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COCYCLIC MORPHISM SETS DEPENDING ON A MORPHISM IN THE CATEGORY OF PAIRS

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Abstract. In this paper, we apply the notion of cocyclic maps to the category of pairs proposed by Hilton and obtain more general concepts. We discuss the concept of cocyclic morphisms with respect to a morphism and find that it is a dual concept of cyclic morphisms with respect to a morphism and a generalization of the notion of cocyclic morphisms with respect to a map. Moreover, we investigate its basic properties including the preservation of cocyclic properties by morphisms and find conditions for which the set of all homotopy classes of cocyclic morphisms with respect to a morphism will have a group structure.

1. Introduction

Given a topological space X, Gottlieb [1,2] introduced and studied the evaluation subgroups $G_n(X)$ of the homotopy groups $\pi_n(X)$ using the concept of cyclic homotopies. The author investigated the relationship between the evaluation subgroups and Euler characteristic, universal bundle, and cohomology groups, etc. Varadarajan [9] extended the concept of cyclic homotopies to that of cyclic maps and introduced the concept of cocyclic maps as a dual concept. Furthermore, the author called the sets of cyclic and cocyclic maps Gottlieb and dual Gottlieb sets, respectively, and studied their properties based on the Whitehead product, Puppe sequence, and Eckmann–Hilton duality [3]. Woo and Kim [10] introduced the generalized Gottlieb groups $G_n(X, A)$ for a topological pair (X, A), while Woo and Lee [11] introduced the relative Gottlieb groups $G_n^{Rel}(X, A)$ for a topological pair (X, A) and G-sequence. It is well-known that G-sequences are not always exact except under certain conditions [6,8,12]. Furthermore, Lee and Woo [5,6] also introduced the concepts of cyclic morphism, cocyclic morphism, and dual G-sequence in the category of pairs. As in the case of G-sequences, the dual G-sequences are not always exact except under certain conditions. In [4], Lee and Kim introduced a more general concept of cyclic morphisms with respect to a map and studied their properties.

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The category of pairs proposed by Hilton[3] is a category in which the objects are maps $(A, *) \to (B, *)$ and a map from $\alpha : A_1 \to A_2$ to $\beta : B_1 \to B_2$ is a pair of maps (f_1, f_2) such that the diagram

$$\begin{array}{ccc} A_1 & \xrightarrow{\alpha} & A_2 \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} \\ B_1 & \xrightarrow{\beta} & B_2 \end{array}$$

is commutative, that is, $\beta f_1 = f_2 \alpha$. We shall call the maps in this category *morphisms* to distinguish them from the maps between spaces. Two morphisms $(f_1, f_2), (g_1, g_2) : \alpha \to \beta$ are said to be homotopic if there exists a morphism $(H_1, H_2) : \alpha \times 1_I \to \beta$ such that H_1 and H_2 are homotopies between f_1 and g_1 and between f_2 and g_2, respectively, where 1_I is the identity map on the unit interval.

The set $\Pi(\alpha, \beta)$ is the set of all homotopy classes of morphisms from α to β in the category of pairs. In particular, $\Pi_n(\alpha, \beta) = \Pi(\Sigma^n \alpha, \beta)$ is a group if $n \geq 1$ and abelian if $n \geq 2$, where $\Sigma^n \alpha : \Sigma^n A_1 \to \Sigma^n A_2$ is the n-fold suspension map. Moreover, if $\alpha = i_n : \Sigma^{n-1} A \to C\Sigma^{n-1} A$ is the natural inclusion, then $\Pi(\alpha, \beta)$ is denoted by $\Pi_n(A, \beta)$. Also, if β is an inclusion and $A = S^0$, then we obtain ordinary relative homotopy groups. Furthermore, if $\beta : * \to B$, $\Pi_n(A, \beta) = \Pi_n(A, B)$ and if $\beta : B \to *$, then $\Pi_n(A, \beta) = \Pi_{n-1}(A, B)$.

In this paper, we extend the concept of cocyclic morphisms to that of cocyclic morphisms with respect to a morphism in the category of pairs and investigate its homotopy properties. Furthermore, we discuss the set of all homotopy classes of cocyclic morphisms with respect to a morphism referred to as the cocyclic morphism set depending on a morphism. The set of cocyclic morphisms from α to β depending the morphism (h_1, h_2) from α to itself is denoted by $DG^{(h_1,h_2)}(\alpha, \beta)$ (Definition 3.3). We investigate whether a cocyclic morphism set depending on a morphism is homotopy invariant or has a group structure. The proof of our main results (Corollary 3.6 and Theorem 3.8) has been presented in Section 3.

