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A DOUBLE INTEGRAL CHARACTERIZATION OF A BERGMAN TYPE SPACE AND ITS MÖBIUS INVARIANT SUBSPACE

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Abstract. This paper shows that if $1 < p < \infty$, $\alpha \geq -n-2$, $\alpha > -1-\frac{p}{2}$ and f is holomorphic on the unit ball $\mathbb{B}_n$, then

$$\int_{\mathbb{B}_n} |Rf(z)|^p (1-|z|^2)^{p+\alpha} \mathrm{d}v_\alpha(z) < \infty$$

if and only if

$$\int_{\mathbb{B}_n}\int_{\mathbb{B}_n} \frac{|f(z)-f(w)|^p}{|1-\langle z,w\rangle|^{n+1+s+t-\alpha}}(1-|w|^2)^s(1-|z|^2)^t \mathrm{d}v(z)\mathrm{d}v(w) < \infty,$$

where $s, t > -1$ with $\min(s,t) > \alpha$.

1. Introduction

The purpose of this note is to give a double integral characterization of a Bergman space, which extends some previous results in [2–4].

Let $\mathbb{B}_n$ be the unit ball of the n-dimensional complex Euclidean space $\mathbb{C}^n$. Let $H(\mathbb{B}_n)$ be the space of holomorphic functions on $\mathbb{B}_n$. For $f \in H(\mathbb{B}_n)$, the radial derivative of f, denoted by Rf, is given by

$$Rf(z) = \sum_{j=1}^{n} z_j \frac{\partial f(z)}{\partial z_j}, \quad z = (z_1, z_2, \ldots, z_n) \in \mathbb{B}_n.$$

We employ the definition of Bergman spaces given in [9]. For $0 < p < \infty$ and $\alpha > -1-p$, the Bergman space $A^p_\alpha := A^p_\alpha(\mathbb{B}_n)$ consists of those holomorphic functions f in $\mathbb{B}_n$ with

(1) $$\|f\|_{A^p_\alpha} = |f(0)| + \|f\|_{\alpha,p}$$

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$$= |f(0)| + \left(\int_{\mathbb{B}_n} |Rf(z)|^p (1-|z|^2)^{p+\alpha} \mathrm{d}v(z)\right)^{\frac{1}{p}} < \infty,$$

where $\mathrm{d}v$ is the normalized volume measure on $\mathbb{B}_n$ so that $v(\mathbb{B}_n) = 1$. When $\alpha > -1$, it is well known that $f \in A^p_\alpha$ if and only if

$$\|f\|_{A^p_{\alpha,*}} = \left(\int_{\mathbb{B}_n} |f(z)|^p (1-|z|^2)^{\alpha} \mathrm{d}v_\alpha(z)\right)^{\frac{1}{p}} < \infty. \tag{2}$$

Here

$$\mathrm{d}v_\alpha(z) = c_\alpha (1-|z|^2)^\alpha \mathrm{d}v(z) = \frac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)} (1-|z|^2)^\alpha \mathrm{d}v(z).$$

When $\alpha = -(ps+1)$ (with $s < 1$), the spaces A^p_α are exactly the diagonal Besov spaces B^s_p. Moreover, $A^2_{-1} = H^2$, the Hardy space on $\mathbb{B}_n$. See [9] for more details of A^p_α.

It is proved in [3,4] that for $\alpha > -1$ and $f \in H(\mathbb{B}_n)$,

(1) (proved in [3]) if $p > n+1+\alpha$, then $f \in A^p_\alpha$ if and only if

$$\int_{\mathbb{B}_n}\int_{\mathbb{B}_n} \frac{|f(w)-f(z)|^p}{|1-\langle z,w\rangle|^p} \left((1-|z|^2)(1-|w|^2)\right)^{\frac{p+\alpha-n-1}{2}} \mathrm{d}v(z)\mathrm{d}v(w) < \infty;$$

(2) (proved in [4]) if $0 < p < n+1+\alpha$, then $f \in A^p_\alpha$ if and only if

$$\int_{\mathbb{B}_n}\int_{\mathbb{B}_n} \frac{|f(w)-f(z)|^p}{|1-\langle z,w\rangle|^p} \mathrm{d}v_\alpha(z)\mathrm{d}v_\alpha(w) < \infty.$$

The main result of this paper is the following:

Theorem 1.1. *Let $1 < p < \infty$, $\alpha \geq -n-2$, $\alpha > -1-\frac{p}{2}$ and let f be holomorphic on $\mathbb{B}_n$. Let $s,t > -1$ such that $\min(s,t) > \alpha$. Then $f \in A^p_\alpha$ if and only if*

$$\int_{\mathbb{B}_n}\int_{\mathbb{B}_n} \frac{|f(z)-f(w)|^p}{|1-\langle z,w\rangle|^{n+1+s+t-\alpha}} (1-|w|^2)^s (1-|z|^2)^t \mathrm{d}v(z)\mathrm{d}v(w) < \infty.$$

