

## ON THE ORBITAL STABILITY OF INHOMOGENEOUS NONLINEAR SCHRÖDINGER EQUATIONS WITH SINGULAR POTENTIAL

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ABSTRACT. We show the existence of ground state and orbital stability of standing waves of nonlinear Schrödinger equations with singular linear potential and essentially mass-subcritical power type nonlinearity. For this purpose we establish the existence of ground state in  $H^1$ . We do not assume symmetry or monotonicity. We also consider local and global well-posedness of Strichartz solutions of energy-subcritical equations. We improve the range of inhomogeneous coefficient in [5, 12] slightly in 3 dimensions.

### 1. Introduction

In this paper we consider the following Cauchy problem:

$$(1.1) \quad \begin{cases} i\partial_t \psi - \Delta \psi = N(x, \psi) & \text{in } \mathbb{R}^{1+n}, \\ \psi(0, x) = \psi_0(x) & \text{in } \mathbb{R}^n. \end{cases}$$

Here  $n \geq 1$ ,  $\psi : \mathbb{R}^{1+n} \rightarrow \mathbb{C}$  and  $N : \mathbb{R}^n \times \mathbb{C} \rightarrow \mathbb{C}$ .

To present our results let us set  $N(x, \psi) = V(x)\psi + g(x)|\psi|^{p-1}\psi$  ( $p > 1$ ) and describe assumptions:

**(A1)**  $V \in C(\mathbb{R}^n \setminus \{0\}, \mathbb{R})$  and for some constant  $A > 0$  and  $a \geq 0$ ,

$$|V(x)| \leq A|x|^{-a}.$$

**(A2)**  $g \in C(\mathbb{R}^n \setminus \{0\}, [0, +\infty))$  and for some constants  $B > 0$  and  $b \geq 0$

$$g(x) \leq B|x|^{-b} \text{ for all } x \in \mathbb{R}^n \setminus \{0\}.$$

**(A3)** There exist  $B_0, R > 0$  and  $b_0 \geq b$  such that

$$g(x) \geq B_0|x|^{-b_0} \text{ if } |x| \geq R, \text{ and } \lim_{|x| \rightarrow \infty} g(x) = 0.$$

The model of (1.1) can be a dilute Bose-Einstein condensate when interactions of the condensate are considered to be inhomogeneous. For this see [2, 18]

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and the references therein. Also it has been considered to study the laser guiding in an axially nonuniform plasma channel. For this see [11, 15, 17]. If  $V = 0$  and  $g = \gamma|x|^{-b}$  for a fixed  $\gamma \in \mathbb{R} \setminus \{0\}$ , then the equation has scaling invariant structure. That is, the scaled function  $u_\lambda(t, x) = \lambda^{-\frac{2-b}{p-1}}u(\frac{t}{\lambda^2}, \frac{x}{\lambda})$  is also the solution of (1.1). If  $p = 1 + \frac{2(2-b)}{n}$ , then the space scaling is  $L^2$ -invariant. We call the equation with this  $p$  mass-critical one. If  $p < 1 + \frac{2(2-b)}{n}$  (if  $>$ ), then we say that it is mass-sub (super) critical.

We define energy functional  $E$  by

$$E(\psi) = \frac{1}{2}\|\nabla\psi\|_{L^2}^2 - \frac{1}{2}\int V(x)|\psi|^2 dx - \frac{1}{p+1}\int g(x)|\psi|^{p+1} dx$$

and also mass  $m$  by  $m(\psi) = \int |\psi|^2 dx$ . By a standing wave of (1.1) we mean a solution  $\psi(t, x)$  of the form  $e^{i\omega t}u$  for some  $\omega \in \mathbb{R}$ , where  $u$  is a solution of the equation

$$(1.2) \quad -\Delta u - \omega u = V(x)u + g(x)|u|^{p-1}u.$$

Many authors have studied the existence of  $u$  and (in)stability of standing waves under suitable conditions on  $V, g$ . For instance see [4, 8, 10] and references therein. For this purpose they showed that if  $(u_k)$  is a minimizing sequence of the problem

$$I_\mu = \inf\{E(u) : u \in S_\mu\}, \quad S_\mu = \{u \in H^1(\mathbb{R}^n, \mathbb{C}) : m(u) = \mu\}$$

with a prescribed positive number  $\mu$ , then  $u_k \rightarrow u$  in  $H^1$  up to a subsequence, where  $u$  is a solution of (1.2) for some  $\omega$ . Here  $H^1$  denotes the usual  $L^2$ -Sobolev space with the norm  $\|u\|_{H^1} = \|u\|_{L^2} + \|\nabla u\|_{L^2}$ . In this paper we will also use the  $L^r$ -Sobolev space  $H_r^1$  ( $1 \leq r \leq \infty$ ) whose norm is defined by  $\|u\|_{H_r^1} = \|u\|_{L^r} + \|\nabla u\|_{L^r}$ .

Now by following the definition of Cazenave-Lions, we set

$$\mathcal{O}_\mu = \{u \in S_\mu : E(u) = I_\mu\}.$$

Our first result is the existence of ground states of case when  $p < 1 + \frac{2(2-b)}{n}$ , which is usually referred as mass-subcritical case.

**Proposition 1.1.** *Let  $n \geq 1$ ,  $0 < b \leq b_0 < \min(n, 2)$ ,  $1 < p < \min(3, 1 + \frac{2(\min(n, 2) - b_0)}{n})$ , and  $\frac{n(p-1)+2b_0}{2} < a < \min(n, 2)$ . Suppose that  $V$ , and  $g$  satisfy the assumptions **(A1)**, and **(A2)** and **(A3)**, respectively. Then  $\mathcal{O}_\mu$  is not empty for any  $\mu > 0$ . If  $b = 0$ , then we have the same conclusion for  $0 < b_0 < \min(n, 2)$ .*

For the proof we use the standard concentration-compactness argument of [14]. The difficulty is coming from the competition between the singularities of linear and nonlinear potentials. We find a room for singularity of linear potential to settle it. For this the lower bound of  $a$  is necessary. The case  $b = 0$  seems new as far as we know. When  $b_0 = b = 0$ , it would be interesting to

show the existence of ground state by assuming  $\lim_{|x| \rightarrow \infty} g(x) = B_0 > 0$ . One may try this issue with the argument of [1].

We say that  $\mathcal{O}_\mu$  is stable if it is not empty and satisfies that for any  $\varepsilon > 0$ , there exists a  $\delta > 0$  such that if  $\psi_0 \in H^1$  with

$$\inf_{u \in \mathcal{O}_\mu} \|\psi_0 - u\|_{H^s} < \delta,$$

then

$$\inf_{u \in \mathcal{O}_\mu} \|\psi(t, \cdot) - u\|_{H^s} < \varepsilon$$

for all  $t \in [-T_1, T_2]$ . Here  $\psi$  is the unique solution to (1.1) in  $C([-T_1, T_2]; H^1)$  with  $m(\psi(t)) = m(\varphi)$  and  $E(\psi(t)) = E(\psi_0)$  for all  $t \in [-T_1, T_2]$ .