Corollary 3.6. *$DG^{(h_1,h_2)}(\alpha, \beta)$ is two-sided homotopy invariant.*

Theorem 3.8. *Let $\alpha : A_1 \to A_2$ be an object and $\beta : B_1 \to B_2$ be an H-group object. If $(h_1, h_2) : \alpha \to \alpha$ is a homotopy equivalent morphism, then $DG^{(h_1,h_2)}(\alpha, \beta)$ is a subgroup of $\Pi(\alpha, \beta)$.*

Throughout this paper, all spaces are pointed, connected and have the homotopy type of a CW-complex. Moreover, all maps and homotopies preserve the base points, and we use the same notation for a map $f : X \to Y$ and its homotopy class in $[X, Y]$.

2. Definitions and notations

In this section, we explain several concepts mentioned in Section 1.

A map $f : A \to X$ is said to be *cyclic* [9] if there exists a map $H : A \times X \to X$ such that the diagram

$$\begin{array}{ccc} A \times X & \xrightarrow{H} & X \\ \uparrow{\scriptstyle j} & & \uparrow{\scriptstyle \nabla} \\ A \vee X & \xrightarrow{f \vee 1} & X \vee X \end{array}$$

is commutative, where j is the inclusion map and ∇ is the folding map.

We denote the set of all homotopy classes of cyclic maps from A to X by $G(A, X)$ (see [9]), that is, $G(A, X) = \{[f] \in \Pi(A, X) \mid f \text{ is a cyclic map}\}$, equivalently, $G(A, X) = \omega_*(\Pi(A, X^X))$, where $\omega : X^X \to X$ is the evaluation map. In particular, $G(\Sigma^n A, X)$ is denoted by $G_n(A, X)$. Clearly, $\omega_*(\Pi_n(A, X^X)) = G_n(A, X)$. The subgroup $G_n(A, X)$ is a generalization of $G(A, X)$ and the Gottlieb group $G_n(X)$. In fact, $G_0(A, X) = G(A, X)$ and $G_n(S^0, X) = G_n(X)$.

A map $f : X \to A$ is said to be *cocyclic* [7] if there exists a map $\phi : X \to X \vee A$ such that the following diagram is homotopy commutative:

$$\begin{array}{ccc} X \times X & \xrightarrow{1 \times f} & X \times A \\ \uparrow{\scriptstyle \triangle} & & \uparrow{\scriptstyle j} \\ X & \xrightarrow{\phi} & X \vee A, \end{array}$$

where j is the inclusion map and $\triangle$ is the folding map.

Such a map ϕ is called a *coassociated map* of f. The set of all homotopy classes of cocyclic maps from X to A is denoted by $DG(X, A)$ (see [7]), that is, $DG(X, A) = \{[f] \in \Pi(X, A) \mid f \text{ is a cocyclic map}\}$.

Let $\alpha : A_1 \to A_2$, $\beta : B_1 \to B_2$, and $\gamma : X \to Y$ be the objects and let $(h_1, h_2) : \gamma \to \beta$ be a morphism in the category of pairs. A map $(f_1, f_2) : \alpha \to \beta$ is called a *cyclic morphism with respect to* (h_1, h_2) [4] if there exists a map $(H_1, H_2) : \alpha \times \gamma \to \beta$ such that $(H_1, H_2)|_\alpha = (f_1, f_2)$ and $(H_1, H_2)|_\gamma = (h_1, h_2)$.

In this case, (H_1, H_2) is called an *affiliated morphism* of (f_1, f_2) with respect to (h_1, h_2). Moreover, if $(h_1, h_2) : \beta \to \beta$ is the identity morphism, then (f_1, f_2) is called a *cyclic morphism*.

The subset $G^{(h_1,h_2)}(\alpha, \beta)$ of $\Pi(\alpha, \beta)$ is defined as the set of homotopy classes of cyclic morphisms with respect to $(h_1, h_2) : \gamma \to \beta$ (see [4]). That is,

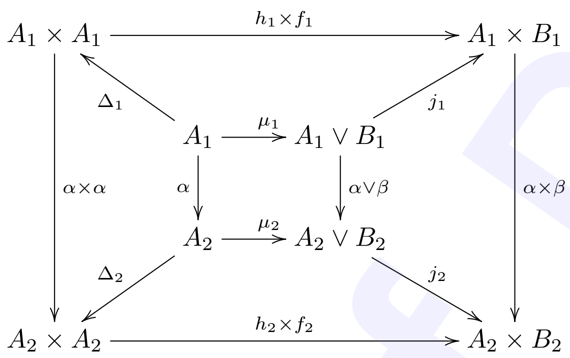
$$G^{(h_1,h_2)}(\alpha, \beta) = \{[f_1, f_2] \in \Pi(\alpha, \beta) | (f_1, f_2) \text{ is a cyclic morphism w.r.t } (h_1, h_2)\}.$$

