Bull. Korean Math. Soc. **0** (0), No. 0, pp. 1–0
https://doi.org/10.4134/BKMS.b190049
pISSN: 1015-8634 / eISSN: 2234-3016

τ_w-LOEWY MODULES AND THEIR APPLICATIONS

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Abstract. In this paper, we study a theory for the structure of τ_w-Loewy series of modules over commutative rings, where τ_w is the hereditary torsion theory induced by the so-called w-operation, and explore the relationship between τ_w-Loewy modules and w-Artinian modules.

1. Introduction

The concept of a Loewy series has received much attention in the literature since this concept was introduced by Krull ([5]). For example, Facchini gave a characterization of Artinian modules in terms of Loewy invariants ([3, Theorem 2.7]). On the other hand, in [2] the authors generalized the concept of the socle of a module by replacing simples with τ-simple modules for a hereditary torsion theory τ, and concerned with the τ-Loewy series to apply to the notions of τ-semiArtinian rings and modules. However, the concept of τ-Loewy series has not fully investigated for results corresponding to the Loewy series due to lack of ideal-theoretic methods and localization techniques (for example, see Theorem 3.9, Corollary 5.13, and Theorem 5.14). Since the concept of semi-divisorial modules, which generalizes both divisorial modules and injective modules, was introduced by Glaz and Vasconcelos ([4]) and was modified to allow the semi-divisorial closure (or w-closure) by the first author, the so-called w-operation has proved to be useful in the study of multiplicative ideal theory and module theory. Therefore, the objective of this paper is to study a theory for the structure of τ_w-Loewy series of modules over commutative rings, where τ_w is the hereditary torsion theory induced by the w-operation, and to explore the relationship between τ_w-Loewy modules and w-Artinian modules.

Throughout this paper, we assume that R is a commutative ring with identity 1. Let J be an ideal of R. Following [12], J is called a *Glaz-Vasconcelos ideal* (a GV*-ideal* for short) if J is finitely generated and the natural homomorphism $\varphi : R \to J^* = \mathrm{Hom}_R(J, R)$ is an isomorphism. Let M be an R-module,

Received January 13, 2019; Revised February 17, 2019; Accepted February 28, 2019.
2010 *Mathematics Subject Classification.* 13E10, 13D30.
Key words and phrases. w-operation, w-Artiniam module, w-Loewy module, w-simple module.

and define

$$\mathrm{tor}_{\mathrm{GV}}(M) = \{x \in M \mid Jx = 0 \text{ for some } J \in \mathrm{GV}(R)\},$$

where $\mathrm{GV}(R)$ is the set of GV-ideals of R. It is clear that $\mathrm{tor}_{\mathrm{GV}}(M)$ is a submodule of M. Now, M is said to be GV-*torsion* (resp., GV-*torsion-free*) if $\mathrm{tor}_{\mathrm{GV}}(M) = M$ (resp., $\mathrm{tor}_{\mathrm{GV}}(M) = 0$). A GV-torsion-free module M is called a *w-module* if $\mathrm{Ext}^1_R(R/J, M) = 0$ for any $J \in \mathrm{GV}(R)$. Projective modules and reflexive modules are w-modules. In the recent paper [15], it was shown that flat modules are w-modules. The notion of w-modules was introduced firstly over a domain [11] in the study of strong Mori domains, and was extended to commutative rings with zero divisors in [12]. Let w-$\mathrm{Max}(R)$ denote the set of maximal w-ideals of R, i.e., w-ideals of R maximal among proper integral w-ideals of R. Following [12, Proposition 3.8], every maximal w-ideal is prime.

For any GV-torsion-free module M,

$$M_w := \{x \in E(M) \mid Jx \subseteq M \text{ for some } J \in \mathrm{GV}(R)\}$$

is a w-submodule of $E(M)$ containing M and is called the *w-envelope* of M, where $E(M)$ denotes the injective hull of M. It is clear that a GV-torsion-free module M is a w-module if and only if $M_w = M$.

Note that the concept of GV-torsion modules (resp., GV-torsion-free modules, w-envelopes, w-modules) is just that of τ_w-torsion modules (resp., τ_w-torsion-free modules, τ_w-injective envelopes, τ_w-closed modules) for the hereditary torsion theory τ_w induced by a (Gabriel) topology

$$\mathcal{F} := \{I \mid I \text{ is an ideal of } R \text{ with } I_w = R\}.$$

For any undefined terminology and notation we refer to [1, 10].

2. τ_w-simple modules

First we recall some terminology and notation in [1]. An R-module M is said to be *τ_w-simple* if M is GV-torsion-free and for any nonzero submodule N of M, M/N is GV-torsion. A submodule N of M is called *τ_w-pure* (resp., *τ_w-dense*) in M if M/N is GV-torsion-free (resp., GV-torsion). Obviously if N is a τ_w-dense submodule of a GV-torsion-free R-module M, then $N_w = M_w$. Set $C^M_{\tau_w}(N) := \{x \in M \mid (N :_R x)_w = R\}$, which is called the *$\tau_w$-closure* of N in M. Then N is called *τ_w-closed* in M if $C^M_{\tau_w}(N) = N$. The set of all τ_w-closed submodules of M is denoted by $\mathscr{C}_{\tau_w}(M)$. It is easy to verify that if M is GV-torsion-free, then $C^M_{\tau_w}(N) = N_w \cap M$; N is τ_w-dense in M if and only if $C^M_{\tau_w}(N) = M$; τ_w-closed submodules of M and its τ_w-pure submodules coincide.

Lemma 2.1. *Let M be a* GV*-torsion-free R-module.*

(1) *If M is a w-module, then τ_w-pure submodules and w-submodules of M coincide.*
(2) *The following statements are equivalent.*

(a) *M is τ_w-simple.*
(b) $\mathscr{C}_{\tau_w}(M) = \{0, M\}$.
(c) $C^M_{\tau_w}(N) = M$ *for any nonzero submodule N of M.*

Proof. (1) Let N be a submodule of M. Then the sequence

$$\mathrm{Hom}_R(R/J, M/N) \to \mathrm{Ext}^1_R(R/J, N) \to \mathrm{Ext}^1_R(R/J, M)$$

is exact for any $J \in \mathrm{GV}(R)$. Note that if N is a τ_w-pure submodule of M, then M/N is GV-torsion-free. So $\mathrm{Hom}_R(R/J, M/N) = 0$. Obviously $\mathrm{Ext}^1_R(R/J, M) = 0$ because M is a w-module. Thus $\mathrm{Ext}^1_R(R/J, N)$=0, and so N is a w-module. Conversely, if N is a w-module, then it is easy to prove that M/N is GV-torsion-free. Thus N is a τ_w-pure submodule of M.

(2) (a) $\Rightarrow$ (b) Assume that M is τ_w-simple. Then for any nonzero proper submodule N of M, $N_w = M_w$. Thus $N_w \cap M = M_w \cap M = M \neq N$. So N is not τ_w-closed in M. Obviously, $C^M_{\tau_w}(0) = 0$. Then $\mathscr{C}_{\tau_w}(M) = \{0, M\}$.

(b) $\Rightarrow$ (c) Assume that $\mathscr{C}_{\tau_w}(M) = \{0, M\}$. Then for any nonzero submodule N of M, $C^M_{\tau_w}(N) = M$. If not, $C^M_{\tau_w}(C^M_{\tau_w}(N)) = C^M_{\tau_w}(N_w \cap M) = (N_w \cap M)_w \cap M = N_w \cap M = C^M_{\tau_w}(N)$. Thus $C^M_{\tau_w}(N)$ is a proper τ_w-closed submodule of M, a contradiction.

(c) $\Rightarrow$ (a) Assume that $C^M_{\tau_w}(N) = M$ for any nonzero submodule N of M. Then $N_w \cap M = M$. Thus $M \subseteq N_w$, and so $N_w = M_w$. Then M/N is GV-torsion. Hence M is τ_w-simple. □

It is well known that an ideal $\mathfrak{m}$ of R is maximal if and only if $R/\mathfrak{m}$ is simple. Next we get the corresponding statement of the τ_w-case.

Lemma 2.2. *If $\mathfrak{m}$ is a w-ideal of R, then $\mathfrak{m}$ is a maximal w-ideal if and only if $R/\mathfrak{m}$ is τ_w-simple.*

Proof. Let $\mathfrak{m}$ be a maximal w-ideal. Then it suffices to show that $\mathscr{C}_{\tau_w}(R/\mathfrak{m}) = \{0, R/\mathfrak{m}\}$. If I is an ideal of R with $\mathfrak{m} \subsetneq I$, then $I_w = R$. Thus $(I/\mathfrak{m})_w \cap R/\mathfrak{m} = (I_w/\mathfrak{m})_w \cap R/\mathfrak{m} = R/\mathfrak{m} \neq I/\mathfrak{m}$, as desired.

Conversely, assume that $\mathfrak{m}$ is not a maximal w-ideal. Then there is a proper w-ideal I properly containing $\mathfrak{m}$. Then $I/\mathfrak{m}$ is a proper nonzero submodule of $R/\mathfrak{m}$. Because $\mathscr{C}_{\tau_w}(R/\mathfrak{m}) = \{0, R/\mathfrak{m}\}$, $(I/\mathfrak{m})_w \cap R/\mathfrak{m} \supsetneq I/\mathfrak{m}$. Choose $0 \neq \overline{r} \in (I/\mathfrak{m})_w \cap R/\mathfrak{m} \setminus I/\mathfrak{m}$, where $r \in R$. Then there is some $J \in \mathrm{GV}(R)$ such that $rJ \subseteq I$. Thus $r \in I_w = I$, a contradiction. □

Proposition 2.3. *Let M be a nonzero* GV*-torsion-free R-module. Then the following statements are equivalent.*

(1) *M is τ_w-simple.*
(2) *$N_w = M_w$ for any nonzero submodule N of M.*
(3) *For any nonzero element $x \in M$, $(Rx)_w = M_w$.*
(4) *Any nonzero submodule N of M is τ_w-simple and the intersection of any two nonzero submodules of M is not zero.*

(5) *For any nonzero element $x \in M$, Rx is τ_w-simple and the intersection of any two nonzero submodules of M is not zero.*

Proof. (1) $\Leftrightarrow$ (2) This follows from the fact that if M/N is GV-torsion, then $M_w = N_w$.

(2) $\Rightarrow$ (3) It is clear.

(3) $\Rightarrow$ (2) For any nonzero submodule N of M, choose a nonzero element $n \in N$. Then $(Rn)_w \subseteq N_w \subseteq M_w$. Note that $(Rn)_w = M_w$. Thus $M_w = N_w$.

(1) + (2) $\Rightarrow$ (4) If N_1 is a nonzero submodule of N (and so of M), then $(N_1)_w = M_w = N_w$ by (2). Thus by (1), N is τ_w-simple. Let M_1 and M_2 be nonzero submodules of M. Then $(M_1)_w = (M_2)_w = M$ again by (2). Thus $(M_1 \cap M_2)_w = M_w \neq 0$. So $M_1 \cap M_2 \neq 0$.

(4) $\Rightarrow$ (5) It is clear.

(5) $\Rightarrow$ (1) Assume that M is not τ_w-simple. Then there is some submodule N of M such that M/N is not GV-torsion. Thus there exists $x \in M \setminus N$ such that $Rx/(Rx \cap N)$ is not GV-torsion, a contradiction to the τ_w-simplicity of Rx. □

Corollary 2.4. *Let M be a nonzero* GV*-torsion-free R-module.*

(1) *If M is τ_w-simple, then for any nonzero element $x \in M$, $\text{ann}_R(x)$ is a maximal w-ideal of R.*

(2) *M is τ_w-simple if and only if M_w is τ_w-simple.*

Proof. (1) Let x be a nonzero element of M and set $I := \text{ann}_R(x)$. Then I is a proper w-ideal of R and $Rx \cong R/I$. By Proposition 2.3, Rx is τ_w-simple, and thus so is R/I. By Lemma 2.2, I is a maximal w-ideal.

(2) Assume that M is τ_w-simple. Then for any nonzero element $x \in M_w$, there exists some $J \in \text{GV}(R)$ such that $xJ \subseteq M$. Hence xJ is τ_w-simple. Thus $(xJ)_w = M_w$ by Proposition 2.3. So $(Rx)_w = M_w$, and hence M_w is τ_w-simple by Proposition 2.3 again. The converse is clear by Proposition 2.3. □

Recall that a nonzero w-module M is called *w-simple* if M has no nontrivial w-submodules [10, Definition 6.5.1]. A w-module M is w-simple if and only if $M = (Rx)_w$ for any nonzero element $x \in M$ [10, Proposition 6.5.2]. Then it is clear that w-simple w-modules and τ_w-simple w-modules coincide by Proposition 2.3. Thus for a GV-torsion-free R-module M, M is τ_w-simple if and only if M_w is w-simple by Corollary 2.4(1). So we have the following corollary.

Corollary 2.5. *Let M be a* GV*-torsion-free module. Then M is τ_w-simple if and only if M_w is w-simple.*

Recall that R is a *DW-ring* if every ideal of R is a w-ideal [10, Definition 6.3.11], if and only if every maximal ideal of R is a w-ideal, if and only if $\text{GV}(R) = \{R\}$ [10, Corollary 6.3.12].

Proposition 2.6. *If a ring R is not a DW-ring, then w-*$\text{Max}(R)$ *is infinite.*

Proof. Since R is not a DW-ring, there exists some maximal ideal $\mathfrak{m}$ of R such that $\mathfrak{m}$ is not a w-ideal of R. Assume that w-Max(R) is finite, say w-Max$(R) = \{\mathfrak{m}_1, \mathfrak{m}_2, \dots, \mathfrak{m}_n\}$. Then $\mathfrak{m} \nsubseteq \bigcup_{i=1}^{n} \mathfrak{m}_i$. If not, by the Prime Avoidance Theorem [10, Theorem 1.4.3], $\mathfrak{m}$ is contained in some $\mathfrak{m}_i$, and thus a w-ideal, a contradiction. Let $x \in \mathfrak{m} \setminus \bigcup_{i=1}^{n} \mathfrak{m}_i$. Then by [10, Exercise 6.11(2)], $(Rx)_w \neq R$. Thus there exists some maximal w-ideal $\mathfrak{m}'$ of R such that $Rx \subseteq \mathfrak{m}'$. Obviously, $\mathfrak{m}' \neq \mathfrak{m}_i$, $i = 1, \dots, n$, a contradiction. So w-Max(R) is infinite. □

Corollary 2.7. *If R is not a DW-ring, then there exist infinitely many w-simple R-modules.*

Proof. Note that for any maximal w-ideal $\mathfrak{m}$ of R, $(R/\mathfrak{m})_w$ is w-simple [10, Proposition 6.5.5]. Now the assertion follows from Proposition 2.6. □

3. τ_w-Loewy modules

First we recall the definition of the Loewy series of a module and related concepts from [8]. Let M be an R-module. The socle of M, denoted by soc(M), is the sum of all simple submodules of M. The *Loewy series* of M, $L_0(M) \subseteq L_1(M) \subseteq L_2(M) \subseteq \cdots$, is defined by transfinite induction for every ordinal α: $L_0(M) = 0$. If submodules $L_\beta(M)$ are defined for all $\beta < \alpha$ and α is a limit ordinal, then $L_\alpha(M)$ is the union of all preceding terms. Otherwise $L_\alpha(M)$ is the inverse image in M of soc$(M/L_{\alpha-1}(M))$. For each ordinal α, the number of summands in a direct sum decomposition of the factor module $L_{\alpha+1}(M)/L_\alpha(M)$ is denoted by $d_\alpha(M)$ and called the α^{th} *Loewy invariant* of M. The Loewy submodule of M, denoted by $L(M)$, is $L_\lambda(M)$, where λ is the least ordinal such that $L_\lambda(M) = L_{\lambda+1}(M)$. If $L(M) = M$ holds, then M is called a *Loewy module*. Facchini proved that an R-module M is Artinian if and only if it is Loewy and all its Loewy invariants are finite ([3, Theorem 2.7]). Moreover, in case M is Loewy, Shores showed that if $d_1(M)$ and $d_2(M)$ are finite, so are all the other Loewy invariants of M ([8, Theorem 4.2]). Sometimes we will use the notation L_{α_R} instead of L_α if confusion arises concerning the ring R.