Let us introduce our main result.

**Theorem 1.2.** *Let  $n \geq 1$ ,  $0 < b \leq b_0 < \min(n, 2)$ ,  $1 < p < \min(3, 1 + \frac{2(\min(n, 2) - b_0)}{n})$ , and  $\frac{n(p-1)+2b_0}{2} < a < \min(n, 2)$ . Suppose that  $V$  satisfies the assumption **(A1)** and  $g$  satisfy **(A2)** and **(A3)**. Let  $\psi$  be a solution in  $C([-T_1, T_2]; H^1)$  with  $m(\psi(t)) = m(\psi_0)$  and  $E(\psi(t)) = E(\psi_0)$  for all  $t \in [-T_1, T_2]$ . Then  $\mathcal{O}_\mu$  is stable.*

In [8, 10] the authors studied the stability when  $p < 1 + \frac{2(2-b)}{n}$  and instability when  $1 + \frac{2(2-b)}{n} < p < 1 + \frac{2(2-b)}{n-2}$  ( $n \geq 3$ ). We only considered the stability result because the approach of instability will be much different from the one used in this paper. We will treat the instability issue in a different place.

We now consider the well-posedness of Strichartz solutions of (1.1). By Duhamel's formula, (1.1) is written as an integral equation

$$(1.3) \quad u = U(t)\psi_0 - i \int_0^t U(t-t')N(x, \psi(t')) dt'.$$

Here we define the linear propagator  $U(t)$  given by the linear problem  $i\partial_t v = \Delta v$  with initial datum  $v(0) = f$ . It is formally given by

$$(1.4) \quad U(t)f = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(x \cdot \xi + t|\xi|^2)} \widehat{f}(\xi) d\xi,$$

where  $\widehat{f} = \mathcal{F}(f)$  denotes the Fourier transform of  $f$  and  $\mathcal{F}^{-1}$  the inverse Fourier transform such that

$$\mathcal{F}(f)(\xi) = \int_{\mathbb{R}^n} e^{-ix \cdot \xi} f(x) dx, \quad \mathcal{F}^{-1}(g)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} g(\xi) d\xi.$$

The well-posedness can be shown by a classical argument of [3] based on the functional analysis. But in this paper we use the standard contraction principle via Strichartz estimates for future work about scattering and blowup.

If a pair  $(q, r)$  satisfies that  $2 \leq q, r \leq \infty$ ,  $\frac{2}{q} + \frac{n}{r} = \frac{n}{2}$ , and  $(n, q, r) \neq (2, 2, \infty)$ , then it is said to be *admissible*. Let  $(q, r)$  and  $(\tilde{q}, \tilde{r})$  be any admissible pair.

Then we have the following Strichartz estimates [13]

$$\begin{aligned} \|U(t)\varphi\|_{L^q(-T,T;L^r)} &\leq C\|\varphi\|_{L_x^2}, \\ \left\| \int_0^t U(t-t')F dt' \right\|_{L^q(-T,T;L^r)} &\leq C\|F\|_{L^{\tilde{q}}(-T,T;L^{\tilde{r}})}, \end{aligned}$$

where the constant  $C$  does not depend on  $T$ .

To simplify our well-posedness result we define the following numbers

$$2_b = \begin{cases} \infty & \text{if } n = 1, 2, \\ 1 + \frac{2(2-b)}{n-2} & \text{if } n \geq 3 \end{cases} \quad \text{and} \quad \tilde{2} = \begin{cases} 1 & \text{if } n = 1, 2, \\ \frac{3}{2} & \text{if } n = 3, \\ 2 & \text{if } n \geq 4. \end{cases}$$

**Theorem 1.3.** *Let  $n \geq 1$ ,  $0 \leq a < \tilde{2}$ ,  $0 \leq b < \tilde{2}$  and  $1 < p < 2_b$ . Let us assume that  $V, g \in C^1(\mathbb{R}^n \setminus \{0\})$  satisfy the assumptions **(A1)** and **(A2)** and that  $\nabla V, \nabla g \in L^\infty(|x| > 1)$ . Suppose that there exist positive constants  $A', B'$  depending on  $n$  such that if  $n = 1$ , then for some  $0 \leq a', b' < 1$*

$$\left| \frac{d}{dx} V(x) \right| \leq A'|x|^{-a'}, \quad \left| \frac{d}{dx} g(x) \right| \leq B'|x|^{-b'}, \quad 0 < |x| \leq 1,$$

and if  $n \geq 2$ , then

$$|\nabla V(x)| \leq A'|x|^{-a-1}, \quad |\nabla g(x)| \leq B'|x|^{-b-1}, \quad 0 < |x| \leq 1.$$

Then for any  $\psi_0 \in H^1$  there exists maximal time interval  $I_* = (-T_*, T^*)$  for  $T_*, T^* \in (0, +\infty]$  such that there exists a unique  $\psi \in C(I_*; H^1)$  and  $\psi \in L^q(-T_1, T_2; H_r^1)$  for any admissible pair  $(q, r)$  and for any  $[-T_1, T_2] \subset I_*$  satisfying that  $m(\psi(t)) = m(\psi_0)$  and  $E(\psi(t)) = E(\psi_0)$  for all  $t \in \mathbb{R}$ . If  $p < 1 + \frac{2(2-b)}{n}$ , then  $I_* = \mathbb{R}$ .

Guzmán [12] and Dinh [5] considered well-posedness in  $H^1$  when  $V = 0$  and  $g = |x|^{-b}$ . When  $n = 3$  they could get the well-posedness for  $0 < b < 1$  and  $1 < p < 2_b$ , and  $1 \leq b < \frac{3}{2}$  and  $p < \frac{5-2b}{2b-1}$ , respectively. We improve the range of  $p$  up to  $2_b$  when  $1 < b < \frac{3}{2}$  by dividing  $\nabla g$  in- and outside the unit ball. The global well-posedness for mass-critical and mass-supercritical case will be interesting. For the case  $V = 0$ , see [6, 7, 9].

Our paper is organized as follows. In Section 2 we will prove the existence of ground states by showing the compactness of the minimizing sequences of the constrained variational problem. This is a key step to show the orbital stability of standing waves. This goal is achieved in Theorem 1.2, which will be shown in Section 3. In the last section, we will discuss the Strichartz solutions of the Cauchy problem for a large class of nonlinearities.