3. Cocyclic morphism with respect to a morphism

In this section, we introduce and discuss the concept of cocyclic morphisms with respect to a morphism in the category of a pair, which is a dual concept of cyclic morphisms with respect to a morphism as discussed in [4]. We begin by introducing the definition of a cocyclic morphism with respect to a morphism.

Let $\alpha : A_1 \to A_2$ and $\beta : B_1 \to B_2$ be objects in the category of pairs and let $(h_1, h_2) : \alpha \to \alpha$ be a morphism.

Definition 3.1. A morphism $(f_1, f_2) : \alpha \to \beta$ is said to be *cocyclic with respect to* (h_1, h_2) if there exists a morphism $(\mu_1, \mu_2) : \alpha \to \alpha \vee \beta$ such that $(j_1, j_2) \circ (\mu_1, \mu_2) : \alpha \to \alpha \times \alpha$ is homotopic to $((h_1 \times f_1) \circ \Delta_1, (h_2 \times f_2) \circ \Delta_2)$, where (Δ_1, Δ_2) is the diagonal morphism and $(j_1, j_2) : \alpha \vee \beta \to \alpha \times \beta$ is the inclusion morphism.



In this case, (μ_1, μ_2) is called a *coaffiliated morphism of* (f_1, f_2) *with respect to* (h_1, h_2). If $(h_1, h_2) = (\mathrm{id}_{A_1}, \mathrm{id}_{A_2})$, then a cocyclic morphism (f_1, f_2) with respect to (h_1, h_2) is called a *cocyclic morphism*.

Lemma 3.1. *If $(f_1, f_2) : \alpha \to \beta$ is a cocyclic morphism, then (f_1, f_2) is a cocyclic morphism with respect to any morphism $(h_1, h_2) : \alpha \to \alpha$.*

Proof. Let (μ_1, μ_2) be a coaffiliated morphism of (f_1, f_2). Then, $((h_1 \vee \mathrm{id}_{B_1}) \circ \mu_1, (h_2 \vee \mathrm{id}_{B_2}) \circ \mu_2)$ is a coaffiliated morphism of (f_1, f_2) with respect to (h_1, h_2). In fact,

$$\begin{aligned}
&(j_1, j_2) \circ ((h_1 \vee \mathrm{id}_{B_1}) \circ \mu_1, (h_2 \vee \mathrm{id}_{B_2}) \circ \mu_2) \\
= &(j_1 \circ (h_1 \vee \mathrm{id}_{B_1}) \circ \mu_1, j_2 \circ (h_2 \vee \mathrm{id}_{B_2}) \circ \mu_2) \\
= &((h_1 \times \mathrm{id}_{B_1}) \circ j_1 \circ \mu_1, (h_2 \times \mathrm{id}_{B_2}) \circ j_2 \circ \mu_2 \\
\simeq &((h_1 \times \mathrm{id}_{B_1})(\mathrm{id}_{A_1} \times f_1) \circ \Delta_1, (h_2 \times \mathrm{id}_{B_2})(\mathrm{id}_{A_2} \times f_2) \circ \Delta_2) \\
= &(h_1 \times f_1, h_2 \times f_2) \circ (\Delta_1, \Delta_2).
\end{aligned}$$

□

Remark. Suppose that $(c_1, c_2) : \alpha \to \alpha$ is the constant morphism, that is, $c_i : A_i \to A_i$ is given by $c_i(a) = *$ for $i = 1, 2$, and $\mu_i : A_i \to A_i \vee B_i$ is defined by $\mu_i(x) = (*, f_i(x))$ for $i = 1, 2$. Then, $(\mu_1, \mu_2) : \alpha \to \alpha \vee \beta$ is a coaffiliated morphism of $(f_1, f_2) : \alpha \to \beta$ with respect to (c_1, c_2). This implies that every morphism $(f_1, f_2) : \alpha \to \beta$ is cocyclic with respect to the constant morphism (c_1, c_2).