Remark 1.2. (1) It is proved in [2] that for $\alpha \geq -2$ and $f \in H(\mathbb{B}_n)$,

$$\int_{\mathbb{B}_n} |Rf(z)|^2 (1-|z|^2)^{2+\alpha} \mathrm{d}v(z)$$

is comparable to

$$\int_{\mathbb{B}_n}\int_{\mathbb{B}_n} \frac{|f(z)-f(w)|^2}{|1-\langle z,w\rangle|^{n+1+s+t-\alpha}} (1-|z|^2)^s (1-|w|^2)^t \mathrm{d}v(z)\mathrm{d}v(w),$$

where $s,t > -1$ with $\min\{s,t\} > \alpha$. The proof in [2] relies on Hilbert space techniques. It is worth to note that our result can not cover the result in [2] when $p = 2$ and $\alpha = -2$, which is contained in [2] but not contained in our theorem.

(2) If $\alpha = -n-1 > -1-\frac{p}{2}$, then $p > 2n$ and A^p_α is the Besov space B_p. It is shown in [10, Theorem 6.28] that if f is holomorphic in $\mathbb{B}_n$, then $f \in B_p$ if and only if

$$\int_{\mathbb{B}_n}\int_{\mathbb{B}_n} \frac{|f(z)-f(w)|^p \mathrm{d}v_t(z)\mathrm{d}v_t(w)}{|1-\langle z,w\rangle|^{2(n+1+t)}} < \infty$$

for $t > -1$. Thus Theorem 1.1 extends [10, Theorem 6.28].

(3) Let $s = t = \frac{p+\alpha-n-1}{2}$ in Theorem 1.1, we obtain the characterization given in [3]. Thus our characterization can be viewed as a generalization of [3]. However, the strategy of the verification is quiet different from [3].

The main idea of the proof of Theorem 1.1 is inspired by [1]. Similar characterization of the Dirichlet type space in the unit disk is given in [7].

Let X be a Banach space of holomorphic functions in $\mathbb{B}_n$. The Möbius invariant subspace of X is defined as

$$M_X = \{f \in X : \sup_{a\in\mathbb{B}_n} \|f\circ\varphi_a\| < \infty\},$$

where

$$\varphi_a(z) = \begin{cases} \dfrac{a - \frac{\langle z,a\rangle a}{|a|^2} - \sqrt{1-|a|^2}\,(z - \frac{\langle z,a\rangle a}{|a|^2})}{1-\langle z,a\rangle}, & \text{if } a \neq 0,\\ -z, & \text{if } a = 0,\end{cases}$$

for $z \in \mathbb{B}_n$. For $a \in \mathbb{B}_n$, the transform φ_a is usually call the involution on $\mathbb{B}_n$. Let $\mathrm{Aut}(\mathbb{B}_n)$ denotes the set of all automorphisms on $\mathbb{B}_n$. It is well known that every automorphism φ of $\mathbb{B}_n$ is of the form

$$\varphi = U\varphi_a = \varphi_b V,$$

where U and V are unitary transformations of $\mathbb{C}^n$, and φ_a and φ_b are involutions.

It is known that $M_{A^p_\alpha}$ is the Bloch space when $\alpha > -1$ and $p \geq 1$. $M_{A^2_{-1}} = BMOA$, the space of holomorphic functions of bounded mean oscillation.

We have the following theorem, which is motivated by [8, Theorem 2.5.2].

Theorem 1.3. *Let* $1 < p < \infty$, $\alpha \geq -n-2$, $\alpha > -1-\frac{p}{2}$ *and let* f *be holomorphic on* $\mathbb{B}_n$. *Let* $s,t > -1$ *such that* $\min(s,t) > \alpha$. *Then* $f \in M_{A^p_\alpha}$ *if and only if*

$$\sup_{a\in\mathbb{B}_n}\int_{\mathbb{B}_n}\int_{\mathbb{B}_n} \frac{|f(z)-f(w)|^p(1-|a|^2)^{n+1+\alpha}(1-|w|^2)^s(1-|z|^2)^t \mathrm{d}v(z)\mathrm{d}v(w)}{|1-\langle z,w\rangle|^{n+1+s+t-\alpha}|1-\langle a,w\rangle|^{n+1+s-t+\alpha}|1-\langle a,z\rangle|^{n+1+t-s+\alpha}} < \infty.$$

Notation. Throughout this paper, we only write $U \lesssim V$ (or $V \gtrsim U$) for $U \leq cV$ for a positive constant c, and moreover $U \approx V$ for both $U \lesssim V$ and $V \lesssim U$. $\square$