## 2. Ground state

### 2.1. Proof of Proposition 1.1

If  $1 < p < 1 + \frac{2(2-b)}{n}$  and  $0 < a < 2$ , then from Hardy-Sobolev's, Gagliardo-Nirenberg's, and then Young's inequalities it follows that for any  $u \in S_\mu$  there

exists a constant  $C_0 > 0$  such that

$$\begin{aligned}
 (2.1) \quad E(u) &= \frac{1}{2} \|\nabla u\|_{L^2}^2 - \frac{1}{2} \int V(x)|u|^2 dx - \frac{1}{p+1} \int g(x)|u|^{p+1} dx \\
 &\geq \frac{1}{2} \|\nabla u\|_{L^2}^2 - \frac{1}{2} A \int |x|^{-a}|u|^2 dx - \frac{B}{p+1} \int |x|^{-b}|u|^{p+1} dx \\
 &\geq \frac{1}{2} \|\nabla u\|_{L^2}^2 - \frac{1}{2} AC_a \|u\|_{L^2}^{2-a} \|\nabla u\|_{L^2}^a \\
 &\quad - \frac{BC_{b,p}}{p+1} \|u\|_{L^2}^{p+1-\frac{n(p-1)+2b}{2}} \|\nabla u\|_{L^2}^{\frac{n(p-1)+2b}{2}} \\
 &\geq \frac{1}{4} \|\nabla u\|_{L^2}^2 - C_0(\mu^2 + \mu^{\theta(p)}),
 \end{aligned}$$

where  $\theta(p) = (p+1 - \frac{n(p-1)+2b}{2}) \cdot \frac{2}{4-n(p-1)-2b}$ . Thus  $I_\mu > -\infty$  for all  $\mu > 0$ .

Now we show that

$$(2.2) \quad I_\mu < 0 \quad \text{for all } \mu > 0.$$

In fact, for  $0 < \lambda \ll 1$  letting  $\varphi_\lambda(x) = \lambda^{\frac{a}{2}} \varphi(\lambda x)$  for a nonnegative, rapidly decreasing radial smooth function  $\varphi$  in  $S_\mu$ , we see that  $\varphi_\lambda \in S_\mu$  and

$$\begin{aligned}
 E(\varphi_\lambda) &= \frac{1}{2} \|\nabla \varphi_\lambda\|_{L^2}^2 - \frac{1}{2} \int V(x)(\varphi_\lambda)^2 dx - \frac{1}{p+1} \int g(x)(\varphi_\lambda)^{p+1} dx \\
 &\leq \frac{1}{2} \|\nabla \varphi_\lambda\|_{L^2}^2 - \frac{1}{2} \int V(x)(\varphi_\lambda)^2 dx - \frac{1}{p+1} \int_{|x| \geq \lambda^{-1}R} g(x)(\varphi_\lambda)^{p+1} dx \\
 &\leq \frac{1}{2} \lambda^2 \|\nabla \varphi\|_{L^2}^2 + \frac{AC_a}{2} \|\varphi_\lambda\|_{L^2}^{2-a} \|\nabla \varphi_\lambda\|_{L^2}^a \\
 &\quad - \frac{B_0}{p+1} \lambda^{\frac{n(p+1)}{2}-n+b_0} \int_{|x| \geq R} |x|^{-b_0} (\varphi(x))^{p+1} dx.
 \end{aligned}$$

Since  $0 < \lambda \ll 1$  and  $\varphi$  is smooth and rapidly decreasing, there exist constants  $C_1, C_2 > 0$  such that

$$E(\varphi_\lambda) \leq \lambda^a C_1 - \lambda^{\frac{n(p-1)+2b_0}{2}} C_2,$$

which is strictly negative from the condition  $a > \frac{n(p-1)+2b_0}{2}$  if  $\lambda$  is sufficiently small.

On the other hand, one can easily show that  $I_\mu$  is continuous on  $(0, \infty)$ . The proof will be given in Section 2.2.

Using the continuity, we deduce that for each  $\mu > 0$  and  $\theta > 1$  there exist  $\varepsilon < -I_\mu(1 - \theta^{-\frac{p-1}{2}})$ , and  $v \in S_\mu$  such that  $I_\mu < E(v) < I_\mu + \varepsilon$ . Then it follows from the definition of  $E$  and  $I_\mu$  that

$$I_{\theta\mu} \leq E(\sqrt{\theta}v) \leq \theta^{\frac{p+1}{2}} E(v) \leq \theta^{\frac{p+1}{2}} (I_\mu + \varepsilon) < \theta I_\mu,$$

which implies that

$$(2.3) \quad I_\mu < I_\nu + I_{\mu-\nu} \quad \text{for all } 0 < \nu < \mu.$$

For the general situation we refer the readers to Lemma II. 1 of [14].

Let  $(u_j) \subset S_\mu$  be a minimizing sequence such that  $E(u_j) \rightarrow I_\mu$ . From (2.1) we deduce that  $(u_j)$  is bounded in  $H^1$ . To show  $\mathcal{O}_\mu \neq \emptyset$  we will use the concentration-compactness (see [14]). Let the concentration function  $\mathfrak{m}_j$  be defined by

$$\mathfrak{m}_j(r) = \sup_{y \in \mathbb{R}^n} \int_{|x-y| < r} |u_j(x)|^2 dx \text{ for } r > 0.$$

Set

$$\nu = \lim_{r \rightarrow \infty} \liminf_{j \rightarrow \infty} \mathfrak{m}_j(r).$$

Then  $0 \leq \nu \leq \mu$  and there exists a subsequence  $u_j$  (still denoted by  $u_j$ ) satisfying the following properties (see [14] or [3]).

- (1) If  $\nu = 0$ , then  $\|u_j\|_{L^q} \rightarrow 0$  as  $j \rightarrow \infty$  for all  $q$  with  $2 < q < 2^*$ ,  $2^* = \frac{2n}{n-2}$  if  $n > 2$  and  $2^* = \infty$  if  $n = 1, 2$ .
- (2) If  $\nu = \mu$ , then there exists a sequence  $(y_j) \subset \mathbb{R}^n$  and  $u \in H^1$  such that for any  $q$  with  $2 \leq q < 2^*$

$$u_j(\cdot + y_j) \rightarrow u \text{ as } j \rightarrow \infty \text{ in } L^q$$

and given  $\varepsilon > 0$  there exist  $j_0(\varepsilon)$  and  $r(\varepsilon)$  such that

$$\int_{|x-y_j| < r(\varepsilon)} |u_j|^2 dx \geq \mu - \varepsilon, \text{ whenever } j \geq j_0(\varepsilon).$$

- (3) If  $0 < \nu < \mu$ , then there exist  $(v_j), (w_j) \subset H^1$  such that

$$(2.4) \quad \text{supp } v_j \cap \text{supp } w_j = \emptyset,$$

$$(2.5) \quad \|v_j\|_{H^1} + \|w_j\|_{H^1} \leq C \|u_j\|_{H^1},$$

$$(2.6) \quad \lim_{j \rightarrow \infty} m(v_j) = \nu, \quad \lim_{j \rightarrow \infty} m(w_j) = \mu - \nu,$$

$$(2.7) \quad \liminf_{j \rightarrow \infty} (\|\nabla u_j\|_{L^2}^2 - \|\nabla v_j\|_{L^2}^2 - \|\nabla w_j\|_{L^2}^2) \geq 0,$$

$$(2.8) \quad \lim_{j \rightarrow \infty} \|u_j - v_j - w_j\|_{L^q} = 0, \quad 2 \leq q < 2^*.$$

If  $\nu = 0$ , then using Hardy-Sobolev's and Gagliardo-Nirenberg's inequality near the origin as in (2.1), we have that for any  $2 < q < \frac{2n}{n-a}$

$$(2.9) \quad \left| \frac{1}{2} \int V(x) |u_j|^2 dx + \frac{1}{p+1} \int g(x) |u_j|^{p+1} dx \right| \\ \leq C \|u_j\|_{L^2(|x| \leq 1)}^{2-a} \|\nabla u_j\|_{L^2}^a + C \| |x|^{-a} \|_{L^{\frac{q}{q-2}}(|x| > 1)} \|u_j\|_{L^q}^2 \\ + C \|u_j\|_{L^2(|x| \leq 1)}^{p+1 - \frac{n(p-1)+2b}{2}} \|\nabla u_j\|_{L^2}^{\frac{n(p-1)+2b}{2}} + C \|u_j\|_{L^{p+1}(|x| > 1)}^{p+1} \rightarrow 0 \text{ as } j \rightarrow \infty.$$

This implies  $I_\mu = \lim_{j \rightarrow \infty} E(u_j) \geq \frac{1}{2} \liminf \|\nabla u_j\|_{L^2}^2 \geq 0$  and contradicts (2.2).