Definition 3.2. A map $\alpha : A_1 \to A_2$ is said to be a *co-H-object with respect to a morphism* $(h_1, h_2) : \alpha \to \alpha$ if there exists a morphism $(\mu_1, \mu_2) : \alpha \to \alpha \vee \alpha$ such that $(j_1, j_2) \circ (\mu_1, \mu_2) : \alpha \to \alpha \times \alpha$ is homotopic to $((h_1, \mathrm{id}_{A_1}), (h_2, \mathrm{id}_{A_2}))$.

If (h_1, h_2) is the identity morphism $(\mathrm{id}_{A_1}, \mathrm{id}_{A_2})$, then a co-$H$-object with respect to (h_1, h_2) is a co-H-object.

Example 1. Every object $\alpha : A_1 \to A_2$ is a co-H-object with respect to the constant morphism $(c_1, c_2) : \alpha \to \alpha$. Let $\mu_i : A_i \to A_i \vee A_i$ be the inclusion $\mu_i(a) = (*, a)$ for $i = 1, 2$. Then, $j_i \circ \mu_i(a) = j_i(*, a) = (*, a) = (c_i, \mathrm{id}_{A_i})(a)$ for $i = 1, 2$.

Example 2. If α is a co-H-object, then it is a co-H-object with respect to any morphism $(h_1, h_2) : \alpha \to \alpha$. Let α be a co-H-object and $(h_1, h_2) : \alpha \to \alpha$ be a morphism. Then, there exists a morphism $(\mu_1, \mu_2) : \alpha \to \alpha \vee \alpha$ such that $(j_1, j_2) \circ (\mu_1, \mu_2) : \alpha \to \alpha \times \alpha$ is homotopic to (Δ_1, Δ_2), where $(j_1, j_2) : \alpha \vee \alpha \to \alpha \times \alpha$ is the inclusion and $(\Delta_1, \Delta_2) : \alpha \to \alpha \times \alpha$ is the diagonal morphism. Define $\overline{\mu_i} = (h_i \vee \mathrm{id}_{A_1}) \circ \mu_i$ for $i = 1, 2$. Then, $(j_1, j_2) \circ (\overline{\mu_1}, \overline{\mu_2}) : \alpha \to \alpha \times \alpha$ is homotopic to $((h_1, \mathrm{id}_{A_1}), (h_2, \mathrm{id}_{A_2}))$.

Example 3. Let (A_1, μ_1) and (A_2, μ_2) be co-H-spaces and $\alpha : A_1 \to A_2$ be a map. If $(\mu_1, \mu_2) : \alpha \to \alpha \vee \alpha$ is a morphism, then α is a co-H-object. Hence, α is a co-H-object with respect to any morphism $(h_1, h_2) : \alpha \to \alpha$.

Suppose α is a co-H-object with respect to a morphism $(h_1, h_2) : \alpha \to \alpha$. Then every morphism $(f_1, f_2) : \alpha \to \beta$ is cocyclic with respect to (h_1, h_2). Moreover, if $(\mu_1, \mu_2) : \alpha \to \alpha \vee \alpha$ is a co-H-structure with respect to (h_1, h_2), then $((1 \vee f_1) \circ \mu_1, (1 \vee f_2) \circ \mu_2)$ is a coaffiliated morphism of (f_1, f_2).

Lemma 3.2. *For a given morphism $(h_1, h_2) : \alpha \to \alpha$, if $(f_1, f_2) : \alpha \to \beta$ is a cocyclic morphism with respect to (h_1, h_2) and $(\theta_1, \theta_2) : \beta \to \gamma$ is an arbitrary morphism, then $(\theta_1, \theta_2) \circ (f_1, f_2) : \alpha \to \gamma$ is a cocyclic morphism with respect to (h_1, h_2).*

Proof. Let $\alpha : A_1 \to A_2$, $\beta : B_1 \to B_2$, and $\gamma : C_1 \to C_2$ be objects. Since $(f_1, f_2) : \alpha \to \beta$ is a cocyclic morphism with respect to (h_1, h_2), then there exists a coaffiliated morphism $(\mu_1, \mu_2) : \alpha \to \alpha \vee \beta$ with respect to (h_1, h_2). Consequently, $(1 \vee \theta_1, 1 \vee \theta_2) \circ (\mu_1, \mu_2) : \alpha \to \alpha \vee \gamma$ is an affiliated morphism of $(\theta_1, \theta_2) \circ (f_1, f_2)$. In fact,