Now we introduce the τ_w-Loewy series of a w-module. To do so, we first introduce the τ_w-socle of a GV-torsion-free module. Let M be a GV-torsion-free module. Write

$$\tau_w\text{-soc}(M) = \sum_{i \in \Omega} \{K_i \mid K_i \text{ is a } \tau_w\text{-simple submodule of } M\}.$$

By [1, Proposition 6.2.2], there is a subset Λ of Ω such that $\sum_{i\in\Lambda} K_i$ is direct and τ_w-dense in τ_w-soc(M). Thus for any nonzero element $x_i \in K_i$ for any $i \in \Lambda$, we have that $\bigoplus_{i\in\Lambda} Rx_i$ is also τ_w-dense in τ_w-soc(M). Indeed, note that K_i/Rx_i is GV-torsion by the definition of τ_w-simple modules, and so $\bigoplus_{i\in\Lambda} K_i/Rx_i$ is GV-torsion. Thus $(\bigoplus_{i\in\Lambda} K_i)/(\bigoplus_{i\in\Lambda} Rx_i) \cong \bigoplus_{i\in\Lambda} K_i/Rx_i$ is

GV-torsion. Then $\bigoplus_{i\in\Lambda} Rx_i$ is also τ_w-dense in τ_w-soc(M). So

$$(\tau_w\text{-soc}(M))_w = \bigoplus_{i\in\Lambda}(Rx_i)_w.$$

Recall that the *w-socle* of a w-module M (denoted by w-soc(M)) is the sum of its all w-simple submodules ([14, Definition 1.1]). Actually the sum is direct ([10, Theorem 6.5.8]). A w-module M is called *w-semisimple* if M is a direct sum of w-simple modules ([10, Definition 6.5.9]). Obviously, the w-socle of a w-module is w-semisimple. Next we show that the relationship between the w-socle and the τ_w-socle. Note that $(Rx_i)_w$ is a w-simple submodule of M_w by Corollary 2.5. Then $(\tau_w\text{-soc}(M))_w \subseteq w\text{-soc}(M_w)$. Actually, we have

$$(\tau_w\text{-soc}(M))_w = w\text{-soc}(M_w). \tag{1}$$

Indeed, write $w\text{-soc}(M_w) = \bigoplus\{S_i \mid S_i \text{ is a } w\text{-simple submodule of } M_w\}$. Then for any S_i, $S_i \cap M \neq 0$. If not, then $S_i = (S_i \cap M)_w = 0$, a contradiction. So choose a nonzero element $s_i \in S_i \cap M$. Then $S_i = (Rx_i)_w$ by the definition of w-simple modules. Note that Rx_i is a τ_w-simple submodule of M again by Corollary 2.5. Thus $\bigoplus Rx_i \subseteq \tau_w$-soc$(M)$. So $w\text{-soc}(M_w) = (\bigoplus Rx_i)_w \subseteq (\tau_w\text{-soc}(M))_w$.

Moreover, if M is a w-module, then τ_w-soc(M) is also a w-submodule of M by Lemma 2.1 and the following result.

Proposition 3.1. *For a* GV*-torsion-free R-module M,* τ_w-soc(M) *is a* τ_w*-pure submodule of M.*

Proof. It suffices to show that $\tau_w\text{-soc}(M) = \tau_w\text{-soc}(M_w) \cap M$. By formula (1) above, we have that $\tau_w\text{-soc}(M) \subseteq w\text{-soc}(M_w) \cap M$. For the reverse inclusion, if A is a w-simple submodule of M_w, then $A \cap M$ is a τ_w-simple submodule of M by Corollary 2.5. Thus $A \cap M \subseteq \tau_w\text{-soc}(M)$. Then $w\text{-soc}(M_w) \cap M \subseteq \tau_w\text{-soc}(M)$. □

Proposition 3.2. *Let M be a w-module, N a* GV*-torsion-free module, and* $f : M \to N$ *be a homomorphism. If* N_1 *is a* τ_w*-pure submodule of N, then* $f^{-1}(N_1)$ *is a w-submodule of M.*

Proof. Since N_1 is a τ_w-pure submodule of N, $N_1 = (N_1)_w \cap N$. Set $M_1 := f^{-1}(N_1)$ and let $Jx \subseteq M_1$, where $x \in M$ and $J \in \text{GV}(R)$. Then $f(Jx) = Jf(x) \subseteq N_1$. Thus $f(x) \in (N_1)_w \cap N$, which implies that $f(x) \in N_1$. So $x \in M_1$. Hence M_1 is a w-submodule of M. □

Now, the τ_w*-Loewy series* of a w-module M, $L^0_{\tau_w}(M) \subseteq L^1_{\tau_w}(M) \subseteq L^2_{\tau_w}(M) \subseteq \cdots$, is defined by transfinite induction for every ordinal α: $L^0_{\tau_w}(M) = 0$. If submodules $L^\beta_{\tau_w}(M)$ are defined for all $\beta < \alpha$ and α is a limit ordinal, then $L^\alpha_{\tau_w}(M)$ is the union of all preceding terms. Otherwise $L^\alpha_{\tau_w}(M)$ is the inverse image in M of $\tau_w\text{-soc}(M/L^{\alpha-1}_{\tau_w}(M))$. Note that $L^1_{\tau_w}(M) = \tau_w\text{-soc}(M)$ is a w-submodule of M. Thus $M/L^1_{\tau_w}(M)$ is GV-torsion-free. By Proposition 3.1

and Proposition 3.2, $L^2_{\tau_w}(M)$ is also a w-submodule of M. As this way, for any ordinal α, $L^{\alpha}_{\tau_w}(M)$ is a w-submodule of M. Note that $(L^{\alpha}_{\tau_w}(M)/L^{\alpha-1}_{\tau_w}(M))_w = (\tau_w\text{-soc}(M/L^{\alpha-1}_{\tau_w}(M)))_w = w\text{-soc}((M/L^{\alpha-1}_{\tau_w}(M))_w)$ if α is not a limit ordinal.

The τ_w-*Loewy submodule* of M is $L_{\tau_w}(M) := L^{\lambda}_{\tau_w}(M)$, where λ is the least ordinal such that $L^{\lambda}_{\tau_w}(M) = L^{\lambda+1}_{\tau_w}(M)$. The ordinal $\lambda = \lambda(M)$ is called the τ_w-*Loewy length* of M. If $L_{\tau_w}(M) = M$, then M is said to be τ_w-*Loewy.* For each ordinal α, the number of summands in a direct sum decomposition of the module $(L^{\alpha+1}_{\tau_w}(M)/L^{\alpha}_{\tau_w}(M))_w = w\text{-soc}((M/L^{\alpha}_{\tau_w}(M))_w)$ is denoted by $d^{\alpha}_{\tau_w}(M)$ and called the α^{th} τ_w-*Loewy invariant* of M. If this number is infinite, then set $d^{\alpha}_{\tau_w}(M) = \infty$. If this number is finite, then by [13, Theorem 2.2] it is independent of any direct sum decomposition of $w\text{-soc}((M/L^{\alpha-1}_{\tau_w}(M))_w)$ by its w-simple submodules. That is why we can define τ_w-Loewy invariants of M.

Let M be a GV-torsion-free R-module and let $\mathfrak{m}$ be a maximal w-ideal of R. By Lemma 2.2, $R/\mathfrak{m}$ is τ_w-simple. Set

$$\tau_w\text{-soc}(M,\mathfrak{m}) = \sum_{i\in\Omega_1}\{K_i \le M \mid K_i \text{ is a } \tau_w\text{-simple submodule isomorphic to } R/\mathfrak{m}\}.$$

Let $w\text{-soc}(M_w,\mathfrak{m})$ denote the sum of all w-simple submodules of M_w which are isomorphic to $(R/\mathfrak{m})_w$. Obviously, the sum is direct. And we can also get that $(\tau_w\text{-soc}(M,\mathfrak{m}))_w = w\text{-soc}(M_w,\mathfrak{m})$.

Let M be a w-module. Replacing $\tau_w\text{-soc}(M)$ (resp., $w\text{-soc}(M)$) by $\tau_w\text{-soc}(M,\mathfrak{m})$ (resp., $w\text{-soc}(M,\mathfrak{m})$), we obtain similar definitions for the τ_w-$\mathfrak{m}$-*Loewy series,* τ_w-$\mathfrak{m}$-*Loewy length,* τ_w-$\mathfrak{m}$-*Loewy module.* More in detail, we denote by $L^{\alpha+1}_{\tau_w}(M,\mathfrak{m})$ the $(\alpha+1)^{th}$ term of the τ_w-$\mathfrak{m}$-Loewy series. Then $(L^{\alpha+1}_{\tau_w}(M,\mathfrak{m})/L^{\alpha}_{\tau_w}(M,\mathfrak{m}))_w$ is the direct sum of all w-simple submodules of $(M/L^{\alpha}_{\tau_w}(M,\mathfrak{m}))_w$ which are isomorphic to $(R/\mathfrak{m})_w$ for any ordinal α.

Next we give some basic properties of the τ_w-Loewy series.

Proposition 3.3. *Let M be a w-module and let N be a w-submodule of M. Then $L^{\alpha}_{\tau_w}(N) = L^{\alpha}_{\tau_w}(M) \cap N$.*

Proof. Note that $w\text{-soc}(N) = w\text{-soc}(M) \cap N$ [14, Corollary 1.11].

Assume that $L^{\beta}_{\tau_w}(N) = L^{\beta}_{\tau_w}(M) \cap N$ for any $\beta < \alpha$. If α is a limit ordinal, then it is clear that $L^{\alpha}_{\tau_w}(N) = L^{\alpha}_{\tau_w}(M) \cap N$. If α is not a limit ordinal, then $N/L^{\alpha-1}_{\tau_w}(N)$ is a submodule of $M/L^{\alpha-1}_{\tau_w}(M)$. Thus

$$\begin{aligned}\left((L^{\alpha}_{\tau_w}(N) + L^{\alpha-1}_{\tau_w}(M))/L^{\alpha-1}_{\tau_w}(M)\right)_w &\cong \left(L^{\alpha}_{\tau_w}(N)/L^{\alpha-1}_{\tau_w}(N)\right)_w \\ &= w\text{-soc}\left((N/L^{\alpha-1}_{\tau_w}(N))_w\right) \\ &\subseteq w\text{-soc}\left((M/L^{\alpha-1}_{\tau_w}(M))_w\right) \\ &= \left(L^{\alpha}_{\tau_w}(M)/L^{\alpha-1}_{\tau_w}(M)\right)_w.\end{aligned}$$

Then $L^{\alpha}_{\tau_w}(N) \subseteq L^{\alpha}_{\tau_w}(M) \cap N$.

On the other hand, note that

$$(L^{\alpha}_{\tau_w}(M)\cap N)/L^{\alpha-1}_{\tau_w}(N))_w = ((L^{\alpha}_{\tau_w}(M)\cap N)/(L^{\alpha-1}_{\tau_w}(M)\cap N))_w$$

$$\begin{aligned}&\subseteq (L^{\alpha}_{\tau_w}(M)/L^{\alpha-1}_{\tau_w}(M))_w\\&= w\text{-soc}\left((M/L^{\alpha-1}_{\tau_w}(M))_w\right).\end{aligned}$$

Then $\left(L^{\alpha}_{\tau_w}(M)\cap N)/L^{\alpha-1}_{\tau_w}(N)\right)_w$ is a w-submodule of the w-semisimple module, and thus it is w-semisimple. By the definition of w-socles, we can get that

$$\begin{aligned}\left((L^{\alpha}_{\tau_w}(M)\cap N)/L^{\alpha-1}_{\tau_w}(N)\right)_w &\subseteq w\text{-soc}\left((N/L^{\alpha-1}_{\tau_w}(N))_w\right)\\&= \left(L^{\alpha}_{\tau_w}(N)/L^{\alpha-1}_{\tau_w}(N)\right)_w.\end{aligned}$$

Then $L^{\alpha}_{\tau_w}(M)\cap N\subseteq L^{\alpha}_{\tau_w}(N)$. □

Proposition 3.4. *Let α be any ordinal and let $\{M_i\}$ be a family of w-modules. Then $L^{\alpha}_{\tau_w}(\bigoplus M_i)=\bigoplus L^{\alpha}_{\tau_w}(M_i)$.*

Proof. Note that w-soc$(\bigoplus M_i)=\bigoplus w$-soc$(M_i)$ ([14, Theorem 1.17]). Then we have $L^{1}_{\tau_w}(\bigoplus M_i)=\bigoplus L^{1}_{\tau_w}(M_i)$.