If  $0 < \nu < \mu$ , then from the support condition (2.4) it follows that

$$\begin{aligned} & E(u_j) - E(v_j) - E(w_j) \\ &= \frac{1}{2} (\|\nabla u_j\|_{L^2}^2 - \|\nabla v_j\|_{L^2}^2 - \|\nabla w_j\|_{L^2}^2) \\ & \quad - \frac{1}{2} \int V(x)(|u_j|^2 - |v_j + w_j|^2) dx - \frac{1}{p+1} \int g(x)(|u_j|^{p+1} - |v_j + w_j|^{p+1}) dx. \end{aligned}$$

From (2.7), (2.8), and estimates in (2.9) we deduce that

$$\liminf_{j \rightarrow \infty} (E(u_j) - E(v_j) - E(w_j)) \geq 0$$

and thus

$$I_\mu = \lim_{j \rightarrow \infty} E(u_j) \geq \liminf_{j \rightarrow \infty} E(v_j) + \liminf_{j \rightarrow \infty} E(w_j).$$

Since  $m(v_j) \rightarrow \nu$  and  $m(w_j) \rightarrow \mu - \nu$ , by the continuity of  $I_\mu$  on  $(0, \infty)$  we get

$$I_\mu \geq I_\nu + I_{\mu-\nu},$$

which contradicts (2.3).

Therefore  $\nu = \mu$ . Set  $\tilde{u}_j(x) = u_j(x + y_j)$ . Then  $u, \tilde{u}_j \in S_\mu$  and  $\tilde{u}_j \rightarrow u$  in  $L^q$  for all  $2 \leq q < 2^*$ . On the other hand,  $(u_j)$  is bounded in  $H^1$ . Hence there is a subsequence (still denoted by  $u_j$ ) converging to  $v$  weakly in  $H^1$  and strongly in  $L^q_{loc}$  for any  $1 \leq q < 2^*$ . If  $(y_j)$  are unbounded, then up to subsequence we may assume that  $|y_j| \rightarrow \infty$ . Since  $\tilde{u}_j \rightarrow u$  in  $L^2$ ,  $u_j - u(\cdot - y_j) \rightarrow 0$  in the sense of distributions. But  $u(\cdot - y_j) \rightarrow 0$  and  $u_j \rightarrow v$  in the sense of distributions and thus  $v = 0$ .

Now for any  $\varepsilon > 0$  we can find  $R_0, j_0 > 1$  such that if  $j \geq j_0$ , then

$$\begin{aligned} & \int_{|x| > R_0} |V(x)|(|u_j|^2 + |v|^2) dx \leq CR_0^{-a} < \frac{\varepsilon}{4}, \\ & \int_{|x| \leq R_0} |V(x)|(|u_j| + |v|)|u_j - v| dx \\ & \leq C \| |x|^{-a} \|_{L^{\frac{q}{q-2}}(|x| \leq R_0)} \| |u_j| + |v| \|_{L^q} \| u_j - v \|_{L^q(|x| \leq R_0)} < \frac{\varepsilon}{4}, \end{aligned}$$

where  $\frac{2n}{n-a} < q < 2^*$ , and also such that

(1) Case:  $b > 0$

$$\begin{aligned} & \int_{|x| > R_0} g(x)(|u_j|^{p+1} + |v|^{p+1}) dx \leq CR_0^{-b} < \frac{\varepsilon}{4}, \\ & \int_{|x| \leq R_0} g(x)(|u_j|^p + |v|^p)|u_j - v| dx \\ & \leq C \| |x|^{-b} \|_{L^{\frac{q}{q-(p+1)}}(|x| \leq R_0)} \| |u_j| + |v| \|_{L^q}^p \| u_j - v \|_{L^q(|x| \leq R_0)} \\ & < \frac{\varepsilon}{4} \quad \text{for } \frac{(p+1)n}{n-b} < q < 2^*. \end{aligned}$$

(2) Case:  $b = 0$  and  $|y_j| \leq R_1$

$$\begin{aligned} & \int_{\{|x-y_j|>R_0\} \cap \{|x|>R_0\}} g(x)(|u_j|^{p+1} + |v|^{p+1}) dx \\ & \leq C \int_{\{|x|>R_0\}} |\tilde{u}_j|^{p+1} dx + C \int_{\{|x|>R_0\}} |v|^{p+1} dx < \frac{\varepsilon}{4}, \\ & \int_{\{|x-y_j|\leq R_0\} \cup \{|x|\leq R_0\}} g(x)(|u_j|^p + |v|^p)|u_j - v| dx \\ & \leq C \| |u_j| + |v| \|_{L^{pq'}}^p \|u_j - v\|_{L^q(|x|\leq R_0+R_1)} < \frac{\varepsilon}{4} \text{ for } 2 \leq pq' \leq 2^*. \end{aligned}$$

(3) Case:  $b = 0$  and  $(y_j)$  are unbounded

$$\begin{aligned} & \int_{\{|x-y_j|>R_0\} \cap \{|x|>R_0\}} g(x)(|u_j|^{p+1} + |v|^{p+1}) dx \leq C \int_{\{|x|>R_0\}} |\tilde{u}_j|^{p+1} dx < \frac{\varepsilon}{4}, \\ & \int_{\{|x-y_j|\leq R_0\} \cup \{|x|\leq R_0\}} g(x)(|u_j|^p + |v|^p)|u_j - v| dx \\ & \leq \int_{\{|x-y_j|\leq R_0\}} g(x)|u_j|^{p+1} dx + C \int_{\{|x|\leq R_0\}} |u_j|^{p+1} dx \\ & \leq C \int_{\{|x|\leq R_0\}} g(x+y_j)|\tilde{u}_j - u|^{p+1} dx + C \int_{\{|x|\leq R_0\}} g(x+y_j)|u|^{p+1} dx + \frac{\varepsilon}{8} < \frac{\varepsilon}{4} \end{aligned}$$

due to the fact  $\tilde{u}_j \rightarrow u$  in  $L^2$  and  $g(x+y_j) \rightarrow 0$ .