$$\begin{aligned}
(j_1, j_2) \circ (1 \vee \theta_1, 1 \vee \theta_2) \circ (\mu_1, \mu_2) &= (1 \times \theta_1, 1 \times \theta_2) \circ (j_1, j_2) \circ (\mu_1, \mu_2) \\
&\simeq (1 \times \theta_1, 1 \times \theta_2)(h_1 \times f_1, h_2 \times f_2) \\
&= (h_1 \times \theta_1 f_1, h_2 \times \theta_2 f_2),
\end{aligned}$$

and we have the following homotopy commutative diagram:

$$\begin{array}{cccccc}
A_1\times A_1 & \xrightarrow{h_1\times f_1} & & A_1\times B_1 & \xrightarrow{1\times\theta_1} & A_1\times C_1 \\
\Big\downarrow{\scriptstyle \alpha\times\alpha} & \nwarrow{\scriptstyle \Delta_1} & & \Big\uparrow{\scriptstyle j_1} & & \Big\uparrow{\scriptstyle j_1} \\
 & A_1 & \xrightarrow{\mu_1} & A_1\vee B_1 & \xrightarrow{1\vee\theta_1} & A_1\vee C_1 \\
 & \Big\downarrow{\scriptstyle \alpha} & & \Big\downarrow{\scriptstyle \alpha\vee\beta} & & \Big\downarrow{\scriptstyle \alpha\vee\gamma} \\
 & A_2 & \xrightarrow[\mu_2]{} & A_2\vee B_2 & \xrightarrow[1\vee\theta_2]{} & A_2\vee C_2 \\
 & \swarrow{\scriptstyle \Delta_2} & & \Big\downarrow{\scriptstyle j_2} & & \Big\downarrow{\scriptstyle j_2} \\
A_2\times A_2 & \xrightarrow[h_2\times f_2]{} & & A_2\times B_2 & \xrightarrow[1\times\theta_2]{} & A_2\times C_2
\end{array}$$

□

From the definition of a cocyclic morphism with respect to a morphism, we can say that if (f_1, f_2) is homotopic to (g_1, g_2) and (f_1, f_2) is a cocyclic morphism with respect to (h_1, h_2), then (g_1, g_2) is a cocyclic morphism with respect to (h_1, h_2).

Definition 3.3. The subset $DG^{(h_1,h_2)}(\alpha, \beta)$ of $\Pi(\alpha, \beta)$ is defined as the set of all elements of $\Pi(\alpha, \beta)$, which is a cocyclic morphism with respect to (h_1, h_2). That is,

$$\begin{aligned} &DG^{(h_1,h_2)}(\alpha, \beta) \\ &= \{[f_1, f_2] \in \Pi(\alpha, \beta) | (f_1, f_2) \text{ is a cocyclic morphism w.r.t. } (h_1, h_2)\}. \end{aligned}$$

$DG^{(h_1,h_2)}(\alpha, \beta)$ is called *the cocyclic morphism set from α to β depending on the morphism* (h_1, h_2).

From Lemma 3.1 and the remark above, we have that

$$\begin{aligned} DG(\alpha, \beta) = DG^{(\mathrm{id}_{A_1}, \mathrm{id}_{A_2})}(\alpha, \beta) &\subseteq DG^{(h_1,h_2)}(\alpha, \beta) \\ &\subseteq DG^{(c_1,c_2)}(\alpha, \beta) = \Pi(\alpha, \beta). \end{aligned}$$

The following result gives a characterization of a co-H-object with respect to a morphism in terms of the cocyclicity of a morphism with respect to the morphism.

Proposition 3.3. *Let $\alpha : A_1 \to A_2$ be an object and $(h_1, h_2) : \alpha \to \alpha$ be a morphism. Then, the following are equivalent*:

(a) *α is a co-H-object with respect to* (h_1, h_2);
(b) $1_\alpha = (id_{A_1}, id_{A_2}) : \alpha \to \alpha$ *is a cocyclic morphism with respect to* (h_1, h_2);
(c) $DG^{(h_1,h_2)}(\alpha, \beta) = \Pi(\alpha, \beta)$.

Proof. (a) $\Leftrightarrow$ (b). This follows immediately from the definitions of the co-H-object with respect to (h_1, h_2) and the cocyclic morphism with respect to (h_1, h_2).

(b) $\Rightarrow$ (c). Let $[f_1, f_2] \in \Pi(\alpha, \beta)$. As $(f_1, f_2) = (f_1, f_2) \circ (\mathrm{id}_{A_1}, \mathrm{id}_{A_2})$, (f_1, f_2) is a cocyclic morphism with respect to (h_1, h_2) by Lemma 3.2. Thus, $[f_1, f_2] \in DG^{(h_1,h_2)}(\alpha, \beta)$.