2. Preliminaries

For $f \in H(\mathbb{B}_n)$ with the homogeneous expansion

$$f(z) = \sum_{k=0}^{\infty} f_k(z),$$

it is easy to check that

$$Rf(z) = \sum_{k=1}^{\infty} k f_k(z), \quad Rf(0) = 0$$

and

$$f(z) - f(0) = \int_0^1 \frac{Rf(tz)}{t} \, \mathrm{d}t$$

for all $z \in \mathbb{B}_n$. According to [10, page 51], for $f \in A^p_\alpha$ with $\alpha > -1$, there exists a sufficiently large β satisfying

$$Rf(z) = \int_{\mathbb{B}_n} \frac{Rf(w)\mathrm{d}v_\beta(w)}{(1 - \langle z, w\rangle)^{n+1+\beta}}, \quad z \in \mathbb{B}_n,$$

where

$$\mathrm{d}v_\beta(z) = \frac{\Gamma(n+1+\beta)}{n!\Gamma(\beta+1)}(1 - |z|^2)^\beta \mathrm{d}v(z).$$

Then

$$Rf(z) = \int_{\mathbb{B}_n} Rf(w) \left(\frac{1}{(1 - \langle z, w\rangle)^{n+1+\beta}} - 1 \right) \mathrm{d}v_\beta(w)$$

since $Rf(0) = 0$. We have

$$f(z) - f(0) = \int_0^1 \frac{Rf(tz)}{t} \, \mathrm{d}t = \int_{\mathbb{B}_n} Rf(w) L(w, z) \mathrm{d}v_\beta(w),$$

where the kernel

$$L(z, w) = \int_0^1 \left(\frac{1}{(1 - t\langle z, w\rangle)^{n+1+\beta}} - 1 \right) \frac{\mathrm{d}t}{t}$$

satisfies

$$|L(z, w)| \lesssim \frac{1}{|1 - \langle z, w\rangle|^{n+\beta}} \tag{3}$$

for all z and w in $\mathbb{B}_n$. So

$$|f(z) - f(0)| \lesssim \int_{\mathbb{B}_n} \frac{|Rf(w)|}{|1 - \langle z, w\rangle|^{n+\beta}} \, \mathrm{d}v_\beta(w). \tag{4}$$

For a holomorphic function f in $\mathbb{B}_n$ we write

$$\nabla f(z) = \left(\frac{\partial f}{\partial z_1}(z), \ldots, \frac{\partial f}{\partial z_n}(z) \right)$$

and call $|\nabla f(z)|$ the holomorphic gradient of f at z. We define

$$\widetilde{\nabla} f(z) = \nabla(f \circ \varphi_z)(0),$$

and call $|\widetilde{\nabla} f(z)|$ the invariant gradient of f at z. It is shown in [10, page 49] that

$$|\widetilde{\nabla}(f \circ \varphi)(z)| = |(\widetilde{\nabla} f) \circ \varphi(z)|$$

for all f and $\varphi \in \mathrm{Aut}(\mathbb{B}_n)$. Moreover,

$$(1-|z|^2)|Rf(z)| \leq (1-|z|^2)|\nabla f(z)| \leq |\widetilde{\nabla} f(z)| \tag{5}$$

for all $z \in \mathbb{B}_n$ and $f \in H(\mathbb{B}_n)$.

Let

$$\mathrm{d}\lambda(z) = \frac{\mathrm{d}v(z)}{(1-|z|^2)^{n+1}}$$

be the Möbius invariant measure on $\mathbb{B}_n$. It is easy to check that

$$\int_{\mathbb{B}_n} f \circ \varphi(z) \mathrm{d}\lambda(z) = \int_{\mathbb{B}_n} f(z) \mathrm{d}\lambda(z), \tag{6}$$

where $\varphi \in \mathrm{Aut}(\mathbb{B}_n)$.

The following lemma is quoted from [10, Exercise 2.4]

Lemma 2.1. *Suppose $0 < p < \infty$, $\alpha > -\frac{p}{2} - 1$, and f is holomorphic in $\mathbb{B}_n$. Then $f \in A^p_\alpha$ if and only if*

$$\int_{\mathbb{B}_n} |\widetilde{\nabla} f(z)|^p (1-|z|^2)^\alpha \mathrm{d}v(z) < \infty.$$

The following lemma is quoted from [5], which is Lemma 2.5 there.

Lemma 2.2. *Suppose $s > -1$ and $r, t > 0$. If $t < s + n + 1 < r$, then*

$$\int_{\mathbb{B}_n} \frac{(1-|w|^2)^s \mathrm{d}v(w)}{|1-\langle z, w\rangle|^r |1-\langle \eta, w\rangle|^t} \lesssim \frac{1}{(1-|z|^2)^{r-s-n-1}|1-\langle \eta, z\rangle|^t}.$$

3. The derivative-free characterization for A^p_α

Inspired by Lemma 2.1 of [1], we have the following lemma.

Lemma 3.1. *Let $1 \leq p < \infty$, and let $\alpha > -1$ and $\beta \geq 0$ with $\beta < n + 1 + \alpha$. Let f be holomorphic on $\mathbb{B}_n$. Then*

$$\int_{\mathbb{B}_n} |f(z) - f(0)|^p \frac{(1-|z|^2)^\alpha}{|1-\langle z, w\rangle|^\beta} \, \mathrm{d}v(z) \lesssim \int_{\mathbb{B}_n} |Rf(z)|^p \frac{(1-|z|^2)^{\alpha+p}}{|1-\langle z, w\rangle|^\beta} \, \mathrm{d}v(z). \tag{7}$$

Proof. The case $\beta = 0$ is contained in [10, Theorem 2.16]. So we assume that $\beta > 0$.