Assume that $L^{\beta}_{\tau_w}(\bigoplus M_i)=\bigoplus L^{\beta}_{\tau_w}(M_i)$ for any $\beta<\alpha$. If α is a limit ordinal, then it is clear that $L^{\alpha}_{\tau_w}(\bigoplus M_i)=\bigoplus L^{\alpha}_{\tau_w}(M_i)$. Now we assume that α is not a limit ordinal. Let $M=\bigoplus M_i$. By Proposition 3.3, we have

$$\begin{aligned}M/L^{\alpha-1}_{\tau_w}(M) &= \bigoplus\left((M_i+L^{\alpha-1}_{\tau_w}(M))/L^{\alpha-1}_{\tau_w}(M)\right)\\&= \bigoplus\left(M_i/L^{\alpha-1}_{\tau_w}(M_i)\right).\end{aligned}$$

Thus we have

$$\begin{aligned}w\text{-soc}\left((M/L^{\alpha-1}_{\tau_w}(M))_w\right) &= w\text{-soc}\left(\bigoplus(M_i/L^{\alpha-1}_{\tau_w}(M_i))_w\right)\\&= \bigoplus w\text{-soc}\left((M_i/L^{\alpha-1}_{\tau_w}(M_i))_w\right).\end{aligned}$$

Then $\left(L^{\alpha}_{\tau_w}(M)/L^{\alpha-1}_{\tau_w}(M)\right)_w=\bigoplus\left(L^{\alpha}_{\tau_w}(M_i)/L^{\alpha-1}_{\tau_w}(M_i)\right)_w$. Thus we have

$$\begin{aligned}\left(L^{\alpha}_{\tau_w}(M)/L^{\alpha-1}_{\tau_w}(M)\right)_w &= \left(\bigoplus L^{\alpha}_{\tau_w}(M_i)/L^{\alpha-1}_{\tau_w}(M_i)\right)_w\\&= \left(\bigoplus(L^{\alpha}_{\tau_w}(M_i)+L^{\alpha-1}_{\tau_w}(M))/L^{\alpha-1}_{\tau_w}(M)\right)_w.\end{aligned}$$

Then for any maximal w-ideal $\mathfrak{m}$,

$$\begin{aligned}\left(L^{\alpha}_{\tau_w}(M))_{\mathfrak{m}}/(L^{\alpha-1}_{\tau_w}(M)\right)_{\mathfrak{m}} &= \left(L^{\alpha}_{\tau_w}(M)/L^{\alpha-1}_{\tau_w}(M)\right)_{\mathfrak{m}}\\&= \left(\bigoplus(L^{\alpha}_{\tau_w}(M_i)+L^{\alpha-1}_{\tau_w}(M))/L^{\alpha-1}_{\tau_w}(M)\right)_{\mathfrak{m}}\\&= \bigoplus\left((L^{\alpha}_{\tau_w}(M_i)+L^{\alpha-1}_{\tau_w}(M))/L^{\alpha-1}_{\tau_w}(M)\right)_{\mathfrak{m}}\\&= \left(\bigoplus(L^{\alpha}_{\tau_w}(M_i))_{\mathfrak{m}})/(L^{\alpha-1}_{\tau_w}(M)\right)_{\mathfrak{m}}.\end{aligned}$$

Hence $(L^{\alpha}_{\tau_w}(M))_{\mathfrak{m}}=\bigoplus(L^{\alpha}_{\tau_w}(M_i))_{\mathfrak{m}}=(\bigoplus L^{\alpha}_{\tau_w}(M_i))_{\mathfrak{m}}$. Therefore we have $L^{\alpha}_{\tau_w}(M)=\bigoplus L^{\alpha}_{\tau_w}(M_i)$. □

Let w-$Sp(M)$ denote the set $\{\mathfrak{m} \in w\text{-Max}(R) \mid (0 :_M \mathfrak{m}) \neq 0\}$. Next our purpose is to prove $L^{\alpha}_{\tau_w}(M) = \bigoplus\limits_{\mathfrak{m}\in w\text{-}Sp(M)} L^{\alpha}_{\tau_w}(M,\mathfrak{m})$ for any ordinal α if w-$Sp(M)$ is finite. First we discuss some basic properties of w-$Sp(M)$.

Lemma 3.5. *Let M be a w-module. Then the following statements hold.*

(1) *If M_1 is the only w-simple submodule of M, then for any nonzero element x in M_1, $\text{ann}_R(x)$ is a maximal w-ideal, $M_1 = (0 :_M \text{ann}_R(x))$, and for other maximal w-ideal $\mathfrak{m}$, $(0 :_M \mathfrak{m}) = 0$.*

(2) *If $\mathfrak{m} \in w$-$Sp(M)$, then $(0 :_M \mathfrak{m}) = w\text{-soc}(M,\mathfrak{m})$. Furthermore, for distinct elements $\mathfrak{m}$ and $\mathfrak{n}$ in w-$Sp(M)$, $w\text{-soc}(M,\mathfrak{m}) \neq w\text{-soc}(M,\mathfrak{n})$.*

(3) *If $M_1 = (Rx)_w$ is a w-simple submodule of M, then $\text{ann}_R(x) \in w$-$Sp(M)$ and M_1 is a direct summand of $w\text{-soc}(M,\text{ann}_R(x))$.*

(4) *$w\text{-soc}(M) = \bigoplus\limits_{\mathfrak{m}\in w\text{-}Sp(M)} w\text{-soc}(M,\mathfrak{m})$.*

Proof. (1) Let $0 \neq x \in M$. Note that $\text{ann}_R(x)$ is a maximal w-ideal. For any nonzero element y in $M_1 = (Rx)_w$, there is some $J \in \text{GV}(R)$ such that $yJ \subseteq Rx$. Then $yJ \cdot \text{ann}_R(x) = 0$. Thus $y \cdot \text{ann}_R(x) = 0$. So $y \in (0 :_M \text{ann}_R(x))$. Conversely, if $0 \neq z \in (0 :_M \text{ann}_R(x))$, then $z \cdot \text{ann}_R(x) = 0$. Thus $\text{ann}_R(x) \subseteq \text{ann}_R(z)$. By the w-maximality of $\text{ann}_R(x)$, $\text{ann}_R(z)$ is a maximal w-ideal. Then $(Rz)_w$ is a w-simple module. Thus $(Rz)_w = M_1$, and so $z \in M_1$.

If there is some $0 \neq y \in (0 :_M \mathfrak{m})$, then we can prove that $(Ry)_w$ is a w-simple submodule of M by the above way. Then $(Ry)_w = M_1$. Thus $\mathfrak{m} = \text{ann}_R(y) = \text{ann}_R(x)$ for any nonzero element x in M_1, a contradiction.

(2) If $0 \neq x \in (0 :_M \mathfrak{m})$, then $x\mathfrak{m} = 0$. Thus $\mathfrak{m} \subseteq \text{ann}_R(x)$. So $\text{ann}_R(x) = \mathfrak{m}$, and hence $\text{ann}_R(x) \in w$-$Sp(M)$. Note that $(Rx)_w$ is w-simple, and $(Rx)_w \cong (R/\text{ann}_R(x))_w = (R/\mathfrak{m})_w$. Since $\sum\limits_{x\in(0:_M\mathfrak{m})} (Rx)_w = (0 :_M \mathfrak{m})$. So $(0 :_M \mathfrak{m}) \subseteq w$-$\text{soc}(M,\mathfrak{m})$. For the reverse inclusion, if the w-simple module $(Rx)_w$ is a direct summand of $w\text{-soc}(M,\mathfrak{m})$, then $(Rx)_w \cong (R/\mathfrak{m})_w$ and $\text{ann}_R(x) = \mathfrak{m}$. Hence $Rx \subseteq (0 :_M \text{ann}_R(x)) = (0 :_M \mathfrak{m})$. Thus $w\text{-soc}(M,\mathfrak{m}) = (0 :_M \mathfrak{m})$.

(3) This follows by the proof of (2).

(4) This follows by (3). □

Let I be an ideal of R. Then I is said to be *T-nilpotent* if for every sequence $a_1, a_2, \ldots$ in I, there is a positive integer n such that $a_1a_2\cdots a_n = 0$. The next result shows that for $\mathfrak{m} \in w$-$Sp(M)$, where M is a w-module, $\mathfrak{m}$ has similar property.

Proposition 3.6. *Let M be a w-module, $\mathfrak{m} \in w$-$Sp(M)$, and $0 \neq m \in M$. Then $m \in L_{\tau_w}(M,\mathfrak{m})$ if and only if for every sequence $t_1, t_2, \ldots$ of elements of $\mathfrak{m}$, there is an integer n such that $mt_1t_2\cdots t_n = 0$. In particular, if n is a positive integer, then $m \in L^{n}_{\tau_w}(M,\mathfrak{m})$ if and only if $m\mathfrak{m}^n = 0$.*

Proof. Note that $L_{\tau_w}(M,\mathfrak{m}) = L^{\alpha}_{\tau_w}(M,\mathfrak{m})$ for some ordinal α. First we show that there is some ordinal $\alpha_1 < \alpha$ such that $mt_1 \in L^{\alpha_1}_{\tau_w}(M,\mathfrak{m})$. If α is a limit ordinal, then $L^{\alpha}_{\tau_w}(M,\mathfrak{m}) = \bigcup_{\beta<\alpha} L^{\beta}_{\tau_w}(M,\mathfrak{m})$. Obviously, there is some $\beta < \alpha$ such that $m \in L^{\beta}_{\tau_w}(M,\mathfrak{m})$, and then $mt_1 \in L^{\beta}_{\tau_w}(M,\mathfrak{m})$. Then set $\alpha_1 = \beta$. If α is not a limit ordinal, then $L^{\alpha}_{\tau_w}(M,\mathfrak{m})/L^{\alpha-1}_{\tau_w}(M,\mathfrak{m}) = \sum\{K_i \mid K_i$ is a τ_w-simple submodule of $M/L^{\alpha-1}_{\tau_w}(M,\mathfrak{m})$ which is isomorphic to $R/\mathfrak{m}\}$ and $L^{\alpha}_{\tau_w}(M,\mathfrak{m})\mathfrak{m} \subseteq L^{\alpha-1}_{\tau_w}(M,\mathfrak{m})$. Then $mt_1 \in L^{\alpha-1}_{\tau_w}(M,\mathfrak{m})$. In this case, set $\alpha_1 = \alpha - 1$. Continue in this fashion and obtain a decreasing sequence of ordinals $\alpha_1, \alpha_2, \ldots$ such that $mt_1t_2\cdots t_k \subseteq L^{\alpha_k}_{\tau_w}(M,\mathfrak{m})$. By the well ordering property, the sequence of ordinals must have a minimal element $\alpha_n = 0$. If not, then $\alpha_n > 0$. By the same way above, we can find another ordinal $\alpha_{n+1} < \alpha_n$ such that $mt_1t_2\cdots t_nt_{n+1} \in L^{\alpha_{n+1}}_{\tau_w}(M,\mathfrak{m})$, a contradiction to the minimality of α_n. Thus $mt_1t_2\cdots t_n \in L^{0}_{\tau_w}(M,\mathfrak{m})$. Then $mt_1t_2\cdots t_n = 0$.

Conversely, if $m \notin L_{\tau_w}(M,\mathfrak{m})$, then there is some t_1 such that $mt_1 \notin L_{\tau_w}(M,\mathfrak{m})$. If not, then $mI \subseteq L_{\tau_w}(M,\mathfrak{m})$. So the image $\overline{m}$ of m in $M/L_{\tau_w}(M,\mathfrak{m})$ satisfies that $\overline{m}\mathfrak{m} = 0$. Then $\mathfrak{m} \subseteq \mathrm{ann}_R(\overline{m})$, and so $\mathfrak{m} = \mathrm{ann}_R(\overline{m})$. Hence $R\overline{m}$ is a τ_w-simple submodule of $M/L_{\tau_w}(M,\mathfrak{m})$ which is isomorphic to $R/\mathfrak{m}$. So $Rm \subseteq L_{\tau_w}(M,\mathfrak{m})$, a contradiction to $m \notin L_{\tau_w}(M,\mathfrak{m})$. Now repeat the above argument and obtain a sequence $t_1, t_1, \ldots$ of elements of $\mathfrak{m}$ such that $mt_1t_2\cdots t_n \notin L_{\tau_w}(M,\mathfrak{m})$ for all n. Thus $mt_1t_2\cdots t_n \neq 0$ for all n, a contradiction to the assumption.

By the first paragraph of the proof, we can get that if $m \in L^{n}_{\tau_w}(M,\mathfrak{m})$, then $m\mathfrak{m}^n = 0$. Conversely, if $m\mathfrak{m} = 0$, then $\mathrm{ann}_R(m) = \mathfrak{m}$. Thus $(Rm)_w \cong (R/\mathfrak{m})_w$. So $(Rm)_w \subseteq L^{1}_{\tau_w}(M,\mathfrak{m})$, which implies $m \in L^{1}_{\tau_w}(M,\mathfrak{m})$. Assume that if $m\mathfrak{m}^{n-1} = 0$, then $m \in L^{n-1}_{\tau_w}(M,\mathfrak{m})$. If $m\mathfrak{m}^n = 0$, then $m\mathfrak{m} \subseteq L^{n-1}_{\tau_w}(M,\mathfrak{m})$. If $m \in L^{n-1}_{\tau_w}(M,\mathfrak{m})$, then $m \in L^{n}_{\tau_w}(M,\mathfrak{m})$. If $m \notin L^{n-1}_{\tau_w}(M,\mathfrak{m})$, then the image $\overline{m}$ of m in $M/L^{n-1}_{\tau_w}(M,\mathfrak{m})$ is not equal to zero and satisfies that $\overline{m}\mathfrak{m} = 0$. Thus $\mathfrak{m} \subseteq \mathrm{ann}_R(\overline{m})$, and so $\mathfrak{m} = \mathrm{ann}_R(\overline{m})$. Hence $R\overline{m}$ is a τ_w-simple submodule of $M/L^{n-1}_{\tau_w}(M,\mathfrak{m})$ which is isomorphic to $R/\mathfrak{m}$. So $Rm \subseteq L^{n}_{\tau_w}(M,\mathfrak{m})$, which implies $m \in L^{n}_{\tau_w}(M,\mathfrak{m})$. □

In order to prove Theorem 3.9, we also need the following lemmas.

Lemma 3.7. *Let M be a w-module. Then for different maximal w-ideals $\mathfrak{m}$ and $\mathfrak{n}$ in w-$Sp(M)$, $\mathrm{Ext}^1_R(S, L^{\alpha}_{\tau_w}(M,\mathfrak{n})) = 0$, where $S \cong R/\mathfrak{m}$ and α is an ordinal.*

Proof. Let $r \in \mathfrak{m} \setminus \mathfrak{n}$. We first show that $rL^{\alpha}_{\tau_w}(M,\mathfrak{n}) = L^{\alpha}_{\tau_w}(M,\mathfrak{n})$.

In the case $\alpha = 1$, since $\mathfrak{n}$ is a maximal w-ideal, $(\mathfrak{n} + Rr)_w = R$. Then there is some $J \in \mathrm{GV}(R)$ such that $J \subseteq \mathfrak{n} + Rr$. Thus for any nonzero element $b \in L^{1}_{\tau_w}(M,\mathfrak{n})$, $bJ \subseteq b\mathfrak{n} + brR \subseteq rL^{1}_{\tau_w}(M,\mathfrak{n})$. Then $b \in (rL^{1}_{\tau_w}(M,\mathfrak{n}))_w$. So $L^{1}_{\tau_w}(M,\mathfrak{n}) = (rL^{1}_{\tau_w}(M,\mathfrak{n}))_w$. To show that $(rL^{1}_{\tau_w}(M,\mathfrak{n}))_w = rL^{1}_{\tau_w}(M,\mathfrak{n})$, it suffices to prove that for any nonzero element x in $L^{1}_{\tau_w}(M,\mathfrak{n})$, $rx \neq 0$. If not, there is some nonzero element x in $L^{1}_{\tau_w}(M,\mathfrak{n})$ such that $rx = 0$. Note that $L^{1}_{\tau_w}(M,\mathfrak{n}) = (\bigoplus_{i\in\Lambda} B_i)_w$, where $B_i \cong R/\mathfrak{n}$. Then there is some $J_1 \in \mathrm{GV}(R)$ such that $xJ_1 \subseteq \bigoplus_{i\in\Lambda} B_i$. Since $xJ_1 \neq 0$ and $xI_2 = 0$, there is some $j \in J_1 \setminus \mathfrak{n}$ such that $xj_1 \neq 0$. We can write $xj_1 = (\overline{r_i}) \in \bigoplus R/\mathfrak{n}$, where $r_i \in R$ and only finitely many $\overline{r_i} \neq 0$. Then $r(\overline{r_i}) = 0$. Thus there is some $r_i \notin \mathfrak{n}$ such that $rr_i \in \mathfrak{n}$. So $r \in \mathfrak{n}$, a contradiction to the choice of r. Thus $(rL^1_{\tau_w}(M,\mathfrak{n}))_w = rL^1_{\tau_w}(M,\mathfrak{n})$, which implies that $L^1_{\tau_w}(M,\mathfrak{n}) = rL^1_{\tau_w}(M,\mathfrak{n})$.

Assume that for any ordinal $\beta < \alpha$, $rL^\beta_{\tau_w}(M,\mathfrak{n}) = L^\beta_{\tau_w}(M,\mathfrak{n})$.