Set  $P(w) := E(w) - \frac{1}{2} \|\nabla w\|_{L^2}^2$ . Then  $P(u_j) \rightarrow P(v)$  as  $j \rightarrow \infty$ . Suppose that  $(y_j)$  is unbounded. Then  $v = 0$  and hence  $P(u_j) \rightarrow 0$  as  $j \rightarrow \infty$ . This implies that  $I_\mu = \lim_{j \rightarrow \infty} E(u_j) \geq 0$ , which contradicts (2.2). Thus  $(y_j)$  is bounded. Now let  $R_1 = \sup_{j \geq 1} |y_j|$ . Then for any  $\varepsilon > 0$  we have

$$\int_{|x|<R_1+r(\varepsilon)} |u_j|^2 dx \geq \int_{|x-y_j|<r(\varepsilon)} |u_j|^2 dx \geq \mu - \varepsilon, \text{ if } j \geq j_0,$$

and thus

$$m(v) \geq \int_{|x|<R_1+r(\varepsilon)} |v|^2 dx \geq \lim_{j \rightarrow \infty} \int_{|x|<R_1+r(\varepsilon)} |u_j|^2 dx \geq \mu - \varepsilon.$$

This means  $m(v) \geq \mu$ , while the semi-continuity of weak limit implies  $m(v) \leq \mu$ . Then  $v \in S_\mu$ . Since  $P(u_j) \rightarrow P(v)$ , we have

$$(2.10) \quad I_\mu \leq E(v) \leq \liminf \frac{1}{2} \|\nabla u_j\|_{L^2}^2 + P(v) = \liminf (E(u_j)) = I_\mu.$$

Therefore  $E(v) = I_\mu$ . This completes the proof of Proposition 1.1.

## 2.2. Proof of continuity of $I_\mu$

For any  $\mu > 0$  let us take sequences  $\mu_j \in (0, \infty)$  and  $u_j \in S_{\mu_j}$  such that  $\mu_j \rightarrow \mu$  and  $I_{\mu_j} < E(u_j) < I_{\mu_j} + \frac{1}{j}$ . From (2.1) it follows that  $\|u_j\|_{H^1} \leq M$  for some constant  $M > 0$ . Then  $\|u_j - \frac{\mu}{\mu_j} u_j\|_{H^1} \leq M|1 - \frac{\mu}{\mu_j}|$  and hence

there exists  $j_0$  such that  $\|u_j - \frac{\mu}{\mu_j} u_j\|_{H^1} \leq M$  for  $j \geq j_0$ . On the other hand,  $E \in C^1(H^1, \mathbb{R})$ ,  $E' \in C(H^1, H^{-1})$  and for any  $v, h \in H^1$

$$\langle E'(v), h \rangle = \langle \nabla v, \nabla h \rangle - \langle Vv, h \rangle - \operatorname{Re} \langle g|v|^{p-1}v, h \rangle.$$

Thus for any  $v \in H^1$  with  $\|v\|_{H^1} \leq 2M$  we have from Hardy-Sobolev inequality that

$$\begin{aligned} (2.11) \quad & |\langle E'(v), h \rangle| \\ & \leq M \|h\|_{H^1} + A \| |x|^{-a/2} v \|_{L^2} \| |x|^{-a/2} h \|_{L^2} + B \| |x|^{-\frac{b}{p+1}} v \|_{L^{p+1}}^p \| |x|^{-\frac{b}{p+1}} h \|_{L^{p+1}} \\ & \leq CM \|h\|_{H^1} + \|v\|_{L^2}^{p(1 - \frac{n(p-1)+2b}{2(p+1)})} \|\nabla v\|_{L^2}^{\frac{p(n(p-1)+2b)}{2(p+1)}} \|h\|_{L^2}^{1 - \frac{n(p-1)+2b}{2(p+1)}} \|\nabla h\|_{L^2}^{\frac{n(p-1)+2b}{2(p+1)}} \\ & \leq C(M + CM^p) \|h\|_{H^1} \end{aligned}$$

and therefore  $\|E'(v)\|_{H^{-1}} \leq C(M + M^p)$ . Using this and mean value theorem we get

$$|E(u_j) - E(\frac{\mu}{\mu_j} u_j)| \leq C(M + M^p) M \left| 1 - \frac{\mu}{\mu_j} \right|$$

if  $j \geq j_0$ . This implies that  $I_\mu \leq \liminf_{j \rightarrow \infty} E(\frac{\mu}{\mu_j} u_j) \leq \liminf_{j \rightarrow \infty} I_{\mu_j}$ .

Now we choose a sequence  $(v_j) \subset S_\mu$  such that  $E(v_j) \rightarrow I_\mu$ . By (2.1) we deduce that there exists  $K > 0$  such that  $\|v_j\|_{H^1} \leq K$ . Thus from (2.11) it follows that

$$I_{\mu_j} \leq E(\frac{\mu_j}{\mu} v_j) \leq |E(\frac{\mu_j}{\mu} v_j) - E(v_j)| + E(v_j) \leq C(K + K^p) K \left| 1 - \frac{\mu_j}{\mu} \right| + E(v_j).$$

This implies that  $\limsup_{j \rightarrow \infty} I_{\mu_j} \leq I_\mu$ . This concludes the proof.

### 3. Proof of Theorem 1.2

The proof proceeds by contradiction. Suppose that  $\mathcal{O}_\mu$  is not stable, then either  $\mathcal{O}_\mu$  is empty or there exist  $w \in \mathcal{O}_\mu$  and a sequence  $\psi_0^j \in H^1$  such that

$$\|\psi_0^j - w\|_{H^1} \rightarrow 0 \text{ as } j \rightarrow \infty$$

but

$$(3.1) \quad \inf_{v \in \mathcal{O}_\mu} \|\psi^j(t_j, \cdot) - v\|_{H^1} \geq \varepsilon_0$$

for some sequence  $t_j \in [-T_1, T_2]$  and  $\varepsilon_0$ , where  $\psi^j(t, \cdot)$  is the solution of (1.1) corresponding to the initial data  $\psi_0^j$ . Let  $w_j = \psi^j(t_j, \cdot)$ . Since  $w \in S_\mu$  and  $E(w) = I_\mu$ , it follows from the continuity of  $L^2$  norm and  $E$  in  $H^1$  that

$$\|\psi_0^j\|_{L^2}^2 \rightarrow \mu \text{ and } E(\psi_0^j) \rightarrow I_\mu.$$

Thus we deduce from the conservation laws that

$$\|w_j\|_{L^2}^2 = \|\psi_0^j\|_{L^2}^2 \rightarrow \mu, \quad E(w_j) = E(\psi_0^j) \rightarrow I_\mu.$$