If $p > 1$, choose $\varepsilon > 0$ with

$$\alpha - \varepsilon \max(1, p-1) > -1 \quad \text{and} \quad \beta + \varepsilon(p-1) < n + 1 + \alpha.$$

Without loss of generality we may assume that the right-hand side of (7) is finite. Then it follows from Hölder's inequality that $Rf \in A^1_{1+\alpha}$. Theorem

2.16 in [10] implies that $f \in A^1_\alpha$. Then it follows from (4) that there exists a sufficiently large t such that

$$\begin{aligned}
&|f(z)-f(0)|^p\\
\lesssim\ &\left(\int_{\mathbb{B}_n}|Rf(u)|\frac{(1-|u|^2)^t}{|1-\langle z,u\rangle|^{n+t}}\,\mathrm{d}v(u)\right)^p\\
\le\ &\left(\int_{\mathbb{B}_n}|Rf(u)|^p\frac{(1-|u|^2)^{t+(p-1)(1+\varepsilon)}}{|1-\langle z,u\rangle|^{n+t}}\,\mathrm{d}v(u)\right)\left(\int_{\mathbb{B}_n}\frac{(1-|u|^2)^{t-1-\varepsilon}}{|1-\langle z,u\rangle|^{n+t}}\,\mathrm{d}v(u)\right)^{p-1}\\
\lesssim\ &\left(\int_{\mathbb{B}_n}|Rf(u)|^p\frac{(1-|u|^2)^{t+(p-1)(1+\varepsilon)}}{|1-\langle z,u\rangle|^{n+t}}\,\mathrm{d}v(u)\right)(1-|z|^2)^{-\varepsilon(p-1)}.
\end{aligned}$$

Now the Fubini's theorem and Lemma 2.2 imply that

$$\begin{aligned}
&\int_{\mathbb{B}_n}|f(z)-f(0)|^p\frac{(1-|z|^2)^\alpha}{|1-\langle z,w\rangle|^\beta}\,\mathrm{d}v(z)\\
\lesssim\ &\int_{\mathbb{B}_n}\left(\int_{\mathbb{B}_n}|Rf(u)|^p\frac{(1-|u|^2)^{t+(p-1)(1+\varepsilon)}}{|1-\langle z,u\rangle|^{n+t}}\,\mathrm{d}v(u)\right)\frac{(1-|z|^2)^{-\varepsilon(p-1)+\alpha}}{|1-\langle z,w\rangle|^\beta}\,\mathrm{d}v(z)\\
=\ &\int_{\mathbb{B}_n}|Rf(u)|^p(1-|u|^2)^{t+(p-1)(1+\varepsilon)}\left(\int_{\mathbb{B}_n}\frac{(1-|z|^2)^{-\varepsilon(p-1)+\alpha}\,\mathrm{d}v(z)}{|1-\langle z,w\rangle|^\beta|1-\langle z,u\rangle|^{n+t}}\right)\mathrm{d}v(u)\\
\lesssim\ &\int_{\mathbb{B}_n}|Rf(u)|^p(1-|u|^2)^{t+(p-1)(1+\varepsilon)}\frac{\mathrm{d}v(u)}{(1-|u|^2)^{t-1-\alpha+\varepsilon(p-1)}|1-\langle u,w\rangle|^\beta}\\
=\ &\int_{\mathbb{B}_n}|Rf(u)|^p\frac{(1-|u|^2)^{p+\alpha}}{|1-\langle u,w\rangle|^\beta}\,\mathrm{d}v(u).
\end{aligned}$$

If $p=1$, by choosing γ big enough and applying (4), we have

$$\begin{aligned}
&\int_{\mathbb{B}_n}|f(z)-f(0)|\frac{(1-|z|^2)^\alpha}{|1-\langle z,w\rangle|^\beta}\,\mathrm{d}v(z)\\
\lesssim\ &\int_{\mathbb{B}_n}|Rf(u)|(1-|u|^2)^\gamma\left(\int_{\mathbb{B}_n}\frac{(1-|z|^2)^\alpha\,\mathrm{d}v(z)}{|1-\langle z,w\rangle|^\beta|1-\langle z,u\rangle|^{n+\gamma}}\right)\mathrm{d}v(u)\\
\lesssim\ &\int_{\mathbb{B}_n}|Rf(u)|(1-|u|^2)^\gamma\frac{\mathrm{d}v(u)}{(1-|u|^2)^{\gamma-1-\alpha}|1-\langle u,w\rangle|^\beta}\\
=\ &\int_{\mathbb{B}_n}|Rf(u)|^p\frac{(1-|u|^2)^{1+\alpha}}{|1-\langle u,w\rangle|^\beta}\,\mathrm{d}v(u).
\end{aligned}$$

The proof is completed. □

We can get the following corollary from Lemma 3.1 and (5).