If α is a limit ordinal, then

$$\begin{aligned} rL^\alpha_{\tau_w}(M,\mathfrak{n}) &= r\left(\bigcup_{\beta<\alpha} L^\beta_{\tau_w}(M,\mathfrak{n})\right) \\ &= \bigcup_{\beta<\alpha} rL^\beta_{\tau_w}(M,\mathfrak{n}) \\ &= \bigcup_{\beta<\alpha} L^\beta_{\tau_w}(M,\mathfrak{n}) \\ &= L^\alpha_{\tau_w}(M,\mathfrak{n}). \end{aligned}$$

Now assume that α is not a limit ordinal. For any nonzero element $b \in L^\alpha_{\tau_w}(M,\mathfrak{n})$, if $b \in L^{\alpha-1}_{\tau_w}(M,\mathfrak{n})$, then $b \in rL^{\alpha-1}_{\tau_w}(M,\mathfrak{n})$ by assumption. So $b \in rL^\alpha_{\tau_w}(M,\mathfrak{n})$. Then we consider the case that b is in $L^\alpha_{\tau_w}(M,\mathfrak{n})$, but not in $L^{\alpha-1}_{\tau_w}(M,\mathfrak{n})$. As the same way of the case $\alpha = 1$, there is some $J \in \mathrm{GV}(R)$ such that $bJ \subseteq b\mathfrak{n}+brR \subseteq L^{\alpha-1}_{\tau_w}(M,\mathfrak{n})+rL^\alpha_{\tau_w}(M,\mathfrak{n}) = rL^{\alpha-1}_{\tau_w}(M,\mathfrak{n})+rL^\alpha_{\tau_w}(M,\mathfrak{n}) \subseteq rL^\alpha_{\tau_w}(M,\mathfrak{n})$. Thus $b \in (rL^\alpha_{\tau_w}(M,\mathfrak{n}))_w$, and so $L^\alpha_{\tau_w}(M,\mathfrak{n}) = (rL^\alpha_{\tau_w}(M,\mathfrak{n}))_w$. Now we prove that $rL^\alpha_{\tau_w}(M,\mathfrak{n}) = (rL^\alpha_{\tau_w}(M,\mathfrak{n}))_w$, i.e., for any nonzero element x in $L^\alpha_{\tau_w}(M,\mathfrak{n})$, $rx \neq 0$. If not, there is some nonzero element x in $L^{\alpha-1}_{\tau_w}(M,\mathfrak{n})$ such that $rx = 0$. Note that by assumption, if $x \in L^{\alpha-1}_{\tau_w}(M,\mathfrak{n})$, then $rx \neq 0$. Hence $x \notin L^{\alpha-1}_{\tau_w}(M,\mathfrak{n})$, and thus $0 \neq \overline{x} \in \frac{L^\alpha_{\tau_w}(M,\mathfrak{n})}{L^{\alpha-1}_{\tau_w}(M,\mathfrak{n})}$. Note that $(\frac{L^\alpha_{\tau_w}(M,\mathfrak{n})}{L^{\alpha-1}_{\tau_w}(M,\mathfrak{n})})_w \cong (\bigoplus_{i\in\Lambda_1} B'_i)_w$, where $B'_i \cong R/\mathfrak{n}$. Note also that $\overline{x}J_1 \neq 0$ for any GV-ideal J_1. Then again as the same way of the case $\alpha = 1$, we can get that $rL^\alpha_{\tau_w}(M,\mathfrak{n}) = (rL^\alpha_{\tau_w}(M,\mathfrak{n}))_w$. Thus $L^\alpha_{\tau_w}(M,\mathfrak{n}) = rL^\alpha_{\tau_w}(M,\mathfrak{n})$.

Note that the mapping

$$\mathrm{Ext}^1_R(1_S, r) : \mathrm{Ext}^1_R(S, L^\alpha_{\tau_w}(M,\mathfrak{n})) \to \mathrm{Ext}^1_R(S, L^\alpha_{\tau_w}(M,\mathfrak{n}))$$

is an isomorphism since $L^\alpha_{\tau_w}(M,\mathfrak{n}) = rL^\alpha_{\tau_w}(M,\mathfrak{n})$. Note also that $\mathrm{Ext}^1_R(1_S, r) = \mathrm{Ext}^1_R(r, 1_{L^\alpha_{\tau_w}(M,\mathfrak{n})})$ ([6, p. 130]). Then $\mathrm{Ext}^1_R(r, 1_{L^\alpha_{\tau_w}(M,\mathfrak{n})})$ is the zero mapping because $rS = 0$. Therefore, we have $\mathrm{Ext}^1_R(S, L^\alpha_{\tau_w}(M,\mathfrak{n})) = 0$. □

Lemma 3.8. *Let M be a w-module. Then for different maximal w-ideals $\mathfrak{m}$ and $\mathfrak{n}$ in w-$Sp(M)$, $L^\alpha_{\tau_w}(M,\mathfrak{m}) \cap L^\alpha_{\tau_w}(M,\mathfrak{n}) = 0$ for any ordinal α.*

Proof. If $n = 1$, then it is clear that $L^1_{\tau_w}(M,\mathfrak{m}) \cap L^1_{\tau_w}(M,\mathfrak{n}) = 0$ since w-$\mathrm{soc}(M,\mathfrak{m}) \cap w$-$\mathrm{soc}(M,\mathfrak{n}) = 0$. Assume that $L^\beta_{\tau_w}(M,\mathfrak{m}) \cap L^\beta_{\tau_w}(M,\mathfrak{n}) = 0$ for any $\beta < \alpha$. Now we show that $L^\alpha_{\tau_w}(M,\mathfrak{m}) \cap L^\alpha_{\tau_w}(M,\mathfrak{n}) = 0$. If not, then there is some nonzero element x in $L^\alpha_{\tau_w}(M,\mathfrak{m}) \cap L^\alpha_{\tau_w}(M,\mathfrak{n})$.

If α is a limit ordinal, then there is an ordinal $\beta < \alpha$ such that $x \in L^{\beta}_{\tau_w}(M,\mathfrak{m}) \cap L^{\beta}_{\tau_w}(M,\mathfrak{n}) = 0$, a contradiction to $x \neq 0$. Thus $L^{\alpha}_{\tau_w}(M,\mathfrak{m}) \cap L^{\alpha}_{\tau_w}(M,\mathfrak{n}) = 0$.

If α is not a limit ordinal, then $x \notin L^{\alpha-1}_{\tau_w}(M,\mathfrak{m})$ or $x \notin L^{\alpha-1}_{\tau_w}(M,\mathfrak{n})$. If not, $x \in L^{\alpha}_{\tau_w}(M,\mathfrak{m}) \cap L^{\alpha}_{\tau_w}(M,\mathfrak{n}) = 0$, a contradiction. Assume that $x \notin L^{\alpha-1}_{\tau_w}(M,\mathfrak{m})$. Then $x\mathfrak{m} \subseteq L^{\alpha-1}_{\tau_w}(M,\mathfrak{m})$. Note that $x\mathfrak{m} \neq 0$, otherwise, $x \in L^{1}_{\tau_w}(M,\mathfrak{m}) \subseteq L^{\alpha-1}_{\tau_w}(M,\mathfrak{m})$ by Proposition 3.6, a contradiction. Then there is some $0 \neq r \in \mathfrak{m}$ such that $0 \neq rx \in L^{\alpha-1}_{\tau_w}(M,\mathfrak{m})$. Now it suffices to consider two cases: one is that $x \in L^{\alpha-1}_{\tau_w}(M,\mathfrak{n})$; the other is that $x \notin L^{\alpha-1}_{\tau_w}(M,\mathfrak{n})$. If $x \in L^{\alpha-1}_{\tau_w}(M,\mathfrak{n})$, then $xr \in L^{\alpha-1}_{\tau_w}(M,\mathfrak{m}) \cap L^{\alpha-1}_{\tau_w}(M,\mathfrak{n}) = 0$, a contradiction to $xr \neq 0$. If $x \notin L^{\alpha-1}_{\tau_w}(M,\mathfrak{n})$, then we can also get that $0 \neq x\mathfrak{n} \subseteq L^{\alpha-1}_{\tau_w}(M,\mathfrak{n})$. By the proof of Lemma 3.7, for any $r_1 \in \mathfrak{n} \setminus \mathfrak{m}$, $r_1 L^{\alpha-1}_{\tau_w}(M,\mathfrak{m}) = L^{\alpha-1}_{\tau_w}(M,\mathfrak{m})$. Thus $xrr_1 \neq 0$. Note that $xr_1 \in L^{\alpha-1}_{\tau_w}(M,\mathfrak{n})$. Thus $0 \neq xrr_1 \in L^{\alpha-1}_{\tau_w}(M,\mathfrak{n}) \cap L^{\alpha-1}_{\tau_w}(M,\mathfrak{m})$, a contradiction again. Then $L^{\alpha}_{\tau_w}(M,\mathfrak{m}) \cap L^{\alpha}_{\tau_w}(M,\mathfrak{n}) = 0$. □

Theorem 3.9. *Let M be a w-module. Assume that w-$Sp(M)$ is a finite set. Then* $L^{\alpha}_{\tau_w}(M) = \bigoplus_{\mathfrak{m}\in w\text{-}Sp(M)} L^{\alpha}_{\tau_w}(M,\mathfrak{m})$ *for any ordinal α.*

Proof. Set w-$Sp(M) = \{\mathfrak{m}_1, \mathfrak{m}_2, \ldots, \mathfrak{m}_n\}$. Note that w-soc$(M) = \bigoplus_{\mathfrak{m}\in w\text{-}Sp(M)} w$-soc$(M,\mathfrak{m})$ by Lemma 3.5(4). Then $L^{1}_{\tau_w}(M) = \bigoplus_{\mathfrak{m}\in w\text{-}Sp(M)} L^{1}_{\tau_w}(M,\mathfrak{m})$.

Assume that $L^{\beta}_{\tau_w}(M) = \bigoplus_{i=1}^{n} L^{\beta}_{\tau_w}(M,\mathfrak{m}_i)$ for any ordinal $\beta < \alpha$. Now we will show that $L^{\alpha}_{\tau_w}(M) = \bigoplus_{i=1}^{n} L^{\alpha}_{\tau_w}(M,\mathfrak{m}_i)$.

If α is a limit ordinal, then for any $x \in L^{\alpha}_{\tau_w}(M)$, there exists some $\beta < \alpha$ such that $x \in L^{\beta}_{\tau_w}(M)$. Thus $x \in \bigoplus_{i=1}^{n} L^{\beta}_{\tau_w}(M,\mathfrak{m}_i) \subseteq \bigoplus_{i=1}^{n} L^{\alpha}_{\tau_w}(M,\mathfrak{m}_i)$. So $L^{\alpha}_{\tau_w}(M) = \bigoplus_{i=1}^{n} L^{\alpha}_{\tau_w}(M,\mathfrak{m}_i)$.

Now assume that α is not a limit ordinal. Set

$$\left(\frac{L^{\alpha}_{\tau_w}(M)}{L^{\alpha-1}_{\tau_w}(M)}\right)_w = \bigoplus_{\Lambda}\left(\frac{H}{L^{\alpha-1}_{\tau_w}(M)}\right)_w,$$

where $\left(\frac{H}{L^{\alpha-1}_{\tau_w}(M)}\right)_w$ is w-simple and H is a w-submodule of M. Thus $\left(\frac{H}{L^{\alpha-1}_{\tau_w}(M)}\right)_w = (Rx)_w$, where $x \in \frac{H}{L^{\alpha-1}_{\tau_w}(M)}$. By Lemma 3.5(3), $\mathrm{ann}_R(Rx) \in w$-$Sp(M)$. Without loss of generality, we assume that $\mathrm{ann}_R(Rx) = \mathfrak{m}_1$ and that

$$H/L^{\alpha-1}_{\tau_w}(M) \cong R/\mathfrak{m}_1.$$

Set $S_i := R/\mathfrak{m}_i$ for each $i = 1, \ldots, n$.

Claim (I). We show that the following exact sequence is split.

$$0 \to \frac{L^{\alpha-1}_{\tau_w}(M)}{L^{\alpha-1}_{\tau_w}(M,\mathfrak{m}_1)} \to \frac{H}{L^{\alpha-1}_{\tau_w}(M,\mathfrak{m}_1)} \to S_1 \to 0. \tag{2}$$

Note that $\frac{L^{\alpha-1}_{\tau_w}(M)}{L^{\alpha-1}_{\tau_w}(M,\mathfrak{m}_1)} = \bigoplus_{i=2}^n L^{\alpha-1}_{\tau_w}(M,\mathfrak{m}_i)$ by assumption. Then by Lemma 3.7

$$\begin{aligned}\mathrm{Ext}^1_R\left(S_1, \frac{L^{\alpha-1}_{\tau_w}(M)}{L^{\alpha-1}_{\tau_w}(M,\mathfrak{m}_1)}\right) &= \mathrm{Ext}^1_R(S_1, \bigoplus_{i=2}^n L^{\alpha-1}_{\tau_w}(M,\mathfrak{m}_i)) \\ &= \bigoplus_{i=2}^n \mathrm{Ext}^1_R(S_1, L^{\alpha-1}_{\tau_w}(M,\mathfrak{m}_i)) \\ &= 0.\end{aligned}$$

Thus the exact sequence (2) above is split.

Claim (II). We show that $L^{\alpha}_{\tau_w}(H,\mathfrak{m}_i) = L^{\alpha-1}_{\tau_w}(H,\mathfrak{m}_i) = L^{\alpha-1}_{\tau_w}(M,\mathfrak{m}_i)$ for $i \in \{2,3,\dots,n\}$.

Obviously $L^{\alpha-1}_{\tau_w}(H,\mathfrak{m}_i) = L^{\alpha-1}_{\tau_w}(M,\mathfrak{m}_i)$.
If $\frac{L^{\alpha}_{\tau_w}(H,\mathfrak{m}_i)}{L^{\alpha-1}_{\tau_w}(H,\mathfrak{m}_i)} \neq 0$, then $(\frac{L^{\alpha}_{\tau_w}(H,\mathfrak{m}_i)}{L^{\alpha-1}_{\tau_w}(H,\mathfrak{m}_i)})_w \cong \bigoplus (S_i)_w$. By Lemma 3.8, we have

$$\begin{aligned}(H/L^{\alpha-1}_{\tau_w}(M))_w &= \left(\frac{H}{\bigoplus\limits_{i=1}^n L^{\alpha-1}_{\tau_w}(H,\mathfrak{m}_i)}\right)_w \\ &\supseteq \left(\frac{L^{\alpha}_{\tau_w}(H,\mathfrak{m}_i)) \oplus (\bigoplus\limits_{j=1,j\neq i}^n L^{\alpha-1}_{\tau_w}(H,\mathfrak{m}_j))}{\bigoplus\limits_{j=1}^n L^{\alpha-1}_{\tau_w}(H,\mathfrak{m}_j)}\right)_w \\ &\cong \left(\frac{L^{\alpha}_{\tau_w}(H,\mathfrak{m}_i)}{L^{\alpha-1}_{\tau_w}(H,\mathfrak{m}_i)}\right)_w.\end{aligned}$$

Hence $(H/L^{\alpha-1}_{\tau_w}(M))_w$ contains an isomorphic copy of $(S_i)_w$, a contradiction to $(H/L^{\alpha-1}_{\tau_w}(M))_w \cong (S_1)_w$.