(c) $\Rightarrow$ (b). By the hypothesis, $(\mathrm{id}_{A_1}, \mathrm{id}_{A_2}) \in \Pi(\alpha, \alpha) = DG^{(h_1,h_2}(\alpha, \alpha)$. Thus, $(\mathrm{id}_{A_1}, \mathrm{id}_{A_2})$ is a cocyclic morphism with respect to (h_1, h_2). □

It has been observed that if $(\theta_1, \theta_2) : \beta \to \gamma$ is a homotopy equivalence, then $(\theta_1, \theta_2)_* : \Pi(\alpha, \beta) \to \Pi(\alpha, \gamma)$ is a one-to-one correspondence.

Let $\alpha : A_1 \to A_2$, $\beta : B_1 \to B_2$ and $\gamma : C_1 \to C_2$ be objects and let $(h_1, h_2) : \alpha \to \alpha$ be a morphism. Then, we have the following lemma.

Lemma 3.4. *If $(\theta_1, \theta_2) : \beta \to \gamma$ is a homotopy equivalence, then $(\theta_1, \theta_2)_*$ maps $DG^{(h_1,h_2)}(\alpha, \beta)$ onto $DG^{(h_1,h_2)}(\alpha, \gamma)$.*

Proof. The proof follows immediately from Lemma 3.2. In fact, if $(\theta_1^{-1}, \theta_2^{-1})$ is the homotopy inverse of (θ_1, θ_2), then

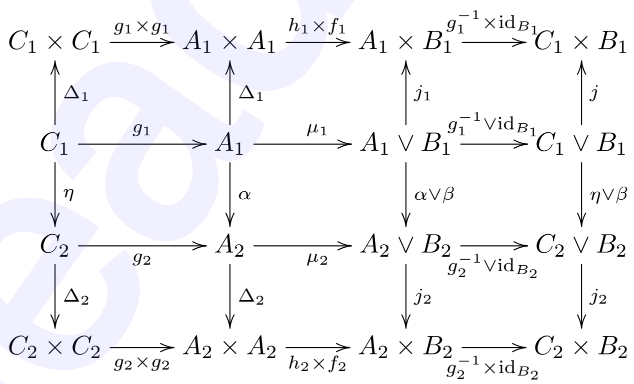
$$(\theta_1^{-1}, \theta_2^{-1})_*(DG^{(h_1,h_2)}(\alpha, \gamma)) \subseteq DG^{(h_1,h_2)}(\alpha, \beta).$$

□

Note that if $(g_1, g_2) : \eta \to \alpha$ is a homotopy equivalence, then $(g_1, g_2)^* : \Pi(\alpha, \beta) \to \Pi(\eta, \beta)$ is a one-to-one correspondence.

Lemma 3.5. *Let $(g_1, g_2) : \eta \to \alpha$ be a homotopy equivalence. Then $(g_1, g_2)^*$ maps $DG^{(h_1,h_2)}(\alpha, \beta)$ onto $DG^{(g_1^{-1}h_1g_1, g_2^{-1}h_2g_2)}(\eta, \beta)$. Moreover, $(g_1^{-1}, g_2^{-1})^*$ maps $DG^{(g_1^{-1}h_1g_1, g_2^{-1}h_2g_2)}(\eta, \beta)$ onto $DG^{(h_1,h_2)}(\alpha, \beta)$.*

Proof. Let $[f_1, f_2] \in DG^{(h_1,h_2)}(\alpha, \beta)$. Then, there exists a coaffiliated map $(\mu_, \mu_2) : \alpha \to \alpha \vee \beta$. That is, $j_i\mu_i \simeq (h_i \times f_i)\Delta_i$, where $j_i : A_i \vee B_i \to A_i \times B_i$ is the inclusion and $\Delta_i : A_i \to A_i \times A_i$ is the diagonal map for $i = 1, 2$. Define $\overline{\mu_i} = (g_i^{-1} \vee \mathrm{id}_{B_i})\mu_i g_i$ for $i = 1, 2$, where (g_1^{-1}, g_2^{-1}) is the inverse morphism of (g_1, g_2). Then, $(\overline{\mu_1}, \overline{\mu_2})$ is a coaffiliated map (f_1g_1, f_2g_2) with respect to $(g_1^{-1}h_1g_1, g_2^{-1}h_2g_2)$.