Corollary 3.2. *Let $1\le p<\infty$, and let $\alpha>-1$ and $\beta\ge 0$ with $\beta<n+1+\alpha$. Let f be holomorphic on $\mathbb{B}_n$. Then we have*

$$\int_{\mathbb{B}_n}|f(z)-f(0)|^p\frac{(1-|z|^2)^\alpha}{|1-\langle z,w\rangle|^\beta}\,\mathrm{d}v(z)\lesssim\int_{\mathbb{B}_n}|\widetilde{\nabla}f(z)|^p\frac{(1-|z|^2)^\alpha}{|1-\langle z,w\rangle|^\beta}\,\mathrm{d}v(z). \tag{8}$$

It is also shown in [10, page 50] that if $f \in A^p_\alpha$ for $\alpha > -1$, then

$$|\widetilde{\nabla} f(z)|^p \lesssim (1-|z|^2)^{n+1+\beta} \int_{\mathbb{B}_n} \frac{|f(w)|^p \mathrm{d}v_\beta(w)}{|1-\langle z,w\rangle|^{2(n+1+\beta)}} \tag{9}$$

for $\beta > \alpha$. We have the following corollary.

Corollary 3.3. *If $p > 0$, $\alpha > -1$ and $f \in A^p_\alpha$, then for $\beta > \alpha$, we have*

$$\int_{\mathbb{B}_n} |\widetilde{\nabla} f(z)|^p \frac{(1-|z|^2)^\alpha}{|1-\langle z,w\rangle|^\beta} \,\mathrm{d}v(z) \lesssim \int_{\mathbb{B}_n} |f(z)-f(0)|^p \frac{(1-|z|^2)^\alpha}{|1-\langle z,w\rangle|^\beta} \,\mathrm{d}v(z). \tag{10}$$

In particular,

$$\int_{\mathbb{B}_n} |Rf(z)|^p \frac{(1-|z|^2)^{p+\alpha}}{|1-\langle z,w\rangle|^\beta} \,\mathrm{d}v(z) \lesssim \int_{\mathbb{B}_n} |f(z)-f(0)|^p \frac{(1-|z|^2)^\alpha}{|1-\langle z,w\rangle|^\beta} \,\mathrm{d}v(z). \tag{11}$$

Proof. We only need to verify (10), since (11) can be easily obtained from (5) and (10).

Replacing f by $f - f(0)$ in (9), we have

$$|\widetilde{\nabla} f(z)|^p \lesssim (1-|z|^2)^{n+1+\beta} \int_{\mathbb{B}_n} \frac{|f(w)-f(0)|^p \mathrm{d}v_\beta(w)}{|1-\langle z,w\rangle|^{2(n+1+\beta)}}.$$

This implies that

$$\begin{aligned}
&\int_{\mathbb{B}_n} |\widetilde{\nabla} f(z)|^p \frac{(1-|z|^2)^\alpha}{|1-\langle z,w\rangle|^\beta} \,\mathrm{d}v(z) \\
\lesssim &\int_{\mathbb{B}_n} \frac{(1-|z|^2)^{n+1+\beta+\alpha}}{|1-\langle z,w\rangle|^\beta} \int_{\mathbb{B}_n} \frac{|f(u)-f(0)|^p \mathrm{d}v_\beta(u)}{|1-\langle z,u\rangle|^{2(n+1+\beta)}} \,\mathrm{d}v(z) \\
= &\int_{\mathbb{B}_n} |f(u)-f(0)|^p \int_{\mathbb{B}_n} \frac{(1-|z|^2)^{n+1+\beta+\alpha} \,\mathrm{d}v(z)}{|1-\langle z,w\rangle|^\beta |1-\langle z,u\rangle|^{2(n+1+\beta)}} \,\mathrm{d}v_\beta(u) \\
\lesssim &\int_{\mathbb{B}_n} |f(u)-f(0)|^p \frac{1}{|1-\langle u,w\rangle|^\beta (1-|u|^2)^{\beta-\alpha}} \,\mathrm{d}v_\beta(u) \\
= &\int_{\mathbb{B}_n} |f(u)-f(0)|^p \frac{(1-|u|^2)^\alpha}{|1-\langle u,w\rangle|^\beta} \,\mathrm{d}v(u).
\end{aligned}$$

The proof is completed. □

Theorem 3.4. *Let $1 \le p < \infty$, $\alpha > -1-p$, $\alpha \ge -n-2$, and let f be holomorphic on $\mathbb{B}_n$. Let $s, t > -1$ such that $\min(s,t) > \alpha$. Then*

$$\int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{|f(z)-f(w)|^p}{|1-\langle z,w\rangle|^{n+1+s+t-\alpha}} (1-|w|^2)^s (1-|z|^2)^t \mathrm{d}v(z) \mathrm{d}v(w)$$

is comparable to

$$\int_{\mathbb{B}_n} |\widetilde{\nabla} f(z)|^p (1-|z|^2)^\alpha \mathrm{d}v(z).$$