Claim (III). We show that $\left(\frac{L^{\alpha}_{\tau_w}(H,\mathfrak{m}_1)}{L^{\alpha-1}_{\tau_w}(M,\mathfrak{m}_1)}\right)_w \cong (S_1)_w$.

It is obvious that $L^{\alpha-1}_{\tau_w}(M,\mathfrak{m}_1) = L^{\alpha-1}_{\tau_w}(H,\mathfrak{m}_1)$. If $\left(\frac{L^{\alpha}_{\tau_w}(H,\mathfrak{m}_1)}{L^{\alpha-1}_{\tau_w}(M,\mathfrak{m}_1)}\right)_w \cong \bigoplus\limits_{\Lambda}(S_1)_w$, where Λ is not a singleton set, then as the same way of (II), we can get that $\left(\frac{L^{\alpha}_{\tau_w}(H,\mathfrak{m}_1)}{L^{\alpha-1}_{\tau_w}(M,\mathfrak{m}_1)}\right)_w \cong \left(\frac{L^{\alpha}_{\tau_w}(H,\mathfrak{m}_1) \bigoplus (\bigoplus\limits_{i=2}^n L^{\alpha-1}_{\tau_w}(H,\mathfrak{m}_i))}{\bigoplus\limits_{i=1}^n L^{\alpha-1}_{\tau_w}(H,\mathfrak{m}_i))}\right)_w \subseteq \left(\frac{H}{L^{\alpha-1}_{\tau_w}(M)}\right)_w$. Thus $\left(\frac{H}{L^{\alpha-1}_{\tau_w}(M)}\right)_w$ is not w-simple, a contradiction.

Claim (IV). We show that $H = \bigoplus\limits_{i=1}^n L^{\alpha}_{\tau_w}(H,\mathfrak{m}_i)$.

By (I) and (III),

$$\left(\frac{H}{L^{\alpha-1}_{\tau_w}(M,\mathfrak{m}_1)}\right)_w \cong \left(\frac{L^{\alpha-1}_{\tau_w}(M)}{L^{\alpha-1}_{\tau_w}(M,\mathfrak{m}_1)}\right)_w \oplus (S_1)_w$$

$$\cong \left(\frac{L^{\alpha-1}_{\tau_w}(M)}{L^{\alpha-1}_{\tau_w}(M,\mathfrak{m}_1)}\right)_w \oplus \left(\frac{L^{\alpha}_{\tau_w}(H,\mathfrak{m}_1)}{L^{\alpha-1}_{\tau_w}(M,\mathfrak{m}_1)}\right)_w.$$

Then

$$\begin{aligned} H &= (L^{\alpha-1}_{\tau_w}(M) + L^{\alpha}_{\tau_w}(H,\mathfrak{m}_1))_w \\ &= (\bigoplus_{i=2}^{n} L^{\alpha-1}_{\tau_w}(M,\mathfrak{m}_i)) \oplus L^{\alpha}_{\tau_w}(H,\mathfrak{m}_1) \\ &\overset{\text{(II)}}{=} (\bigoplus_{i=2}^{n} L^{\alpha}_{\tau_w}(H,\mathfrak{m}_i)) \oplus L^{\alpha}_{\tau_w}(M,\mathfrak{m}_1) \\ &= \bigoplus_{i=1}^{n} L^{\alpha}_{\tau_w}(H,\mathfrak{m}_i). \end{aligned}$$

Claim (V). We show that $L^{\alpha}_{\tau_w}(M) = \bigoplus_{i=1}^{n} L^{\alpha}_{\tau_w}(M,\mathfrak{m}_i)$.

$$\begin{aligned} L^{\alpha}_{\tau_w}(M) &= (\sum_{\Lambda} H)_w \\ &= \left(\sum_{\Lambda} \bigoplus_{i=1}^{n} L^{\alpha}_{\tau_w}(H,\mathfrak{m}_i)\right)_w \\ &\subseteq \bigoplus_{i=1}^{n} \left(\sum_{\Lambda} L^{\alpha}_{\tau_w}(H,\mathfrak{m}_i)\right)_w \\ &\subseteq \bigoplus_{i=1}^{n} \left(L^{\alpha}_{\tau_w}(\sum_{\Lambda} H,\mathfrak{m}_i)\right)_w \\ &\subseteq \bigoplus_{i=1}^{n} L^{\alpha}_{\tau_w}(M,\mathfrak{m}_i) \\ &\subseteq L^{\alpha}_{\tau_w}(M). \end{aligned}$$

Therefore we have $L^{\alpha}_{\tau_w}(M) = \bigoplus_{i=1}^{n} L^{\alpha}_{\tau_w}(M,\mathfrak{m}_i)$. □

4. The relationship between τ_w-Loewy and Loewy series

In this section, we study the relationship between $L^{\alpha}_{\tau_w}(M)_{\mathfrak{m}}$ and $L_{\alpha_{R_{\mathfrak{m}}}}(M_{\mathfrak{m}})$, where M is a w-module, $\mathfrak{m} \in w\text{-}Sp(M)$, α is an ordinal, and $L_{\alpha_{R_{\mathfrak{m}}}}(M_{\mathfrak{m}})$ denotes the α^{th} term of Loewy series of $M_{\mathfrak{m}}$ as an $R_{\mathfrak{m}}$-module.

Lemma 4.1. *Let M be a w-module. Then M is w-simple if and only if there is some $\mathfrak{m} \in w$-Max(R) such that $M_{\mathfrak{m}}$ is simple as an $R_{\mathfrak{m}}$-module, and $M_{\mathfrak{n}} = 0$ for any other maximal w-ideal $\mathfrak{n}$.*

Proof. Assume that M is w-simple. Then there is a maximal w-ideal $\mathfrak{m}$ of R such that $M \cong (R/\mathfrak{m})_w$ [10, Proposition 6.5.6]. Since $R/\mathfrak{m}$ is GV-torsion-free, $M_{\mathfrak{m}} \cong ((R/\mathfrak{m})_w)_{\mathfrak{m}} \cong R_{\mathfrak{m}}/\mathfrak{m}R_{\mathfrak{m}}$. Then $M_{\mathfrak{m}}$ is simple as an $R_{\mathfrak{m}}$-module because $R_\mathfrak{m}/\mathfrak{m}R_\mathfrak{m}$ is a field. For any other maximal w-ideal $\mathfrak{n}$, note that $M_\mathfrak{n} \cong (R/\mathfrak{m})_\mathfrak{n} \cong R_\mathfrak{n}/\mathfrak{m}R_\mathfrak{n}$ and $\mathfrak{m}R_\mathfrak{n} = R_\mathfrak{n}$. Then $M_\mathfrak{n} = 0$.

For the converse, let N be a w-submodule of M. Since $M_\mathfrak{m}$ is simple as an $R_\mathfrak{m}$-module, $N_\mathfrak{m} = 0$ or $N_\mathfrak{m} = M_\mathfrak{m}$. Note that $M_\mathfrak{n} = 0$ for any other maximal w-ideal $\mathfrak{n}$. Then $N_\mathfrak{n} = 0$. Thus N is GV-torsion or $N = M$, which implies that $N = 0$ or $N = M$. Hence M is w-simple as an R-module. □

Denote by $\mathrm{soc}_{R_\mathfrak{m}}(M_\mathfrak{m})$ the socle of $M_\mathfrak{m}$ as an $R_\mathfrak{m}$-module.

Proposition 4.2. *Let M be a w-module. Then $\mathrm{soc}_{R_\mathfrak{m}}(M_\mathfrak{m})$, $(w\text{-soc}(M,\mathfrak{m}))_\mathfrak{m}$, and $(w\text{-soc}(M_\mathfrak{m},\mathfrak{m}))_\mathfrak{m}$ coincide for any $\mathfrak{m} \in w\text{-}Sp(M)$. Furthermore, if $\mathfrak{m} \in w\text{-}\mathrm{Max}(R)\backslash w\text{-}Sp(M)$, then $\mathrm{soc}_{R_\mathfrak{m}}(M_\mathfrak{m}) = (w\text{-soc}(M,\mathfrak{m}))_\mathfrak{m} = (w\text{-soc}(M_\mathfrak{m},\mathfrak{m}))_\mathfrak{m} = 0$.*

Proof. Assume that $\mathfrak{m} \in w\text{-}Sp(M)$. If M_1 is a w-submodule of M as an R-module and $M_1 \cong (R/\mathfrak{m})_w$, then $(M_1)_\mathfrak{m}$ is simple over $R_\mathfrak{m}$ by the proof of Lemma 4.1. So $(w\text{-soc}(M,\mathfrak{m}))_\mathfrak{m} \subseteq \mathrm{soc}_{R_\mathfrak{m}}(M_\mathfrak{m})$. If a submodule N of $M_\mathfrak{m}$ is simple over $R_\mathfrak{m}$, then $N \cong R_\mathfrak{m}/\mathfrak{m}R_\mathfrak{m}$. Note that $(R/\mathfrak{m})_w$ is the quotient field $R_\mathfrak{m}/\mathfrak{m}R_\mathfrak{m}$ of $R/\mathfrak{m}$ ([10, Proposition 6.5.5]). Then $N \cong (R/\mathfrak{m})_w$. Thus N is w-simple over R. So $N \subseteq w\text{-soc}(M_\mathfrak{m},\mathfrak{m})$. Since $N_\mathfrak{m} = N$, we have $N \subseteq (w\text{-soc}(M_\mathfrak{m},\mathfrak{m}))_\mathfrak{m}$, which implies that $\mathrm{soc}_{R_\mathfrak{m}}(M_\mathfrak{m}) \subseteq (w\text{-soc}(M_\mathfrak{m},\mathfrak{m}))_\mathfrak{m}$. If a submodule M' of $M_\mathfrak{m}$ is w-simple over R and $M' \cong (R/\mathfrak{m})_w$, then for any nonzero element $x \in M'$, $M' \cong (Rx)_w$. Thus there is some nonzero element $s \in R \setminus \mathfrak{m}$ such that $0 \neq xs \in M \cap M'$. So $M' = (Rxs)_w \subseteq M \cap M'$, implying that $M' \subseteq M$. Thus $M' \subseteq w\text{-soc}(M,\mathfrak{m})$. Hence $(w\text{-soc}(M_\mathfrak{m},\mathfrak{m}))_\mathfrak{m} \subseteq (w\text{-soc}(M,\mathfrak{m}))_\mathfrak{m}$.

Assume that $\mathfrak{m} \notin w\text{-}Sp(M)$. As the same way above, we can prove that $\mathrm{soc}_{R_\mathfrak{m}}(M_\mathfrak{m}) \subseteq (w\text{-soc}(M_\mathfrak{m},\mathfrak{m}))_\mathfrak{m}$. If there is some nonzero submodule M' of $M_\mathfrak{m}$ such that M' is w-simple over R and that $M' \cong (R/\mathfrak{m})_w$, then again by the same way above, we can prove that $M' \subseteq M$, and thus $\mathfrak{m} \in w\text{-}Sp(M)$, a contradiction. Thus $w\text{-soc}(M_\mathfrak{m},\mathfrak{m}) = 0$. □

Theorem 4.3. *Let M be a w-module. Then $L^{\alpha}_{\tau_w}(M,\mathfrak{m})_\mathfrak{m} = L_{\alpha_{R_\mathfrak{m}}}(M_\mathfrak{m})$ for any $\mathfrak{m} \in w\text{-}\mathrm{Max}(R)$ and any ordinal α.*

Proof. It is clear that for any $\mathfrak{m} \in w\text{-}\mathrm{Max}(R) \setminus w\text{-}Sp(M)$, $L^{\alpha}_{\tau_w}(M,\mathfrak{m})_\mathfrak{m} = L_{\alpha_{R_\mathfrak{m}}}(M_\mathfrak{m}) = 0$ by Proposition 4.2. Then we show that for any $\mathfrak{m} \in w\text{-}Sp(M)$, $L^{a}_{\tau_w}(M,\mathfrak{m})_\mathfrak{m} = L_{\alpha_{R_\mathfrak{m}}}(M_\mathfrak{m})$. By Proposition 4.2, $\mathrm{soc}_{R_\mathfrak{m}}(M_\mathfrak{m}) = (w\text{-soc}(M,\mathfrak{m}))_\mathfrak{m}$. So $(L^{1}_{\tau_w}(M,\mathfrak{m}))_\mathfrak{m} = L_{1_{R_\mathfrak{m}}}(M_\mathfrak{m})$. Assume that $(L^{\beta}_{\tau_w}(M,\mathfrak{m}))_\mathfrak{m} = L_{\beta_{R_\mathfrak{m}}}(M_\mathfrak{m})$ for any $\beta < \alpha$. If α is a limit ordinal, then $L^{\alpha}_{\tau_w}(M,\mathfrak{m})_\mathfrak{m} = (\bigcup_{\beta<\alpha} L^{\beta}_{\tau_w}(M,\mathfrak{m}))_\mathfrak{m} = \bigcup_{\beta<\alpha}(L^{\beta}_{\tau_w}(M,\mathfrak{m}))_\mathfrak{m} = \bigcup_{\beta<\alpha} L_{\beta_{R_\mathfrak{m}}}(M_\mathfrak{m}) = L_{\alpha_{R_\mathfrak{m}}}(M_\mathfrak{m})$. If α is not a limit ordinal, then

$$\frac{L_{\alpha_{R_\mathfrak{m}}}(M_\mathfrak{m})}{L_{(\alpha-1)_{R_\mathfrak{m}}}(M_\mathfrak{m})} = \mathrm{soc}_{R_\mathfrak{m}}\left(\frac{M_\mathfrak{m}}{L_{(\alpha-1)_{R_\mathfrak{m}}}(M_\mathfrak{m})}\right)$$

$$= \mathrm{soc}_{R_\mathfrak{m}}\left(\left((\frac{M}{L^{\alpha-1}_{\tau_w}(M,\mathfrak{m})})_w\right)_\mathfrak{m}\right)$$
$$= \left(w\text{-soc}\left(\left(\frac{M}{L^{\alpha-1}_{\tau_w}(M,\mathfrak{m})}\right)_w,\mathfrak{m}\right)\right)_\mathfrak{m}$$
$$= \left(\left(\frac{L^{\alpha}_{\tau_w}(M,\mathfrak{m})}{L^{\alpha-1}_{\tau_w}(M,\mathfrak{m})}\right)_w\right)_\mathfrak{m}$$
$$= \frac{(L^{\alpha}_{\tau_w}(M,\mathfrak{m}))_\mathfrak{m}}{(L^{\alpha-1}_{\tau_w}(M,\mathfrak{m}))_\mathfrak{m}}$$
$$= \frac{(L^{\alpha}_{\tau_w}(M,\mathfrak{m}))_\mathfrak{m}}{L_{(\alpha-1)_{R_\mathfrak{m}}}(M_\mathfrak{m})}.$$

Thus $L_{\alpha_{R_\mathfrak{m}}}(M_\mathfrak{m}) = (L^{\alpha}_{\tau_w}(M,\mathfrak{m}))_\mathfrak{m}$. □

Next we give the relationship between $L^{\alpha}_{\tau_w}(M)_\mathfrak{m}$ and $L_{\alpha_{R_\mathfrak{m}}}(M_\mathfrak{m})$, where $\mathfrak{m} \in w\text{-Max}(R)$ and α is an ordinal.