Therefore, $(g_1, g_2)^*[f_1, f_2] = [f_1g_1, f_2g_2] \in DG^{(g_1^{-1}h_1g_1, g_2^{-1}h_2g_2)}(\eta, \beta)$. Similarly, if $[k_1, k_2] \in DG^{((g_1^{-1})h_1g_1, g_2^{-1}h_2g_2)}(\eta, \beta)$, then

$$\begin{aligned}
&(g_1^{-1}, g_2^{-1})^*[k_1, k_2] = [k_1g_1^{-1}, k_2g_2^{-1}] \\
&\in DG^{((g_1^{-1})^{-1}g_1^{-1}h_1g_1g_1^{-1}, (g_2^{-1})^{-1}g_2^{-1}h_2g_2g_2^{-1})}(\alpha, \beta) = DG^{(h_1,h_2)}(\alpha, \beta).
\end{aligned}$$

It follows that $(g_1, g_2)^*$ maps $DG^{(h_1,h_2)}(\alpha, \beta)$ onto $DG^{(g_1^{-1}h_1g_1, g_2^{-1}h_2g_2)}(\eta, \beta)$.
Similarly, $(g_1^{-1}, g_2^{-1})^*$ maps $DG^{(g_1^{-1}h_1g_1, g_2^{-1}h_2g_2)}(\eta, \beta)$ onto $DG^{(h_1,h_2)}(\alpha, \beta)$.
This completes the proof of the lemma. □

From Lemmas 3.4 and 3.5, we have the following corollary.

Corollary 3.6. *$DG^{(h_1,h_2)}(\alpha, \beta)$ is two-sided homotopy invariant.*

Definition 3.4 (Lee and Woo [6])**.** We say that $\alpha : A_1 \to A_2$ is an *H-object* if there exists a morphism $(m_1, m_2) : \alpha \times \alpha \to \alpha$ such that $(m_1, m_2) \circ (j_1, j_2) : \alpha \vee \alpha \to \alpha$ is homotopic to (∇_1, ∇_2), where $(j_1, j_2) : \alpha \vee \alpha \to \alpha \times \alpha$ is the inclusion map and $\nabla_i : A_i \vee A_i \to A_i$ is the folding map for $i = 1, 2$.

Here, we recall a well-known fact.

Fact. For any space X, let $e_X : X \to \Omega\Sigma X$ be the usual map given by $e_X(x)(s) = \langle x, s\rangle$. If $\alpha : X \to Y$ is an object, then $(e_X, e_Y) : \alpha \to \Omega\Sigma\alpha$ is a morphism. In fact,

$$e_Y(\alpha(x))(s) = \langle \alpha(x), s\rangle = \Sigma\alpha\langle x, s\rangle = \Sigma\alpha(e_X(x)(s)) = (\Omega\Sigma\alpha \circ e_X(x))(s).$$

In particular, if X is an H-space, then there exists a map $s_X : \Omega\Sigma X \to X$ such that $s_X \circ e_X \simeq 1_X$. Moreover, $(s_X, s_Y) : \Omega\Sigma\alpha \to \alpha$ is a morphism if α is an H-object.

Lemma 3.7. *Let $\alpha : A_1 \to A_2$ be an object, $\beta : B_1 \to B_2$ be an H-object, and $(h_1, h_2) : \alpha \to \alpha$ be a morphism. Then, $(f_1, f_2) : \alpha \to \beta$ is a cocyclic morphism with respect to (h_1, h_2) if and only if $(e_{B_1}, e_{B_2}) \circ (f_1, f_2)$ is a cocyclic morphism with respect to (h_1, h_2).*

Proof. The "only if" part follows from Lemma 3.2.

Let $(e_{B_1}, e_{B_2}) \circ (f_1, f_2)$ be a cocyclic morphism with respect to (h_1, h_2) with respect to (h_1, h_2) and (s_{B_1}, s_{B_2}) be the morphism mentioned above. From Lemma 3.2, $(s_{B_1}, s_{B_2}) \circ (e_{B_1}, e_{B_2}) \circ (f_1, f_2)$ is a cocyclic morphism with respect to (h_1, h_2). Since $(s_{B_1}, s_{B_2}) \circ (e_{B_1}, e_{B_2}) \circ (f_1, f_2) \simeq (f_1, f_2)$, (f_1, f_2) is cocyclic with respect to (h_1, h_2). □