Proof. It is suffices to consider the case $s = t$ since $s \le t$ implies

$$\frac{2^{s-t}(1-|z|^2)^t(1-|w|^2)^t}{|1-\langle z,w\rangle|^{n+1+2t-\alpha}} \le \frac{(1-|z|^2)^s(1-|w|^2)^t}{|1-\langle z,w\rangle|^{n+1+s+t-\alpha}} \le \frac{(1-|z|^2)^s(1-|w|^2)^s}{2^{s-t}|1-\langle z,w\rangle|^{n+1+2s-\alpha}}.$$

Let $\zeta = \varphi_w(z)$ and recall that

$$1-\langle\varphi_w(\zeta),\varphi_w(a)\rangle = \frac{(1-|w|^2)(1-\langle\zeta,a\rangle)}{(1-\langle\zeta,w\rangle)(1-\langle w,a\rangle)}, \quad w,a,\zeta\in\mathbb{B}_n. \tag{12}$$

It follows from Corollary 3.2 that

$$\begin{aligned}
&\int_{\mathbb{B}_n}\int_{\mathbb{B}_n}\frac{|f(z)-f(w)|^p}{|1-\langle z,w\rangle|^{n+1+2s-\alpha}}(1-|w|^2)^s(1-|z|^2)^s\mathrm{d}v(z)\mathrm{d}v(w)\\
=&\int_{\mathbb{B}_n}\int_{\mathbb{B}_n}\frac{|f(\varphi_w(\zeta))-f(\varphi_w(0))|^p(1-|\varphi_w(\zeta)|^2)^s}{|1-\langle\varphi_w(\zeta),\varphi_w(0)\rangle|^{n+1+2s-\alpha}(1-|w|^2)^{-s}}\,\mathrm{d}v(\varphi_w(\zeta))\mathrm{d}v(w)\\
=&\int_{\mathbb{B}_n}(1-|w|^2)^\alpha\int_{\mathbb{B}_n}\frac{|f(\varphi_w(\zeta))-f(\varphi_w(0))|^p(1-|\zeta|^2)^s}{|1-\langle\zeta,w\rangle|^{n+1+\alpha}}\,\mathrm{d}v(\zeta)\mathrm{d}v(w)\\
\lesssim&\int_{\mathbb{B}_n}(1-|w|^2)^\alpha\int_{\mathbb{B}_n}|\widetilde{\nabla}(f\circ\varphi_w)(\zeta)|^p\frac{(1-|\zeta|^2)^s}{|1-\langle\zeta,w\rangle|^{n+1+\alpha}}\,\mathrm{d}v(\zeta)\mathrm{d}v(w)\\
=&\int_{\mathbb{B}_n}(1-|w|^2)^\alpha\int_{\mathbb{B}_n}|\widetilde{\nabla}(f)\circ\varphi_w(\zeta)|^p\frac{(1-|\zeta|^2)^{s+n+1}}{|1-\langle\zeta,w\rangle|^{n+1+\alpha}}\,\mathrm{d}\lambda(\zeta)\mathrm{d}v(w)\\
=&\int_{\mathbb{B}_n}(1-|w|^2)^\alpha\int_{\mathbb{B}_n}|\widetilde{\nabla}(f)(z)|^p\frac{(1-|\varphi_w(z)|^2)^{s+n+1}}{|1-\langle\varphi_w(z),w\rangle|^{n+1+\alpha}}\,\mathrm{d}\lambda(z)\mathrm{d}v(w)\\
=&\int_{\mathbb{B}_n}(1-|w|^2)^s\int_{\mathbb{B}_n}|\widetilde{\nabla}(f)(z)|^p\frac{(1-|z|^2)^s}{|1-\langle z,w\rangle|^{n+1+2s-\alpha}}\,\mathrm{d}v(z)\mathrm{d}v(w)\\
=&\int_{\mathbb{B}_n}|\widetilde{\nabla}(f)(z)|^p(1-|z|^2)^s\int_{\mathbb{B}_n}\frac{(1-|w|^2)^s}{|1-\langle z,w\rangle|^{n+1+2s-\alpha}}\,\mathrm{d}v(w)\mathrm{d}v(z)\\
=&\int_{\mathbb{B}_n}|\widetilde{\nabla}(f)(z)|^p(1-|z|^2)^\alpha\mathrm{d}v(z).
\end{aligned}$$