Theorem 4.4. *Let M be a w-module. Then $L^{\alpha}_{\tau_w}(M)_\mathfrak{m} = L^{\alpha}_{\tau_w}(M,\mathfrak{m})_\mathfrak{m} = L_{\alpha_{R_\mathfrak{m}}}(M_\mathfrak{m})$ for any $\mathfrak{m} \in w\text{-Max}(R)$ and any ordinal α.*

Proof. By Lemma 4.1, it is clear that $w\text{-soc}(M)_\mathfrak{m} = w\text{-soc}(M,\mathfrak{m})_\mathfrak{m}$. Assume that $L^{\beta}_{\tau_w}(M)_\mathfrak{m} = L^{\beta}_{\tau_w}(M,\mathfrak{m})_\mathfrak{m}$ for any $\beta < \alpha$. If α is a limit ordinal, then $(L^{\alpha}_{\tau_w}(M))_\mathfrak{m} = (\bigcup_{\beta<\alpha} L^{\beta}_{\tau_w}(M))_\mathfrak{m} = \bigcup_{\beta<\alpha}(L^{\beta}_{\tau_w}(M))_\mathfrak{m} = \bigcup_{\beta<\alpha}(L^{\beta}_{\tau_w}(M,\mathfrak{m}))_\mathfrak{m} = (\bigcup_{\beta<\alpha} L^{\beta}_{\tau_w}(M,\mathfrak{m}))_\mathfrak{m} = L^{\alpha}_{\tau_w}(M,\mathfrak{m})_\mathfrak{m}$. If α is not a limit ordinal, then

$$w\text{-soc}\left(\left(\frac{M}{L^{\alpha-1}_{\tau_w}(M)}\right)_w\right)_\mathfrak{m} = \left(\left(\frac{L^{\alpha}_{\tau_w}(M)}{L^{\alpha-1}_{\tau_w}(M)}\right)_w\right)_\mathfrak{m}$$
$$= \frac{L^{\alpha}_{\tau_w}(M)_\mathfrak{m}}{L^{\alpha-1}_{\tau_w}(M)_\mathfrak{m}}$$
$$= \frac{L^{\alpha}_{\tau_w}(M)_\mathfrak{m}}{L^{\alpha-1}_{\tau_w}(M,\mathfrak{m})_\mathfrak{m}}$$
$$= \left(\frac{L^{\alpha}_{\tau_w}(M)}{L^{\alpha-1}_{\tau_w}(M,\mathfrak{m})}\right)_\mathfrak{m}.$$

Again by Lemma 4.1, $w\text{-soc}\left(\left(\frac{M}{L^{\alpha-1}_{\tau_w}(M)}\right)_w\right)_\mathfrak{m} = w\text{-soc}\left(\left(\frac{M}{L^{\alpha-1}_{\tau_w}(M)}\right)_w,\mathfrak{m}\right)_\mathfrak{m}$.

Note that $w\text{-soc}\left(\left(\frac{M}{L^{\alpha-1}_{\tau_w}(M)}\right)_w,\mathfrak{m}\right) = \left(\frac{L^{\alpha}_{\tau_w}(M,\mathfrak{m})}{L^{\alpha-1}_{\tau_w}(M,\mathfrak{m})}\right)_w$.

Then $\left(\frac{L^{\alpha}_{\tau_w}(M)}{L^{\alpha-1}_{\tau_w}(M,\mathfrak{m})}\right)_\mathfrak{m} = \left(\frac{L^{\alpha}_{\tau_w}(M,\mathfrak{m})}{L^{\alpha-1}_{\tau_w}(M,\mathfrak{m})}\right)_\mathfrak{m}$. Thus $L^{\alpha}_{\tau_w}(M)_\mathfrak{m} = L^{\alpha}_{\tau_w}(M,\mathfrak{m})_\mathfrak{m}$. By Theorem 4.3, $L^{\alpha}_{\tau_w}(M)_\mathfrak{m} = L^{\alpha}_{\tau_w}(M,\mathfrak{m})_\mathfrak{m} = L_{\alpha_{R_\mathfrak{m}}}(M_\mathfrak{m})$. □

Lemma 4.5. *Let M be a w-module and $\mathfrak{m} \in w\text{-}Sp(M)$. Then $L^{\alpha}_{\tau_w}(M,\mathfrak{m})_\mathfrak{n} = 0$ for any other maximal w-ideal $\mathfrak{n}$ and any ordinal α.*

Proof. By the proof of Lemma 4.1, $w\text{-soc}(M,\mathfrak{m})_\mathfrak{n} = 0$. Assume that for any ordinal $\beta < \alpha$, $L^{\beta}_{\tau_w}(M,\mathfrak{m})_\mathfrak{n} = 0$. If α is a limit ordinal, then $L^{\alpha}_{\tau_w}(M,\mathfrak{m})_\mathfrak{n} =$

$(\bigcup_{\beta<\alpha} L^{\beta}_{\tau_w}(M,\mathfrak{m}))_{\mathfrak{n}} = \bigcup_{\beta<\alpha}(L^{\beta}_{\tau_w}(M,\mathfrak{m}))_{\mathfrak{n}} = 0$. If α is not a limit ordinal, then $0 = \left(\left(\frac{L^{\alpha}_{\tau_w}(M,\mathfrak{m})}{L^{\alpha-1}_{\tau_w}(M,\mathfrak{m})}\right)_w\right)_{\mathfrak{n}} = \frac{L^{\alpha}_{\tau_w}(M,\mathfrak{m})_{\mathfrak{n}}}{L^{\alpha-1}_{\tau_w}(M,\mathfrak{m})_{\mathfrak{n}}}$. Thus $L^{\alpha}_{\tau_w}(M,\mathfrak{m})_{\mathfrak{n}} = 0$. □

Remark 4.6. In Theorem 4.4, if w-$Sp(M)$ is finite, then its proof is simple by Lemma 4.1. Indeed, if w-$Sp(M)$ is finite, then $L^{\alpha}_{\tau_w}(M) = \bigcup_{\mathfrak{m}\in w\text{-}Sp(M)} L^{\alpha}_{\tau_w}(M,\mathfrak{m})$ by Theorem 3.9. Then for any $\mathfrak{n} \in w$-$Sp(M)$,

$$L^{\alpha}_{\tau_w}(M)_{\mathfrak{n}} = \bigcup_{\mathfrak{m}\in w\text{-}Sp(M)} L^{\alpha}_{\tau_w}(M,\mathfrak{m})_{\mathfrak{n}} = L^{\alpha}_{\tau_w}(M,\mathfrak{n})_{\mathfrak{n}}$$

by Lemma 4.5. For any $\mathfrak{n} \in w\text{-Max}(R) \setminus w\text{-}Sp(M)$,

$$L^{\alpha}_{\tau_w}(M)_{\mathfrak{n}} = \bigcup_{\mathfrak{m}\in w\text{-}Sp(M)} L^{\alpha}_{\tau_w}(M,\mathfrak{m})_{\mathfrak{n}} = 0$$

again by Lemma 4.5.

5. τ_w-Loewy length

Recall some concepts from [9]. Let M and N be R-modules and let $f \in \mathrm{Hom}_R(M,N)$. If for each maximal w-ideal $\mathfrak{m}$ of R, $f_{\mathfrak{m}} : M_{\mathfrak{m}} \to N_{\mathfrak{m}}$ is a monomorphism (resp., an epimorphism, an isomorphism), we call f a *w-monomorphism* (resp., a *w-epimorphism*, a *w-isomorphism*). A module M is said to be *of finite type* if there exist a finitely generated free module F and a w-epimorphism $f : F \to M$ ([9, Definition 1.3]). Obviously, a finitely generated R-module is of finite type. A GV-torsion-free module M is of finite type if and only if there exists a finitely generated submodule B of M such that $M_w = B_w$ ([9, Proposition 1.2]).

Let τ be a torsion theory. A *τ-composition series* for an R-module M is a chain $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$ of submodules of M such that M_{k+1}/M_k is τ-simple for $k = 0, 1, \ldots, n-1$. Here n is referred to as the *length of the τ-composition series*, and if M has a τ-composition series, we say that M has *τ-finite length* ([1, Definition 3.1.1]). For the w-case, if M is a w-module, M has *w-finite length* if there is a chain $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$ of submodules of M such that M_{k+1}/M_k is τ_w-simple for $k = 0, 1, \ldots, n-1$. It is easy to prove that M has w-finite length if and only if there is a chain $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$ of w-submodules of M such that $(M_{k+1}/M_k)_w$ is w-simple for $k = 0, 1, \ldots, n-1$. An R-module M is said to be *τ-Artinian* (resp., *τ-Noetherian*) if M satisfies the descending (resp., ascending) chain condition on τ-pure submodules of M ([1, Definition 2.4.1, Definition 2.3.1]). Also, a w-module M is called *w-Artinian* if M satisfies the descending chain condition on w-submodules of M ([10, Definition 6.9.1]). An R-module M is called *w-Noetherian* if every submodule of M is of finite type ([10, Definition 6.8.9]). Then it is known that a w-module M is w-Noetherian if and only if M satisfies the ascending chain condition on w-submodules of M ([10, Theorem 6.8.4]). By Lemma 2.1(1), if M is a w-module, then M is w-Artinian (resp.,

w-Noetherian) if and only if M is τ_w-Artinian (resp., τ_w-Noetherian). Then a w-module M has w-finite length if and only if M is w-Noetherian and w-Artinian by [1, Proposition 3.1.3].

Proposition 5.1. *Let M be a w-module. If M is w-semisimple, then the following statements are equivalent.*

(1) *M is w-Noetherian.*
(2) *M is w-Artinian.*
(3) *M has w-finite length.*
(4) *M is of finite type.*

Proof. Let $M = \bigoplus_{i\in\Gamma} M_i$, where each M_i is a w-simple submodule of M.

(1) $\Rightarrow$ (4) This follows from the fact that w-Noetherian modules are of finite type ([10, Theorem 6.9.5]).

(4) $\Rightarrow$ (3) Because M is of finite type, Γ is a finite set. Thus it is easy to see that M has w-finite length.

(3) $\Rightarrow$ (1) and (3) $\Rightarrow$ (2) both follow from [1, Proposition 3.1.3].

(2) $\Rightarrow$ (3) It suffices to show that Γ is a finite set. If not, then M has a w-submodule $N = \bigoplus_{i=1}^{\infty} M_i$. Thus $\bigoplus_{i=1}^{\infty} M_i \supsetneq \bigoplus_{i=2}^{\infty} M_i \supsetneq \cdots$ is a strict descending chain of w-submodules of M, a contradiction to M being w-Artinian. □

Recall that an R-module M has *finite length* if M has composition series, i.e., there is a chain $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$ of submodules of M such that M_{k+1}/M_k is simple for $k = 0, 1, \dots, n-1$.

Lemma 5.2 ([10, Theorem 6.2.21])**.** *Let $\mathfrak{m}$ be a maximal w-ideal of R, $\overline{R} = R/\mathfrak{m}$, and M be an $\overline{R}$-module. Then M is a w-module as an R-module if and only if M is a torsion-free injective module as an $\overline{R}$-module.*

Lemma 5.3. *If a* GV*-torsion-free R-module M is semisimple, then M is a w-module.*

Proof. It suffices to show that M is a w-module if M is GV-torsion-free and simple. If M is simple, then $M = Rx$ for any nonzero element in M and $M \cong R/\mathrm{ann}_R(x)$. Thus $\mathfrak{m} := \mathrm{ann}_R(x)$ is a maximal ideal. Then $R/\mathfrak{m}$ as an $R/\mathfrak{m}$-module is torsion-free and injective, and hence so is M as an $R/\mathfrak{m}$-module. Since M is GV-torsion-free, then it is easy to prove that $\mathfrak{m}$ is w-ideal. By Lemma 5.2, M as an R-module is a w-module. □

Proposition 5.4. *Let M be a w-module. Then the following statements hold.*

(1) *$Sp(M) \subseteq w$-$Sp(M)$.*
(2) *$L_\alpha(M)$ is a w-module and $L_\alpha(M) \subseteq L^{\alpha}_{\tau_w}(M)$ for any ordinal α.*

Proof. (1) Note that simple submodules of M are w-simple by Lemma 5.3. Thus it is easy to show that $Sp(M) \subseteq w$-$Sp(M)$.

(2) Note that $\mathrm{soc}(M)$ is w-semisimple by Lemma 5.3. Then $\mathrm{soc}(M) \subseteq w$-$\mathrm{soc}(M)$. Since M is a w-module, $L_2(M)/\mathrm{soc}(M)$ is GV-torsion-free, and thus

it is w-semisimple by its semisimple property. By the exact sequence $0 \to \mathrm{soc}(M) \to L_2(M) \to L_2(M)/\mathrm{soc}(M) \to 0$, $L_2(M)$ is a w-module. Assume that $L_\beta(M)$ is a w-module for any ordinal $\beta < \alpha$. If α is a limit ordinal, then $L_\alpha(M) = \bigcup_{\beta<\alpha} L_\beta(M)$. Since $L_\beta(M)$ is a w-module, so is $L_\alpha(M)$. If α is not a limit ordinal, then we have the exact sequence $0 \to L_{\alpha-1}(M) \to L_\alpha(M) \to L_\alpha(M)/L_{\alpha-1}(M) \to 0$. Since $L_\alpha(M)/L_{\alpha-1}(M)$ is GV-torsion-free and semisimple, it is a w-module again by Lemma 5.3. Then $L_\alpha(M)$ is a w-module. Thus for any $\mathfrak{m} \in Sp(M)$, $L_\alpha(M, \mathfrak{m})$ is a w-module. Then for any ordinal α, $L_\alpha(M, \mathfrak{m}) = L^\alpha_{\tau_w}(M, \mathfrak{m})$ by (1). Thus $L_\alpha(M) \subseteq L^\alpha_{\tau_w}(M)$ by Theorem 3.9. □

Theorem 5.5. *Let M be a w-module. Assume that there is a chain $(*)$ $0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M$ with M_n/M_{n-1} being simple, where n is a positive integer. Then M has w-finite length if and only if M has finite length.*

Proof. Assume that M has w-finite length. If M does not have finite length, then the chain $(*)$ is not stationary. Note that M_1 is GV-torsion-free and simple. Then M_1 is a w-module by Lemma 5.3. Thus M_1 is a w-simple module and M_2/M_1 is GV-torsion-free. Since M_2/M_1 is simple, M_2/M_1 is a w-module, and thus it is w-simple. Also M_2 is a w-module. As the same way, we can get that for each positive integer n, both M_n and M_n/M_{n-1} are w-modules and M_n/M_{n-1} is w-simple. So by the chain $(*)$, M does not have w-finite length, a contradiction. By the proof above, we can easily get that if M has finite length, then it has w-finite length. □

Note that if M is a GV-torsion-free module with a w-submodule N, then M/N is M/N is GV-torsion-free, since M_w/N is GV-torsion-free and M/N is a submodule of M_w/N. In the proof of Theorem 5.5, we can get the following corollary.

Corollary 5.6. *Let M be* GV*-torsion-free. If M has finite length, then M is a w-module.*

The proof of the following proposition is essentially the same as that of [8, Theorem 3.3]. For completeness, we will give its proof.