Therefore  $(w_j)$  has a subsequence converging to an element  $v' \in H^1$  such that  $\|v'\|_{L^2}^2 = \mu$  and  $E(v) = I_\mu$ . This shows that  $v' \in \mathcal{O}_\mu$  but

$$\inf_{v \in \mathcal{O}_\mu} \|\psi^j(t_j, \cdot) - v\|_{H^1} \leq \|w_j - v'\|_{H^1},$$

which contradicts (3.1). Since  $\mathcal{O}_\mu$  is not empty, to show the orbital stability of  $\mathcal{O}_\mu$  one has to prove that any sequence  $(w_j) \subset H^1$  with

$$(3.2) \quad \|w_j\|_{L^2}^2 \rightarrow \mu \quad \text{and} \quad E(w_j) \rightarrow I_\mu$$

is relatively compact in  $H^1$ . Let  $\mu_j = \|w_j\|_{L^2}^2$  and  $u_j = \frac{\mu}{\mu_j} w_j$ . Then  $u_j \in S_\mu$ , and since  $I_\mu$  is finite for all  $\mu \in (0, \infty)$  and  $p < 1 + \frac{2(2-b)}{n}$ , by the arguments in the proof of Proposition 1.1 we may assume that  $(u_j)$  is bounded in  $H^1$  and also verify from all argument around (2.10) that by passing to a subsequence there exists  $v \in H^1$  such that

$$(3.3) \quad u_j \rightharpoonup v \quad \text{in} \quad H^1 \quad \text{and} \quad \lim_{j \rightarrow \infty} \|\nabla u_j\|_{L^2} = \|\nabla v\|_{L^2}.$$

This implies  $w_j \rightarrow v$  in  $H^1$  and thus the relative compactness.

#### 4. Well-posedness

In this section we prove Theorem 1.3. Let us first consider the local well-posedness on  $[-T, T]$ . Let  $(X_T^\rho, d_X)$  be a metric space with metric  $d_X$  defined by

$$X_T^\rho = \{\psi \in C([-T, T]; H^1) \cap L_T^{q_0} H_{r_0}^1 : \|\psi\|_{L_T^\infty H^1 \cap L_T^{q_0} H_{r_0}^1} \leq \rho\},$$

$$d_X(\psi, \psi') = \|\psi - \psi'\|_{L_T^\infty H^1 \cap L_T^{q_0} H_{r_0}^1},$$

where  $L_T^q$  denotes  $L^q([-T, T])$  and  $(q_0, r_0)$  is an admissible pair, which will be chosen later. Then  $X_T^\rho$  is clearly complete metric space. We define a mapping  $\Phi$  on  $X_T^\rho$  by

$$(4.1) \quad \Phi(\psi)(t) = U(t)\psi_0 - i \int_0^t U(t-t')[N(\cdot, \psi)](t') dt'.$$

We have from Strichartz estimates with admissible pairs  $(q_i, r_i)$ ,  $i = 0, 1, \dots, 8$  that

$$(4.2) \quad \|\Phi(\psi)\|_{L_T^\infty H^1 \cap L_T^{q_0} H_{r_0}^1} \leq C(\|\varphi\|_{H^1} + \sum_{i=1}^8 \mathcal{N}_i),$$

where

$$\begin{aligned}\mathcal{N}_1 &= \|V(\psi, \nabla\psi)\|_{L_T^{q'_1} L^{r'_1}(|x|\leq 1)}, & \mathcal{N}_2 &= \|V(\psi, \nabla\psi)\|_{L_T^{q'_2} L^{r'_2}(|x|> 1)}, \\ \mathcal{N}_3 &= \|\nabla V\psi\|_{L_T^{q'_3} L^{r'_3}(|x|\leq 1)}, & \mathcal{N}_4 &= \|\nabla V\psi\|_{L_T^{q'_4} L^{r'_4}(|x|> 1)}, \\ \mathcal{N}_5 &= \|g|\psi|^{p-1}(\psi, \nabla\psi)\|_{L_T^{q'_5} L^{r'_5}(|x|\leq 1)}, & \mathcal{N}_6 &= \|g|\psi|^{p-1}(\psi, \nabla\psi)\|_{L_T^{q'_6} L^{r'_6}(|x|> 1)}, \\ \mathcal{N}_7 &= \|\nabla g|\psi|^p\|_{L_T^{q'_7} L^{r'_7}(|x|\leq 1)}, & \mathcal{N}_8 &= \|\nabla g|\psi|^p\|_{L_T^{q'_8} L^{r'_8}(|x|> 1)}.\end{aligned}$$

Here we used the notation  $\|(f, F)\|_{L^r} = \|f\|_{L^r} + \|F\|_{L^r}$  for  $F = (f_1, \dots, f_n)$ .

Let  $\nu = 1$  or  $p$ . Set  $s_\nu = 1$  if  $n = 1$ ,  $1 < s_\nu < \min(\frac{1}{a}, \frac{1}{b})$  if  $n = 2$ , and  $s_\nu = \frac{2n}{n+2-(n-2)\nu}$  if  $n \geq 3$ . We proceed by dividing the sum  $\sum_{i=1}^8 \mathcal{N}_i$  into two parts: linear part ( $i = 1, 2, 3, 4$ ) and nonlinear part ( $i = 5, 6, 7, 8$ ).

#### 4.1. Linear part

Here we set  $(q_0, r_0) = (\frac{4s_1}{n}, \frac{2s_1}{s_1-1})$  and take  $(q_i, r_i) = (q_0, r_0)$  for  $i = 1, 3$  and  $(q_i, r_i) = (\infty, 2)$  for  $i = 2, 4$ . Let  $\psi \in X_T^\rho$ . Then we have that for  $n = 1, 2$

$$\begin{aligned}\mathcal{N}_1 + \mathcal{N}_2 + \mathcal{N}_4 &\leq C(T^{1-\frac{2}{q_0}} \|V\|_{L^{s_1}(|x|\leq 1)} \|\psi\|_{L_T^{q_0} H_{r_0}^1} + T \|V\|_{L^\infty(|x|> 1)} \|\psi\|_{L_T^\infty H^1} \\ &\quad + T \|\nabla V\|_{L^\infty(|x|> 1)} \|\psi\|_{L_T^\infty L^2}) \\ &\leq C(T + T^{1-\frac{2}{q_0}}) \|\psi\|_{L_T^\infty H^1 \cap L_T^{q_0} H_{r_0}^1} \leq C(T + T^{1-\frac{2}{q_0}}) \rho.\end{aligned}$$

If  $n \geq 3$ , then we choose  $\frac{2n}{n+2-2a} < r < r_0 = 2^*$  and let  $(q, r)$  be the corresponding admissible pair. Then we get

$$\begin{aligned}\mathcal{N}_1 + \mathcal{N}_2 + \mathcal{N}_4 &\leq C(T^{\frac{1}{2}-\frac{1}{q}} \| |x|^{-a} \|_{L^{\frac{2nr}{r(n+2)-2n}}(|x|\leq 1)} \|\psi\|_{L_T^q H_r^1} \\ &\quad + T \|V\|_{L^\infty(|x|> 1)} \|\psi\|_{L_T^\infty H^1} + T \|\nabla V\|_{L^\infty(|x|> 1)} \|\psi\|_{L_T^\infty L^2}) \\ &\leq C(T + T^{\frac{1}{2}-\frac{1}{q}}) \|\psi\|_{L_T^\infty H^1 \cap L_T^{q_0} H_{r_0}^1} \leq C(T + T^{\frac{1}{2}-\frac{1}{q}}) \rho.\end{aligned}$$