For the converse direction, recall that if $f\in H(\mathbb{B}_n)$, then (see [10, page 87])

$$|\widetilde{\nabla}(f)(z)|^p \lesssim \int_{\mathbb{B}_n}|f\circ\varphi_z(w)-f(z)|^p\mathrm{d}v_\beta(w) \tag{13}$$

for all $z\in\mathbb{B}_n$ and $\beta>-1$. Choose $\beta = 1+s$ and let $w=\varphi_z(\eta)$ in (13) we get

$$\begin{aligned}
&\int_{\mathbb{B}_n}|\widetilde{\nabla}(f)(z)|^p\mathrm{d}v_\alpha(z) \lesssim \int_{\mathbb{B}_n}\int_{\mathbb{B}_n}|f\circ\varphi_z(w)-f(z)|^p\mathrm{d}v_\beta(w)\mathrm{d}v_\alpha(z)\\
=&\int_{\mathbb{B}_n}\int_{\mathbb{B}_n}|f(\eta)-f(z)|^p(1-|\varphi_z(\eta)|^2)^{\beta+n+1}\mathrm{d}\lambda(\varphi_z(\eta))\mathrm{d}v_\alpha(z)\\
=&\int_{\mathbb{B}_n}\int_{\mathbb{B}_n}|f(\eta)-f(z)|^p\frac{(1-|z|^2)^{\beta+n+1}(1-|\eta|^2)^{\beta+n+1}}{|1-\langle z,\eta\rangle|^{2(\beta+n+1)}}\mathrm{d}\lambda(\eta)\mathrm{d}v_\alpha(z)
\end{aligned}$$

$$= \int_{\mathbb{B}_n}\int_{\mathbb{B}_n} \frac{|f(\eta)-f(z)|^p(1-|z|^2)^s(1-|\eta|^2)^s}{|1-\langle z,\eta\rangle|^{n+1+2s-\alpha}}\frac{(1-|z|^2)^{\alpha+n+2}(1-|\eta|^2)}{|1-\langle z,\eta\rangle|^{n+3+\alpha}}\,\mathrm{d}v(\eta)\mathrm{d}v(z)$$
$$\lesssim \int_{\mathbb{B}_n}\int_{\mathbb{B}_n} \frac{|f(\eta)-f(z)|^p(1-|z|^2)^s(1-|\eta|^2)^s}{|1-\langle z,\eta\rangle|^{n+1+2s-\alpha}}\,\mathrm{d}v(\eta)\mathrm{d}v(z),$$

where the last inequality follows from the assumption $\alpha \geq -n-2$, which implies that

$$\frac{(1-|z|^2)^{\alpha+n+2}(1-|\eta|^2)}{|1-\langle z,\eta\rangle|^{n+3+\alpha}} \lesssim 1.$$

This gives the desired result. □

Proof of Theorem 1.1. Put Theorem 3.4 and Lemma 2.1 together, we get Theorem 1.1. □

A slight modification of the proof of Theorem 3.4 can give the following corollary.

Corollary 3.5. (1) *Let $1 \leq p < \infty$ and f be holomorphic on $\mathbb{B}_n$. Let $s, t > -1$, and $\gamma > t+n+1$. Then*

$$\int_{\mathbb{B}_n}\int_{\mathbb{B}_n} \frac{|f(z)-f(w)|^p}{|1-\langle z,w\rangle|^\gamma}(1-|w|^2)^s(1-|z|^2)^t\mathrm{d}v(z)\mathrm{d}v(w)$$
$$\lesssim \begin{cases} \int_{\mathbb{B}_n}|\widetilde{\nabla}f(z)|^p(1-|z|^2)^{t+s+n+1-\gamma}\mathrm{d}v(z), & \text{if } \gamma > n+1+s;\\ \int_{\mathbb{B}_n}|\widetilde{\nabla}f(z)|^p(1-|z|^2)^t\log\frac{1}{1-|z|^2}\,\mathrm{d}v(z), & \text{if } \gamma = n+1+s;\\ \int_{\mathbb{B}_n}|\widetilde{\nabla}f(z)|^p(1-|z|^2)^t\,\mathrm{d}v(z), & \text{if } \gamma < n+1+s.\end{cases}$$

(2) *Let $0 < p < \infty$ and f be holomorphic on $\mathbb{B}_n$. Let $\beta > -1$ and α is real. Then*

$$\int_{\mathbb{B}_n}|\widetilde{\nabla}f(z)|^p(1-|z|^2)^\alpha\mathrm{d}v(z)$$
$$\lesssim \int_{\mathbb{B}_n}\int_{\mathbb{B}_n}|f(z)-f(w)|^p\frac{(1-|z|^2)^{\alpha+\beta+n+1}(1-|w|^2)^\beta}{|1-\langle z,w\rangle|^{2(\beta+n+1)}}\,\mathrm{d}v(z)\mathrm{d}v(w)$$
$$\lesssim \int_{\mathbb{B}_n}\int_{\mathbb{B}_n}\frac{|f(z)-f(w)|^p}{|1-\langle z,w\rangle|^\gamma}(1-|z|^2)^t(1-|w|^2)^s\mathrm{d}v(z)\mathrm{d}v(w)$$

for all $s \leq \beta$, $t \leq \alpha+\beta+n+1$ and $\gamma \geq n+1+t+s-\alpha$.