Proposition 5.7. *Let M be a w-module, s be a positive integer, and $\mathfrak{m} \in w$-$Sp(M)$. If w-$\mathrm{soc}(M, \mathfrak{m})$ and $(L^{s+1}_{\tau_w}(M, \mathfrak{m})/L^s_{\tau_w}(M, \mathfrak{m}))_w$ have w-finite length, then so does $(L^{n+1}_{\tau_w}(M, \mathfrak{m})/L^n_{\tau_w}(M, \mathfrak{m}))_w$ for all integers $n > s$.*

Proof. It suffices to show that $(L^{s+1}_{\tau_w}(M, \mathfrak{m})/L^s_{\tau_w}(M, \mathfrak{m}))_w$ is w-Noetherian by Proposition 5.1. Assume that $c_1, c_2, \ldots, c_n$ are elements of $L^{s+1}_{\tau_w}(M, \mathfrak{m})$ such that $(R\overline{c_1} + R\overline{c_2} + \cdots + R\overline{c_n})_w = (L^{s+1}_{\tau_w}(M, \mathfrak{m})/L^s_{\tau_w}(M, \mathfrak{m}))_w$, and we can also require that the image $R\overline{c_i}$ of Rc_i in $L^{s+1}_{\tau_w}(M, \mathfrak{m})/L^s_{\tau_w}(M, \mathfrak{m})$ satisfies that $(R\overline{c_i})_w$ is w-simple. Set $K = \bigcap_{i=1}^n (0 : c_i)$. It is easy to prove that the natural homomorphism $f : R \to \bigoplus_{i=1}^n R/(0 : c_i)$ is an epimorphism and $\ker(f) = K$. Thus R/K can be regarded as a submodule of $\bigoplus_{i=1}^n R/(0 : c_i)$. For each i,

$R/(0:c_i) \cong Rc_i$ is a submodule of M, and so w-soc$((R/(0:c_i))_w, \mathfrak{m}) \subseteq w$-soc$(M, \mathfrak{m})$. Since w-soc$(M, \mathfrak{m})$ has w-finite length, again by Proposition 5.1, w-soc$(M, \mathfrak{m})$ is w-Noetherian, and thus so is w-soc$((R/(0:c_i))_w, \mathfrak{m})$. Note that w-soc$(\bigoplus_{i=1}^n (R/(0:c_i))_w, \mathfrak{m}) = \bigoplus_{i=1}^n w$-soc$((R/(0:c_i))_w, \mathfrak{m})$ by Proposition 3.4. Thus w-soc$((\bigoplus_{i=1}^n R/(0:c_i))_w, \mathfrak{m})$ is w-Noetherian, and hence so is w-soc$((R/K)_w, \mathfrak{m})$. Since $c_i \in L_{\tau_w}^{s+1}(M, \mathfrak{m})$, we have $c_i \mathfrak{m}^{s+1} = 0$ by Proposition 3.6. Then $\mathfrak{m}^{s+1} \subseteq (0:c_i)$, and so $((\mathfrak{m}^s + K)/K)\mathfrak{m} = (\mathfrak{m}^{s+1} + K)/K = 0$. Then $(\mathfrak{m}^s + K)/K$ is an $R/\mathfrak{m}$-module. Thus $((\mathfrak{m}^s + K)/K)_w$ is an $(R/\mathfrak{m})_w$-module (actually, $(R/\mathfrak{m})_w$ is the quotient field of $R/\mathfrak{m}$ by the proof of [10, Proposition 6.5.5]). So $((\mathfrak{m}^s + K)/K)_w$ is isomorphic to some copies of $(R/\mathfrak{m})_w$. Thus $((\mathfrak{m}^s + K)/K)_w \subseteq w$-soc$((R/K)_w, \mathfrak{m})$. Then $((\mathfrak{m}^s + K)/K)_w$ is of w-finite type. So there exist elements $y_1, y_2, \ldots, y_n$ in $\mathfrak{m}^s$ such that $(\mathfrak{m}^s + K)_w = (y_1 R + y_2 R + \cdots + y_n R + K)_w$. Thus there is a finite number of elements of $\mathfrak{m}$, say $z_1, z_2, \ldots, z_t$, such that $(\mathfrak{m}^s + K)_w = ((z_1 R + z_2 R + \cdots + z_t R)^s + K)_w$. Then

$$(\mathfrak{m}^s)_w = ((z_1 R + z_2 R + \cdots + z_t R)^s + (K \cap \mathfrak{m}^s))_w. \tag{3}$$

Because $z_i \in \mathfrak{m}$, $z_i L_{\tau_w}^{s+1}(M, \mathfrak{m}) \subseteq L_{\tau_w}^{s}(M, \mathfrak{m})$. Set $U_j := L_{\tau_w}^{j+1}(M, \mathfrak{m})/L_{\tau_w}^{j}(M, \mathfrak{m})$ for any positive integer j. Then multiplication by the element z_i induces the R-homomorphism

$$z_i^* : U_{s+1} \to U_s, i = 1, 2, \ldots, t.$$

Set $T_i := U_{s+1}/\ker z_i^*$. Then T_i is a submodule of U_s. Thus $(T_i)_w$ is w-Noetherian. Set $B = \bigcap_{i=1}^t \ker z_i^*$. Then U_{s+1}/B is isomorphic to a submodule of $\bigoplus_{i=1}^t T_i$. Hence $(U_{s+1}/B)_w$ is w-Noetherian.

If $B = 0$, then $(L_{\tau_w}^{s+2}(M, \mathfrak{m})/L_{\tau_w}^{s+1}(M, \mathfrak{m}))_w$ is w-Noetherian.

Next we show that B must be zero. If not, there is some nonzero element $c \in L_{\tau_w}^{s+2}(M, \mathfrak{m}) \setminus L_{\tau_w}^{s+1}(M, \mathfrak{m})$ and the image of c in $L_{\tau_w}^{s+2}(M, \mathfrak{m})/L_{\tau_w}^{s+1}(M, \mathfrak{m})$ belongs to B. By (3), we can get that

$$(c\mathfrak{m}^{s+1})_w = (c\mathfrak{m}(z_1 R + z_2 R + \cdots + z_t R)^s + c\mathfrak{m}(K \cap \mathfrak{m}^s))_w.$$

Note that $c\mathfrak{m}(z_1 R + z_2 R + \cdots + z_t R)^s$ is generated by elements of the form $u := czz_{i_1} z_{i_2} \cdots z_{i_s}$, where $z \in \mathfrak{m}$, each $z_{i_j} \in \{z_1, z_2, \ldots, z_t\}$, $j = 1, 2, \ldots, s$. Note that $cz_{i_1} \in L_{\tau_w}^{s}(M, \mathfrak{m})$, since $c + L_{\tau_w}^{s+1}(M, \mathfrak{m}) \in \ker z_{i_1}^*$. Furthermore $zz_{i_2} \cdots z_{i_s} \in \mathfrak{m}^s$, which implies that $u = 0$. So $c\mathfrak{m}(z_1 R + z_2 R + \cdots + z_t R)^s = 0$. On the other hand, $c\mathfrak{m} \subseteq L_{\tau_w}^{s+1}(M, \mathfrak{m})$. By the choice of the c_i's, we have that $L_{\tau_w}^{s+1}(M, \mathfrak{m}) = (c_1 R + c_2 R + \cdots + c_n R + L_{\tau_w}^{s}(M, \mathfrak{m}))_w$. Then $(c_1 R + c_2 R + \cdots + c_n R + L_{\tau_w}^{s}(M, \mathfrak{m}))(K \cap \mathfrak{m}^s) = 0$, since K annihilates the c_i's and $\mathfrak{m}^s$ annihilates $L_{\tau_w}^{s}(M, \mathfrak{m})$. So $(L_{\tau_w}^{s+1}(M, \mathfrak{m})(K \cap \mathfrak{m}^s))_w = (c_1 R + c_2 R + \cdots + c_n R + L_{\tau_w}^{s}(M, \mathfrak{m}))(K \cap \mathfrak{m}^s))_w = 0$. So $(c\mathfrak{m}(K \cap \mathfrak{m}^s))_w = 0$. Consequently $(c\mathfrak{m}^{s+1})_w = 0$, implying that $c\mathfrak{m}^{s+1} = 0$. Thus $c \in L_{\tau_w}^{s+1}(M, \mathfrak{m})$, which contradicts our choice of c. □

Corollary 5.8. *Let M be a w-module and $\mathfrak{m} \in w$-$Sp(M)$. If w-soc$(M, \mathfrak{m})$ and $(L_{\tau_w}^{2}(M, \mathfrak{m})/L_{\tau_w}^{1}(M, \mathfrak{m}))_w$ have w-finite length, then $(L_{\tau_w}^{n+1}(M, \mathfrak{m})/L_{\tau_w}^{n}(M, \mathfrak{m}))_w$*

has w-finite length for all integers $n > 1$. That is, if the 0^{th} and 1^{st} τ_w-$\mathfrak{m}$-Loewy invariants are finite, then so are all positive integral τ_w-$\mathfrak{m}$-Loewy invariants.

Theorem 5.9. *Let M be a w-module and let s be a positive integer. If the 0^{th} and s^{th} τ_w-Loewy invariants are finite, then so is the n^{th} τ_w-Loewy invariant for all integers $n > s$.*

Proof. Note that $w\text{-soc}(M) = \bigoplus\limits_{\mathfrak{m}\in w\text{-}Sp(M)} w\text{-soc}(M,\mathfrak{m})$ by Lemma 3.5(4). Then $w\text{-}Sp(M)$ is finite and the 0^{th} τ_w-$\mathfrak{m}$-Loewy invariant is finite by Proposition 5.1, since the 0^{th} τ_w-Loewy invariant is finite, where $\mathfrak{m} \in w\text{-}Sp(M)$. By Theorem 3.9,

$$\left(\frac{L^{s+1}_{\tau_w}(M)}{L^{s}_{\tau_w}(M)}\right)_w = \left(\frac{\bigoplus\limits_{\mathfrak{m}\in w\text{-}Sp(M)} L^{s+1}_{\tau_w}(M,\mathfrak{m})}{\bigoplus\limits_{\mathfrak{m}\in w\text{-}Sp(M)} L^{s}_{\tau_w}(M,\mathfrak{m})}\right)_w = \bigoplus_{\mathfrak{m}\in w\text{-}Sp(M)} \left(\frac{L^{s+1}_{\tau_w}(M,\mathfrak{m})}{L^{s}_{\tau_w}(M,\mathfrak{m})}\right)_w.$$

Then it follows from the assumption that the s^{th} τ_w-Loewy invariant is finite. Hence the s^{th} τ_w-$\mathfrak{m}$-Loewy invariant is finite, and thus so is the n^{th} τ_w-$\mathfrak{m}$-Loewy invariant for all integers $n > s$ by Proposition 5.7. Again by Theorem 3.9,

$$\left(\frac{L^{n+1}_{\tau_w}(M)}{L^{n}_{\tau_w}(M)}\right)_w = \bigoplus_{\mathfrak{m}\in w\text{-}Sp(M)} \left(\frac{L^{n+1}_{\tau_w}(M,\mathfrak{m})}{L^{n}_{\tau_w}(M,\mathfrak{m})}\right)_w.$$

Then the n^{th} τ_w-Loewy invariant is finite since $w\text{-}Sp(M)$ is finite. □

Corollary 5.10. *Let M be a w-module. If the 0^{th} and 1^{st} τ_w-Loewy invariants are finite, then so are all positive integral τ_w-Loewy invariants.*

Actually, the next result shows that all the integral τ_w-Loewy invariants are relative to $L^2_{\tau_w}(M)$.

Proposition 5.11. *Let M be a w-module. Then the following statements hold.*

(1) *Let $\mathfrak{m} \in w\text{-}Sp(M)$. Then $L^2_{\tau_w}(M,\mathfrak{m})$ has w-finite length if and only if all its integral τ_w-$\mathfrak{m}$-Loewy invariants are finite.*
(2) *If $L^2_{\tau_w}(M)$ has w-finite length, then all its integral τ_w-Loewy invariants are finite.*
(3) *If M has w-finite length if and only if M is a τ_w-Loewy module, its τ_w-Loewy length is finite, and all its τ_w-Loewy invariants are finite.*

Proof. (1) ($\Rightarrow$) Note that $L^2_{\tau_w}(M,\mathfrak{m})$ has w-finite length if and only if $L^2_{\tau_w}(M,\mathfrak{m})$ is a w-Noetherian and w-Artinian module by [10, Exercise 6.48]. Since the sequence $0 \to L^1_{\tau_w}(M,\mathfrak{m}) \to L^2_{\tau_w}(M,\mathfrak{m}) \to \left(\frac{L^2_{\tau_w}(M,\mathfrak{m})}{L^1_{\tau_w}(M,\mathfrak{m})}\right)_w \to 0$ is w-exact, i.e., the localization of the sequence at every maximal w-ideal of R is exact, it follows that both $L^1_{\tau_w}(M,\mathfrak{m})$ and $\left(\frac{L^2_{\tau_w}(M,\mathfrak{m})}{L^1_{\tau_w}(M,\mathfrak{m})}\right)_w$ are w-Noetherian and w-Artinian modules by [10, Theorems 6.8.2 and 6.9.4]. Then they have w-finite length again by [10, Exercise 6.48]. Thus the 0^{th} and 1^{st} τ_w-$\mathfrak{m}$-Loewy invariants are finite. Therefore, all integral τ_w-$\mathfrak{m}$-Loewy invariants are finite by Corollary 5.8.

($\Leftarrow$) If the 1^{st} τ_w-$\mathfrak{m}$-Loewy invariant is finite, then $\left(\frac{L^2_{\tau_w}(M,\mathfrak{m})}{L^1_{\tau_w}(M,\mathfrak{m})}\right)_w$ has w-finite length. Since the 0^{th} τ_w-$\mathfrak{m}$-Loewy invariant is finite, we get that $L^1_{\tau_w}(M,\mathfrak{m})$ has w-finite length. Thus by the same way above, $L^2_{\tau_w}(M,\mathfrak{m})$ has w-finite length.

(2) This follows by the same way of the proof of (1).

(3) Assume that M has w-finite length. Then M is w-Artinian, and so M is a τ_w-Loewy module. Note that the w-finite length of M is not less than the τ_w-Loewy one. Then the τ_w-Loewy length of M is finite. Thus all integral τ_w-Loewy invariants of M are finite, since $L^2_{\tau_w}(M)$ has w-finite length.

Conversely, assume that the τ_w-Loewy length of M is n, and the t^{th} τ_w-Loewy invariant of M is m_t, where n and t are positive integers. Let

$$(L^{t+1}_{\tau_w}(M)/L^t_{\tau_w}(M))_w = \bigoplus_{j=1}^{m_t}(M_{tj}/L^t_{\tau_w}(M)))_w,$$

where $(M_{t_j}/L^t_{\tau_w}(M,\mathfrak{m})))_w$ is w-simple. Then there is a chain of submodules of M

$$\begin{aligned}
0 &\subseteq (M_{01})_w \subseteq (M_{01})_w \oplus (M_{02})_w \subseteq \cdots \subseteq \bigoplus_{j=1}^{m_0}(M_{0j})_w = w\text{-soc}(M) \\
&\subseteq (M_{11})_w \subseteq (M_{11}+M_{12})_w \subseteq \cdots \subseteq (M_{11}+\cdots+M_{1m_1})_w = L^2_{\tau_w}(M) \\
&\subseteq (M_{(n-1)1})_w \subseteq (M_{(n-1)1}+M_{(n-1)2})_w \\
&\subseteq \cdots \subseteq (M_{(n-1)1}+\cdots+M_{(n-1)m_{n-1}})_w \\
&= L^n_{\tau_w}(M) = M.
\end{aligned}$$

Obviously, the w-closure of factor modules of adjacent modules in the chain above is w-simple. Thus M has w-finite length. □

Theorem 5.12. *Let M be a w-module, $\mathfrak{m} \in w\text{-}Sp(M)$, and ω be the first infinite ordinal. If M is a τ_w-$\mathfrak{m}$-Loewy module and all its integral τ_w-$\mathfrak{m}$-Loewy invariants are finite, then $M = L^{\omega}_{\tau_w}(M,\mathfrak{m})$.*

Proof. Let $0 \neq x \in M$. If $\mathfrak{n} \in w\text{-Max}(R)$ with $\mathfrak{n} \neq \mathfrak{m}$, then $(Rx)_{\mathfrak{n}} = 0$ by Lemma 4.5. Note that $(Rx)_{\mathfrak{m}} \cong (R/\text{ann}_R(x))_{\mathfrak{m}} \cong R_{\mathfrak{m}}/(\text{ann}_R(x))_{\mathfrak{m}}$. If $\text{ann}_R(x) \nsubseteq \mathfrak{m}$, then $(\text{ann}_R(x))_{\mathfrak{m}} = R_{\mathfrak{m}}$. Thus $(Rx)_{\mathfrak{m}} = 0$.