To treat  $\mathcal{N}_3$  we need to restrict the range of  $a$ . If  $n = 1$ , then

$$\mathcal{N}_3 \leq CT^{\frac{1}{2}} \| |x|^{-a'} \|_{L^1(|x|\leq 1)} \|\psi\|_{L_T^4 L^\infty} \leq CT^{\frac{1}{2}} \rho.$$

If  $n = 2$ , then since  $s_1 < \frac{1}{a}$  for  $n = 2$

$$\begin{aligned}\mathcal{N}_3 &\leq CT^{1-\frac{2}{q_0}} \| |x|^{-a-1} \|_{L^{\frac{2s_1}{s_1+1}}(|x|\leq 1)} \|\psi\|_{L_T^{q_0} L^\infty} \\ &\leq CT^{1-\frac{1}{s_1}} \|\psi\|_{L_T^{q_0} H_{r_0}^1} \leq CT^{1-\frac{1}{s_1}} \rho.\end{aligned}$$

If  $n = 3$ , then since  $a < \frac{3}{2}$ , we can choose  $r$  such that  $\frac{6}{3-2a} < r < \infty$  (and hence  $\frac{6r(a+1)}{5r-6}$ ) to get

$$\mathcal{N}_3 \leq CT^{\frac{1}{12}} \| |x|^{-a-1} \|_{L^{\frac{6r}{5r-6}}(|x|\leq 1)} \|\psi\|_{L_T^{\frac{12}{5}} L^r} \leq CT^{\frac{1}{12}} \|\psi\|_{L_T^{\frac{12}{5}} H_{\frac{3}{5}}^1} \leq CT^{\frac{1}{12}} \rho.$$

If  $n = 4$ , then we choose  $\frac{4}{3} < r < \min(2, \frac{4}{a+1})$  and get

$$\begin{aligned} \mathcal{N}_3 &\leq T^{\frac{3s-4}{2s}} \| |x|^{-a-1} \|_{L^r(|x|\leq 1)} \| \psi \|_{L_T^{\frac{2s}{4-s}} L^{\frac{4s}{3s-4}}} \\ &\leq CT^{\frac{3s-4}{2s}} \| \psi \|_{L_T^{\frac{s}{2-s}} H_{s-1}^1} \leq CT^{\frac{3s-4}{2s}} \rho. \end{aligned}$$

If  $n \geq 5$ , then we choose  $\frac{n}{2} < r < \min(\frac{n}{a}, n)$  (hence  $\tilde{r} := \frac{2rn}{(n+2)r-2n} < \frac{2n}{n-2}$ ) to get

$$\begin{aligned} \mathcal{N}_3 &\leq CT^{\frac{nr-n+r}{2nr}} \| |x|^{-a} \|_{L^r(|x|\leq 1)} \| |x|^{-1} \psi \|_{L_T^{\frac{2nr}{n-r}} L^{\tilde{r}}} \\ &\leq CT^{\frac{nr-n+r}{2nr}} \| \psi \|_{L_T^{\frac{2nr}{n-r}} H_{\tilde{r}}^1} \leq CT^{\frac{nr-n+r}{2nr}} \rho. \end{aligned}$$

Therefore any  $\psi \in X_T^\rho$  we obtain that for some  $\theta > 0$

$$(4.3) \quad \sum_{i=1}^4 \mathcal{N}_i \leq C(T + T^\theta) \rho.$$

#### 4.2. Nonlinear part

Now we move onto  $\mathcal{N}_i, i = 5, 6, 8$ . Let  $(q_0, r_0) = (\frac{4s_p}{n}, \frac{2s_p}{s_p-1})$  if  $n = 1, 2$  and  $(2, 2^*)$  if  $n \geq 3$ . Since  $g \leq B|x|^{-b}$  with  $0 < b < \min(2, n)$  and  $p < 2b$ . For  $i = 5, 6, 8$  we take  $(q_5, r_5) = (q_0, r_0)$ , and  $(q_6, r_6) = (q_8, r_8) = (\infty, 2)$  for  $n = 1, 2$  and  $(\frac{4(p+1)}{n(p-1)}, p+1)$  for  $n \geq 3$ . If  $n = 1$ , then

$$\begin{aligned} \mathcal{N}_5 + \mathcal{N}_6 + \mathcal{N}_8 &\leq C(T^{\frac{1}{2}} + T) (\|g\|_{L^{s_p}(|x|\leq 1)} + \|(g, g')\|_{L^\infty(|x|>1)}) \\ &\quad \times (\|\psi\|_{L_T^\infty H^1}^{p-1} \|\psi\|_{L_T^4 H_\infty^1} + \|\psi\|_{L_T^{q_6} H_{r_6}^1}^p) \\ &\leq C(T^{\frac{1}{2}} + T) \rho^p. \end{aligned}$$

If  $n = 2$ , then we choose  $\varepsilon$  with  $\frac{s_p}{1-\varepsilon s_p} < \frac{2}{b}$  (this is possible because  $0 < b < 1$  and  $s_p < \frac{1}{b}$ ). Let  $r = \frac{p-1}{\varepsilon}$ . Then we get

$$\begin{aligned} \mathcal{N}_5 + \mathcal{N}_6 + \mathcal{N}_8 &\leq C(T^{1-\frac{2}{q_0}} + T^{1-\frac{1}{q_0}}) (\|g\|_{L^{\frac{s_p}{1-\varepsilon s_p}}(|x|\leq 1)} + \|(g, \nabla g)\|_{L^\infty(|x|>1)}) \\ &\quad \times (\|\psi\|_{L_T^\infty L^r}^{p-1} + \|\psi\|_{L_T^\infty L^{2(p-1)r_0 r_0-2}}^{p-1}) \|\psi\|_{L_T^{q_0} H_{r_0}^1} \\ &\leq C(T^{1-\frac{2}{q_0}} + T^{1-\frac{1}{q_0}}) \rho^p. \end{aligned}$$

If  $n \geq 3$ , then let us invoke  $s_p = \frac{2n}{n+2-(n-2)p}$  and choose a small  $\varepsilon$  with  $p+\varepsilon 2^* < 2b$ . Let  $\frac{1}{r} = \frac{n-2}{2n} + \varepsilon$  and  $(q, r)$  be corresponding admissible pair. Then we have

$$\mathcal{N}_5 \leq CT^{\frac{\varepsilon n}{2}} \|g\|_{L^{\frac{s_p}{1-\varepsilon s_p}}} \|\psi\|_{L_T^\infty L^{r_0}}^{p-1} \|\psi\|_{L_T^q H_r^1} \leq CT^{\frac{\varepsilon n}{2}} \rho^p.$$

As for  $i = 6, 8$  let  $(q, r) = (\frac{4(p+1)}{n(p-1)}, p+1)$ . Then  $q > 2$  and we have that

$$\mathcal{N}_6 + \mathcal{N}_8 \leq C(T^{1-\frac{2}{q}}) \|(g, \nabla g)\|_{L^\infty(|x|>1)} (\|\psi\|_{L_T^\infty L^r}^{p-1} \|\psi\|_{L_T^q H_x^1}) \leq CT^{1-\frac{2}{q}} \rho^p.$$

