_2}, \overline{z_1})$ for $z_1, z_2 \in \mathbb{C}$. Since $|T| = \frac{1}{\sqrt{2}}\left(\begin{smallmatrix} 1 & 1 \\ 1 & 3 \end{smallmatrix}\right)$, it follows that

$$C|T|C|T| = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} \text{ and } |T|C|T|C = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}.$$

Hence T is not binormal by Theorem 2.16.

Example 2.19. Let $\mathcal{H} = \ell^2$ and let C be the canonical conjugation given by $C(\sum_{n=0}^{\infty} x_n e_n) = \sum_{n=0}^{\infty} \overline{x_n} e_n$ with $Ce_n = e_n$ for all n. Assume that $T = \left(\begin{smallmatrix} S^* & I \\ 0 & S \end{smallmatrix}\right)$ on $\mathcal{H} \oplus \mathcal{H}$, where $S \in \mathcal{L}(\mathcal{H})$ is the unilateral shift. Then S and S^* commute with the conjugation C. Denote the conjugation $\mathcal{C}$ given by $\mathcal{C} = \left(\begin{smallmatrix} 0 & C \\ C & 0 \end{smallmatrix}\right)$. Then we obtain that

$$\mathcal{C}T^* - T\mathcal{C} = \begin{pmatrix} C & CS^* \\ CS & 0 \end{pmatrix} - \begin{pmatrix} C & S^*C \\ SC & 0 \end{pmatrix} = 0.$$

Hence T is a complex symmetric operator (cf. [9]). Moreover, since $T = \left(\begin{smallmatrix} S^* & I \\ 0 & S \end{smallmatrix}\right)$, it follows that

$$T^*T = \begin{pmatrix} SS^* & S \\ S^* & 2I \end{pmatrix} \text{ and } TT^* = \begin{pmatrix} 2I & S^* \\ S & SS^* \end{pmatrix}.$$

Therefore we have

$$TT^*T^*T = \begin{pmatrix} 2SS^* + S^{*2} & 2S + 2S^* \\ S^2S^* + SS^{*2} & S^2 + 2SS^* \end{pmatrix}$$

and

$$T^*TTT^* = \begin{pmatrix} S^2 + 2SS^* & SS^{*2} + S^2S^* \\ 2S + 2S^* & S^{*2} + 2SS^* \end{pmatrix}.$$

Hence T is not binormal. On the other hand, if S is the unilateral shift on $\mathcal{H}$, then $T = S^* \oplus S$ is binormal and complex symmetric. Indeed, in this case, we have $T^*T = \left(\begin{smallmatrix} SS^* & 0 \\ 0 & I \end{smallmatrix}\right)$, $|T|\mathcal{C} = \left(\begin{smallmatrix} 0 & C \\ CSS^* & 0 \end{smallmatrix}\right)$, and $\mathcal{C}|T| = \left(\begin{smallmatrix} 0 & SS^*C \\ C & 0 \end{smallmatrix}\right)$. Hence $[|T|, \mathcal{C}|T|\mathcal{C}] = 0$ and so $T \in \mathcal{C}(|T|)$. Therefore T is binormal by Theorem 2.16.

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References

[1] J. W. Bunce, *Models for n-tuples of noncommuting operators*, J. Funct. Anal. **57** (1984), no. 1, 21–30. https://doi.org/10.1016/0022-1236(84)90098-3

[2] S. L. Campbell and C. D. Meyer, *EP operators and generalized inverses*, Canad. Math. Bull **18** (1975), no. 3, 327–333. https://doi.org/10.4153/CMB-1975-061-4

[3] M. Chō, J. E. Lee, and H. Motoyoshi, *On [m, C]-isometric operators*, Filomat **31** (2017), no. 7, 2073–2080. https://doi.org/10.2298/FIL1707073C

[4] M. Chō, J. E. Lee, K. Tanahashi, and J. Tomiyama, *On [m, C]-symmetric operators*, Kyungpook Math. J. **58** (2018), no. 4, 637–650. https://doi.org/10.5666/KMJ.2018.58.4.637

[5] C. Foiaş, I. B. Jung, E. Ko, and C. Pearcy, *Complete contractivity of maps associated with the Aluthge and Duggal transforms*, Pacific J. Math. **209** (2003), no. 2, 249–259. https://doi.org/10.2140/pjm.2003.209.249

[6] T. Furuta, *Invitation to Linear Operators*, Taylor & Francis, Ltd., London, 2001. https://doi.org/10.1201/b16820

[7] S. R. Garcia and M. Putinar, *Complex symmetric operators and applications*, Trans. Amer. Math. Soc. **358** (2006), no. 3, 1285–1315. https://doi.org/10.1090/S0002-9947-05-03742-6

[8] ______, *Complex symmetric operators and applications. II*, Trans. Amer. Math. Soc. **359** (2007), no. 8, 3913–3931. https://doi.org/10.1090/S0002-9947-07-04213-4

[9] S. R. Garcia and W. R. Wogen, *Some new classes of complex symmetric operators*, Trans. Amer. Math. Soc. **362** (2010), no. 11, 6065–6077. https://doi.org/10.1090/S0002-9947-2010-05068-8

[10] V. Godič and I. E. Lucenko, *On the representation of a unitary operator in the form of a product of two involutions*, Uspehi Mat. Nauk **20** (1965), no. 6 (126), 64–65.

[11] S. Jung, E. Ko, and J. E. Lee, *On complex symmetric operator matrices*, J. Math. Anal. Appl. **406** (2013), no. 2, 373–385. https://doi.org/10.1016/j.jmaa.2013.04.056

[12] M. Saji, *On the study of some problems in spectral sets, Duggal transformations and Aluthge transformations*, (Doctoral dissertation), Retrieved from Shodhganga, 2008.

[13] J. L. Taylor, *A joint spectrum for several commuting operators*, J. Functional Analysis **6** (1970), 172–191. https://doi.org/10.1016/0022-1236(70)90055-8

[14] F.-H. Vasilescu, *On pairs of commuting operators*, Studia Math. **62** (1978), no. 2, 203–207. https://doi.org/10.4064/sm-62-2-203-207

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