Assume that $\text{ann}_R(x) \subseteq \mathfrak{m}$. By Theorem 4.4,

$$(L^2_{\tau_w}(M,\mathfrak{m}) \cap (Rx)_w)_{\mathfrak{m}} = L_{2_{R_{\mathfrak{m}}}}(M_{\mathfrak{m}}) \cap (Rx)_{\mathfrak{m}} = L_{2_{R_{\mathfrak{m}}}}(M_{\mathfrak{m}}, \mathfrak{m}R_{\mathfrak{m}}) \cap (Rx)_{\mathfrak{m}}.$$

Note the ring $T := R_{\mathfrak{m}}/\text{ann}_R(x)_{\mathfrak{m}}$ is $R_{\mathfrak{m}}$-isomorphic to a submodule of $M_{\mathfrak{m}}$. Since $R_{\mathfrak{m}}$- and T-module structure of T coincide,

$$L_{2_{R_{\mathfrak{m}}}}(M_{\mathfrak{m}}, \mathfrak{m}R_{\mathfrak{m}}) \cap (Rx)_{\mathfrak{m}} = L_{2_T}(T, \mathfrak{m}R_{\mathfrak{m}}/(\text{ann}_R(x))_{\mathfrak{m}}).$$

By [7, Proposition 3.3], $L_{2_T}(T, \mathfrak{m}R_{\mathfrak{m}}/\text{ann}_R(x)_{\mathfrak{m}}) = (0 :_T (\mathfrak{m}R_{\mathfrak{m}}/(\text{ann}_R(x))_{\mathfrak{m}})^2)$. Since $L^2_{\tau_w}(M,\mathfrak{m})$ has w-finite length by Proposition 5.11, $L^2_{\tau_w}(M,\mathfrak{m}) \cap (Rx)_w$

has w-finite length. Hence it is easy to prove that $(L^2_{\tau_w}(M,\mathfrak{m}) \cap (Rx)_w)_{\mathfrak{m}}$ has finite length by Lemma 4.1. Thus $(0 :_T (\mathfrak{m}R_{\mathfrak{m}}/(\mathrm{ann}_R(x))_{\mathfrak{m}})^2)$ has finite length. Next we show that $\mathfrak{m}R_{\mathfrak{m}}/(\mathrm{ann}_R(x))_{\mathfrak{m}}$ is a T-nilpotent ideal of $R_{\mathfrak{m}}$. Let $\overline{x_1}, \overline{x_2}, \dots$ be a sequence of elements of $\mathfrak{m}R_{\mathfrak{m}}/\mathrm{ann}_R(x)$, where $x_i \in \mathfrak{m}R_{\mathfrak{m}}$. Set $x_i = \frac{t_i}{s_i}$, where $t_i \in \mathfrak{m}$ and $s_i \in R \setminus \mathfrak{m}$. Since $x \in L_{\tau_w}(M,\mathfrak{m})$, there is a positive integer n such that $xt_1t_2\cdots t_n = 0$ by Proposition 3.6. So $\frac{x}{1}\frac{t_1}{s_1}\frac{t_2}{s_2}\cdots\frac{t_n}{s_n} = 0$. Then $\frac{t_1}{s_1}\frac{t_2}{s_2}\cdots\frac{t_n}{s_n} \in \mathrm{ann}_{R_{\mathfrak{m}}}(\frac{x}{1}) = (\mathrm{ann}_R(x))_{\mathfrak{m}}$, which implies that $\overline{x_1 x_2}\cdots\overline{x_n} = 0$. Thus $\mathfrak{m}R_{\mathfrak{m}}/(\mathrm{ann}_R(x))_{\mathfrak{m}}$ is T-nilpotent over $R_{\mathfrak{m}}$. By [8, Corollary 3.7], T has finite length. So $(Rx)_{\mathfrak{m}}$ has finite length. Thus $(Rx)_{\mathfrak{m}} \subseteq L_{\omega_{R_{\mathfrak{m}}}}(M_{\mathfrak{m}})$. Again by Theorem 4.4, $L_{\omega_{R_{\mathfrak{m}}}}(M_{\mathfrak{m}}) = L^{\omega}_{\tau_w}(M,\mathfrak{m})_{\mathfrak{m}}$. Thus for any maximal w-ideal $\mathfrak{n}$, $(Rx)_{\mathfrak{n}} \subseteq L^{\omega}_{\tau_w}(M,\mathfrak{m})_{\mathfrak{n}}$. So $(Rx)_w \subseteq L^{\omega}_{\tau_w}(M,\mathfrak{m})$. Hence $M = L^{\omega}_{\tau_w}(M,\mathfrak{m})$. □

Corollary 5.13. *Let M be a w-module. If M is a τ_w-Loewy module and all its integral τ_w-Loewy invariants are finite, then $M = L^{\omega}_{\tau_w}(M)$, i.e., the τ_w-Loewy length of M is at most ω.*

Proof. Since 0^{th} τ_w-Loewy length is finite, w-soc(M) is of w-finite type. Thus w-$Sp(M)$ is finite. Note that all integral τ_w-$\mathfrak{m}$-Loewy invariants of M are finite by assumption. Thus $L_{\tau_w}(M,\mathfrak{m}) = L^{\omega}_{\tau_w}(M,\mathfrak{m})$ by Theorem 5.12. Then

$$\begin{aligned} M &= L_{\tau_w}(M) \\ &= \bigoplus_{\mathfrak{m}\in w\text{-}Sp(M)} L_{\tau_w}(M,\mathfrak{m}) \\ &= \bigoplus_{\mathfrak{m}\in w\text{-}Sp(M)} L^{\omega}_{\tau_w}(M,\mathfrak{m}) \\ &= L^{\omega}_{\tau_w}(M) \end{aligned}$$

by Theorem 3.9. □

We say that an R-module M is a *w-locally Artinian module* if $M_{\mathfrak{m}}$ is Artinian over $R_{\mathfrak{m}}$ for any $\mathfrak{m} \in w$-Max(R). A theorem of Facchini states that an R-module M is Artinian if and only if it is Loewy and all its Loewy invariants are finite ([3, Theorem 2.7]). Now we give the w-version of this result.

Theorem 5.14. *Let M be a w-module. Then the following statements are equivalent.*

(1) *M is w-Artinian.*
(2) *M is a w-locally Artinian module and w-$Sp(M)$ is finite.*
(3) *M is a τ_w-Loewy R-module and all its τ_w-Loewy invariants are finite.*

Proof. (1) ⇒ (2) It suffices to show that w-$Sp(M)$ is finite. If M is w-Artinian, then w-soc(M) is w-Artinian. So by Proposition 5.1 w-soc(M) has a finite number of direct summands. Thus w-$Sp(M)$ is finite.

(2) ⇒ (3) Since $M_{\mathfrak{m}}$ is Artinian over $R_{\mathfrak{m}}$ for any $\mathfrak{m} \in w$-Max(R), by [7, Theorem 2.7] $M_{\mathfrak{m}}$ is a Loewy $R_{\mathfrak{m}}$-module. Then by [8, Theorem 3.8] $M_{\mathfrak{m}} = L_{\omega_{R_{\mathfrak{m}}}}(M_{\mathfrak{m}})$. By Theorem 4.4, $L_{\omega_{R_{\mathfrak{m}}}}(M_{\mathfrak{m}}) = L^{\omega}_{\tau_w}(M)_{\mathfrak{m}}$. Then $M = L^{\omega}_{\tau_w}(M)$.

Thus M is a τ_w-Loewy R-module and its τ_w-Loewy length is at most ω. For $\mathfrak{m} \in w\text{-}Sp(M)$, again by Theorem 4.4,

$$\left(\left(\frac{L^{n+1}_{\tau_w}(M,\mathfrak{m})}{L^{n}_{\tau_w}(M,\mathfrak{m})}\right)_w\right)_{\mathfrak{m}} = \frac{L_{(n+1)_{R_\mathfrak{m}}}(M_\mathfrak{m})}{L_{n_{R_\mathfrak{m}}}(M_\mathfrak{m})},$$

where n is a positive integer. Since the n^{th} Loewy invariant of $M_\mathfrak{m}$ is finite, so is the n^{th} τ_w-$\mathfrak{m}$-Loewy invariant of M. Note that

$$\begin{aligned}\left(\frac{L^{n+1}_{\tau_w}(M)}{L^{n}_{\tau_w}(M)}\right)_w &= \left(\frac{\bigoplus_{\mathfrak{m}\in w\text{-}Sp(M)} L^{n+1}_{\tau_w}(M,\mathfrak{m})}{\bigoplus_{\mathfrak{m}\in w\text{-}Sp(M)} L^{n}_{\tau_w}(M,\mathfrak{m})}\right)_w \\ &= \bigoplus_{\mathfrak{m}\in w\text{-}Sp(M)} \left(\frac{L^{n+1}_{\tau_w}(M,\mathfrak{m})}{L^{n}_{\tau_w}(M,\mathfrak{m})}\right)_w\end{aligned}$$

by Theorem 3.9. Since $w\text{-}Sp(M)$ is finite, the n^{th} τ_w-Loewy invariant of M is finite.

(3) $\Rightarrow$ (2) By Corollary 5.13, the τ_w-Loewy length of M is at most ω. It is easy to get that $w\text{-}Sp(M)$ is finite, since the 0^{th} τ_w-Loewy invariant is finite. By Theorem 4.4, $L^{\omega}_{\tau_w}(M)_\mathfrak{m} = L_{\omega_{R_\mathfrak{m}}}(M_\mathfrak{m}) = L^{\omega}_{\tau_w}(M,\mathfrak{m})_\mathfrak{m}$ for any $\mathfrak{m} \in w\text{-Max}(R)$. Then $M_\mathfrak{m} = L_{\omega_{R_\mathfrak{m}}}(M_\mathfrak{m})$, which implies that $M_\mathfrak{m}$ is Loewy over $R_\mathfrak{m}$ and its Loewy length is at most ω. Thus if $\mathfrak{m} \in w\text{-Max}(R) \setminus w\text{-}Sp(M)$, then $M_\mathfrak{m} = 0$ by Lemma 4.5. If $\mathfrak{m} \in w\text{-}Sp(M)$, then

$$\frac{L_{(n+1)_{R_\mathfrak{m}}}(M_\mathfrak{m})}{L_{n_{R_\mathfrak{m}}}(M_\mathfrak{m})} = \left(\left(\frac{L^{n+1}_{\tau_w}(M)}{L^{n}_{\tau_w}(M)}\right)_w\right)_{\mathfrak{m}}.$$

Since the n^{th} τ_w-Loewy invariant of M is finite, so is the n^{th} Loewy invariant of $M_\mathfrak{m}$. Then $M_\mathfrak{m}$ is Artinian over $R_\mathfrak{m}$ by [3, Theorem 2.7].

(2) + (3) $\Rightarrow$ (1) By (2), for any $\mathfrak{m} \in w\text{-}Sp(M)$, $M_\mathfrak{m}$ is Artinian. Let $M \supseteq M_1 \supseteq M_2 \supseteq \cdots$ be a descending chain of w-submodules of M. Then for every $\mathfrak{m} \in w\text{-}Sp(M)$, the descending chain $M_\mathfrak{m} \supseteq (M_1)_\mathfrak{m} \supseteq (M_2)_\mathfrak{m} \supseteq \cdots$ is stationary. Thus there exists some positive integer $n_\mathfrak{m}$ such that $(M_{n_\mathfrak{m}})_\mathfrak{m} = (M_{n_\mathfrak{m}+i})_\mathfrak{m}$ for all $i \geq 1$. Note that $w\text{-}Sp(M)$ is finite. Then there exits a positive integer n such that $(M_n)_\mathfrak{m} = (M_{n+i})_\mathfrak{m}$ for any $\mathfrak{m} \in w\text{-}Sp(M)$ and all $i \geq 1$. By the proof of (3) $\Rightarrow$ (2), for any $\mathfrak{m} \in w\text{-Max}(R) \setminus w\text{-}Sp(M)$, $M_\mathfrak{m} = 0$. Thus for any $\mathfrak{m} \in w\text{-Max}(R)$, $(M_n)_\mathfrak{m} = (M_{n+i})_\mathfrak{m}$ for all $i \geq 1$. Then $M_n = M_{n+i}$ for all $i \geq 1$ by [10, Theorem 6.2.17]. So M is w-Artinian. □

The following example shows that even if M is a w-Artinian module (and so $w\text{-}Sp(M)$ is finite), its τ_w-Loewy length can be ω (cf., Corollary 5.13).

Example 5.15. Let $\mathbb{Z}$ denote the ring of integers, p be a prime number, and $\mathbb{Q}$ denote the field of rational numbers. Obviously, $\mathbb{Z}$ is a DW-ring, and hence

w-simple modules are simple and w-Artinian modules are Artinian. Set

$$\mathbb{Q}_p := \{x \in \mathbb{Q} \mid p^n x \in \mathbb{Z} \text{ for some positive integer } n\}$$

and $M := \mathbb{Q}_p/\mathbb{Z}$. Let $\overline{x}$ be the image of x in M. Define $M_n := \{\overline{x} \in M \mid p^n x \in \mathbb{Z}\}$. Then by [10, Example 2.8.16], $M = \bigcup_{n=1}^{\infty} M_n$ is an Artinian module, but not Noetherian. Actually $M_1 \subset M_2 \subset \cdots \subset M_n \subset \cdots$ is a strictly ascending chain. Thus M is a Loewy $\mathbb{Z}$-module by [3, Theorem 2.7]. Note that $M_1 \cong \mathbb{Z}_p$ is the only simple submodule of M; $M_n/M_{n-1} \cong M_1$ for any positive integer $n \geq 2$; $Sp(M) = \{p\mathbb{Z}\}$. Then $0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_n \subset \cdots \subset M_\omega = M$ is the (τ_w-)Loewy series of M and its (τ_w-)Loewy length is ω.

We end this paper with an example of a w-locally Artinian module, but not w-Artinian.

Example 5.16. Let R be not a DW-ring. Then w-Max(R) is infinite by Proposition 2.6. Set $M = \bigoplus_{\mathfrak{m} \in w\text{-Max}(R)} (R/\mathfrak{m})_w$. Then M is w-locally Artinian, but not w-Artinian.

Acknowledgements. The authors thank the anonymous referee for several helpful suggestions, which have significantly improved the original version of the manuscript. This research was supported by the Academic Research Fund of Hoseo University in 2016 (20160026).

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