Let us now consider  $\mathcal{N}_7$ . We take  $(q_7, r_7) = (q_0, r_0)$ . If  $n = 1$ , then we get

$$\mathcal{N}_7 \leq CT^{\frac{1}{2}} \| |x|^{-b'} \|_{L^1(|x|\leq 1)} \|\psi\|_{L_T^4 L^\infty}^p \leq CT^{\frac{1}{2}} \rho^p.$$

If  $n = 2$ , then we choose  $s_p > 1$  and very close to 1. For a small  $\varepsilon > 0$  so that  $\frac{s_p(b+\varepsilon)}{1-2\varepsilon s_p} < 1$  we get that

$$\begin{aligned} \mathcal{N}_7 &\leq CT^{1-\frac{1}{q_0}} \| |x|^{-b-\varepsilon} \|_{L^{\frac{2s_0}{1-\varepsilon s_p}}(|x|\leq 1)} \|p\|_{L_T^\infty L^{\frac{p-1}{\varepsilon}}}^{p-1} \| |x|^{-1+\varepsilon} p \|_{L_T^\infty L^\infty} \\ &\leq CT^{1-\frac{1}{q_0}} \|\psi\|_{L_T^\infty H^1}^p \leq CT^{1-\frac{1}{q_0}} \rho^p. \end{aligned}$$

If  $n = 3$ , then we divide the range of  $p$  into two parts, (i)  $1 < p < 2$  and (ii)  $2 \leq p < 2b = 5 - 2b$ . For both cases we need the condition  $b < \frac{3}{2}$ .

Case (i): We take a small  $\varepsilon > 0$  with  $\frac{6(b+1)}{5-\varepsilon} < 3$ . Then we get

$$\begin{aligned} \mathcal{N}_7 &\leq CT^{\frac{2-p}{4}} \| |x|^{-b-1} \|_{L^{\frac{6}{5-\varepsilon}}(|x|\leq 1)} \| \|\psi\|_{L^{\frac{6p}{\varepsilon}}}^p \|_{L_T^{\frac{4}{p}}} \leq CT^{\frac{2-p}{4}} \| \|\psi\|_{H^3} \|_{L_T^{\frac{4}{p}}} \\ &\leq CT^{\frac{2-p}{4}} \| \|\psi\|_{H^1}^{\frac{p}{2}} \| \|\psi\|_{H^6}^{\frac{p}{2}} \|_{L_T^{\frac{4}{p}}} \\ &\leq CT^{\frac{2-p}{4}} \| \|\psi\|_{L_T^\infty H^1}^{\frac{p}{2}} \| \|\psi\|_{L_T^2 H^6}^{\frac{p}{2}} \| \\ &\leq CT^{\frac{2-p}{4}} \rho^p. \end{aligned}$$

Case (ii): We take a small  $\varepsilon > 0$  with  $\frac{6(b+1)}{7-p-\varepsilon} < 3$  (hence  $p + \varepsilon < 2b$ ). Then we get for  $\frac{1}{r} = \frac{\varepsilon}{12} - \frac{1}{3}$

$$\begin{aligned} \mathcal{N}_7 &\leq C \| |x|^{-b-1} \|_{L^{\frac{6}{7-p-\varepsilon}}(|x|\leq 1)} \| \|\psi\|_{L^6}^{p-2} \|\psi\|_{L^{\frac{12}{\varepsilon}}}^2 \|_{L_T^2} \leq C \| \|\psi\|_{H^1}^{p-2} \|\psi\|_{H^1}^2 \|_{L_T^2} \\ &\leq C \| \|\psi\|_{H^1}^{p-2} \|\psi\|_{H^1}^{\frac{2+\varepsilon}{2}} \| \|\psi\|_{H^6}^{\frac{2-\varepsilon}{2}} \|_{L_T^2} \\ &\leq CT^{\frac{\varepsilon}{4}} \rho^p. \end{aligned}$$

If  $n \geq 4$ , then we choose  $\varepsilon$  with  $\frac{2n(b+\varepsilon)}{4-(n-2)(p-1)} < n$ . We have from Hardy-Sobolev's inequality that

$$\begin{aligned} \mathcal{N}_7 &\leq C \| |x|^{-b-\varepsilon} \|_{L^{\frac{2n}{4-(n-2)(p-1)}}} \| \|\psi\|_{r_0}^{p-1} \| |x|^{-1+\varepsilon} \psi \|_{L^{r_0}} \|_{L_T^2} \\ &\leq C \| \|\psi\|_{r_0}^{p-1} \| \|\psi\|_{L^{r_0}}^{1-\varepsilon} \| \|\nabla \psi\|_{L^{r_0}}^\varepsilon \|_{L_T^2} \\ &\leq CT^{\frac{1-\varepsilon}{2}} \| \|\psi\|_{L_T^\infty H^1}^{p-\varepsilon} \| \|\psi\|_{L_T^{q_0} H_{r_0}^1}^\varepsilon \| \\ &\leq CT^{\frac{1-\varepsilon}{2}} \rho^p. \end{aligned}$$

Therefore we can find  $\theta' > 0$  such that

$$(4.4) \quad \sum_{i=5}^8 \mathcal{N}_i \leq C(T + T^{\theta'})\rho^p.$$

Taking  $\rho, T$  such that  $\rho \geq 2C\|\psi_0\|_{H^1}$  and  $2C(2T + T^\theta + T^{\theta'})\rho^p \leq \rho$ , from (4.3) and (4.4) we deduce that  $\Phi$  is self-mapping on  $X_T^p$ .

### 4.3. Contraction

By direct calculation we have that for  $p \geq 2$

$$\begin{aligned} |\nabla(|u|^{p-1}u) - \nabla(|v|^{p-1}v)| &\leq C(|u|^{p-2} + |v|^{p-2})(|\nabla u| + |\nabla v|)|u - v| \\ &\quad + C(|u|^{p-1} + |v|^{p-1})|\nabla u - \nabla v|, \end{aligned}$$

and for  $1 < p < 2$

$$\begin{aligned} |\nabla(|u|^{p-1}u) - \nabla(|v|^{p-1}v)| &\leq C(|u|^{p-1} + |v|^{p-1})|\nabla u - \nabla v| \\ &\quad + C(|\nabla u| + |\nabla v|)|u - v|^{p-1}. \end{aligned}$$

Applying the above estimates of self-mapping one can easily obtain the contraction

$$(4.5) \quad d(\Phi(u), \Phi(v)) \leq \frac{1}{2}d(u, v),$$

provided  $T$  is a little smaller than one of self-mapping. This shows the local well-posedness of (1.1). One can also show the conservation laws by the argument for Strichartz solutions of [16] or classical argument of [3]. The global well-posedness follows easily from the conservations and (2.1), which give us uniform bound of  $\|\nabla\psi\|_{L^2}$ . We omit the detail.

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