4. The Möbius invariant subspace of A^p_α

Proof of Theorem 1.3. For $a \in \mathbb{B}_n$, an easily application of (12) and (6) gives that

$$\int_{\mathbb{B}_n}\int_{\mathbb{B}_n}\frac{|f\circ\varphi_a(z)-f\circ\varphi_a(w)|^p}{|1-\langle z,w\rangle|^{n+1+s+t-\alpha}}(1-|w|^2)^s(1-|z|^2)^t\mathrm{d}v(z)\mathrm{d}v(w)$$
$$= \int_{\mathbb{B}_n}\int_{\mathbb{B}_n}\frac{|f(u)-f(\xi)|^p(1-|\varphi_a(\xi)|^2)^{s+n+1}(1-|\varphi_a(u)|^2)^{t+n+1}}{|1-\langle\varphi_a(u),\varphi_a(\xi)\rangle|^{n+1+s+t-\alpha}}\,\mathrm{d}\lambda(u)\mathrm{d}\lambda(\xi)$$

$$= \int\limits_{\mathbb{B}_n}\int\limits_{\mathbb{B}_n} \frac{|f(u)-f(\xi)|^p(1-|a|^2)^{n+1+\alpha}(1-|\xi|^2)^{n+1+s}(1-|u|^2)^{n+1+t}\,\mathrm{d}\lambda(u)\mathrm{d}\lambda(\xi)}{|1-\langle u,\xi\rangle|^{n+1+s+t-\alpha}|1-\langle a,\xi\rangle|^{n+1+\alpha+s-t}|1-\langle a,u\rangle|^{n+1+\alpha+t-s}}$$

$$= \int\limits_{\mathbb{B}_n}\int\limits_{\mathbb{B}_n} \frac{|f(u)-f(\xi)|^p(1-|a|^2)^{n+1+\alpha}(1-|\xi|^2)^{s}(1-|u|^2)^{t}\,\mathrm{d}v(u)\mathrm{d}v(\xi)}{|1-\langle u,\xi\rangle|^{n+1+s+t-\alpha}|1-\langle a,\xi\rangle|^{n+1+\alpha+s-t}|1-\langle a,u\rangle|^{n+1+\alpha+t-s}}.$$

The proof is completed by taking supremum over $a \in \mathbb{B}_n$. □

The holomorphic function spaces Q_s on the unit ball is introduced in [6]. For $s > 0$, Q_s is defined by

$$Q_s = \left\{ f \in H(\mathbb{B}_n) : \sup_{a\in\mathbb{B}_n} \int_{\mathbb{B}_n} \left|\widetilde{\nabla} f(z)\right|^2 G(z,a)^s \mathrm{d}\lambda(z) < \infty \right\},$$

where $G(z,a)$ is invariant Green's function of $\mathbb{B}_n$ defined by $G(z,a) = g(\varphi_a(z))$, and

$$g(z) = \frac{n+1}{2n} \int_{|z|}^1 (1-t^2)^{n-1} t^{-2n+1} \mathrm{d}t.$$

It is shown in [6, Proposition 3.4] that when $0 < s \leq 1$, a holomorphic function f on $\mathbb{B}_n$ is belonging to Q_s if and only if

$$\sup_{a\in\mathbb{B}_n} \int_{\mathbb{B}_n} |\widetilde{\nabla} f(z)|^2 (1-|\varphi_a(z)|^2)^{ns} \mathrm{d}\lambda(z) < \infty.$$

Moreover, when $1 < s < \frac{n}{n-1}$, $f \in Q_s$ if and only if f is in the Bloch space, or equivalently,

$$\sup_{a\in\mathbb{B}_n} \int_{\mathbb{B}_n} |\widetilde{\nabla} f(z)|^2 (1-|\varphi_a(z)|^2)^{nq} \mathrm{d}\lambda(z) < \infty$$

for all $q > 1$. Therefore, when $0 < s \leq 1$, Q_s can be viewed as the Möbius invariant subspace of a holomorphic function space on $\mathbb{B}_n$ satisfying

$$\int_{\mathbb{B}_n} |\widetilde{\nabla} f(z)|^2 (1-|z|^2)^{ns} \mathrm{d}\lambda(z) < \infty.$$

On the other hand, it is well known that Q_s is nontrivial (i.e., contains all polynomials) if and only if $\frac{n-1}{n} < s < \frac{n}{n-1}$. Thus, we have the following characterization of Q_s on $\mathbb{B}_n$.

Corollary 4.1. *Let $\frac{n-1}{n} < s < \frac{n}{n-1}$ and let f be holomorphic on $\mathbb{B}_n$. Let $\beta, \gamma > -1$ such that $\min(\beta,\gamma) > ns - n - 1$. Then $f \in Q_s$ if and only if*

$$\sup_{a\in\mathbb{B}_n} \int\limits_{\mathbb{B}_n}\int\limits_{\mathbb{B}_n} \frac{|f(z)-f(w)|^2(1-|a|^2)^{ns}(1-|w|^2)^{\beta}(1-|z|^2)^{\gamma}\mathrm{d}v(z)\mathrm{d}v(w)}{|1-\langle z,w\rangle|^{2(n+1)+\beta+\gamma-ns}|1-\langle a,w\rangle|^{ns+\beta-\gamma}|1-\langle a,z\rangle|^{ns-\beta+\gamma}} < \infty.$$

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