Now, we discuss the group structure of $DG^{(h_1,h_2)}(\alpha, \beta)$. Let $\alpha : A_1 \to A_2$ and $\beta : B_1 \to B_2$ be objects. Then, the set $\Pi(\alpha, \beta)$ has a group structure if β is an H-group object. Let (m_1, m_2) be an H-structure with inverse structure (ν_1, ν_2). If $[f_1, f_2], [g_1, g_2] \in \Pi(\alpha, \beta)$, then $[f_1, f_2]+[g_1, g_2]$ is defined by $[m_1(f_1 \times g_1)\Delta_1, m_2(f_2 \times g_2)\Delta_2]$, where $(\Delta_1, \Delta_2) : \alpha \to \alpha \times \alpha$ is the diagonal morphism. Moreover, $[f_1, f_2]^{-1}$ is given by $[\nu_1 f_1, \nu_2 f_2]$. In [6], it was shown that $DG(\alpha, \beta)$

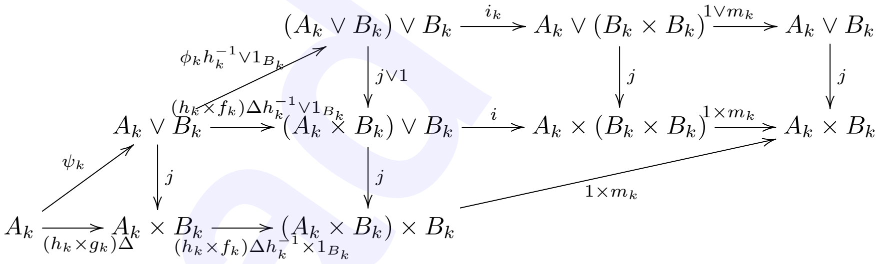
is a subgroup of $\Pi(\alpha,\beta)$ if β is an H-group object. Therefore, if β is an H-group object, then $DG^{(h_1,h_2)}(\alpha,\beta)$ is a subgroup of $\Pi(\alpha,\beta)$ provided $(h_1,h_2) = (1_{A_1},1_{A_2})$ or $(h_1,h_2) = (c_1,c_2)$, where $(1_{A_1},1_{A_2}) : \alpha \to \alpha$ is the identity morphism and $(c_1,c_2) : \alpha \to \alpha$ is the constant morphism. This follows from the fact that $DG^{(1_{A_1},1_{A_2})}(\alpha,\beta) = DG(\alpha,\beta)$ and $DG^{(c_1,c_2)}(\alpha,\beta) = \Pi(\alpha,\beta)$.

These facts naturally raise the following questions. If β is an H-group object, does $DG^{(h_1,h_2)}(\alpha,\beta)$ have a group structure for any morphism $(h_1,h_2) : \alpha \to \alpha$? If not, is there any morphism (h_1,h_2) different from the identity and constant morphisms mentioned above, such that $DG^{(h_1,h_2)}(\alpha,\beta)$ is a subgroup of $\Pi(\alpha,\beta)$? Theorem 3.8 below provides an affirmative answer to the last question.

Theorem 3.8. *Let $\alpha : A_1 \to A_2$ be an object and $\beta : B_1 \to B_2$ be an H-group object. If $(h_1,h_2) : \alpha \to \alpha$ is a homotopy equivalent morphism, then $DG^{(h_1,h_2)}(\alpha,\beta)$ is a subgroup of $\Pi(\alpha,\beta)$.*

Proof. Let (m_1,m_2) be the H-structure and (ν_1,ν_2) be the inverse structure on β. Then, the inverse of $[f_1,f_2]$ in the group $\Pi(\alpha,\beta)$ is the homotopy class of $[\nu_1 f_1,\nu_2 f_2]$. According to Lemma 3.2, $(\nu_1 f_1,\nu_2 f_2) = (\nu_1,\nu_2)\circ(f_1,f_2)$ is a cocyclic morphism with respect to (h_1,h_2) if (f_1,f_2) is cocyclic with respect to (h_1,h_2). Thus, $DG^{(h_1,h_2)}(\alpha,\beta)$ is closed under the inverse structure.

Next, we show that $DG^{(h_1,h_2)}(\alpha,\beta)$ is closed under the additive operation. Let $[f_1,f_2], [g_1,g_2] \in DG^{(h_1,h_2)}(\alpha,\beta)$. Then, we have two affiliated morphisms $(\phi_1,\phi_2) : \alpha \to \alpha \vee \beta$ and $(\psi_1,\psi_2) : \alpha \to \alpha\vee\beta$ such that $j\phi_1 \simeq (h_1\times f_1)\Delta$, $j\phi_2 \simeq (h_1\times f_2)\Delta$, $j\psi_1 \simeq (h_1 \times g_1)\Delta$ and $j\psi_2 \simeq (h_2\times g_2)\Delta$. Let $i_k : (A_k \vee B_k)\vee B_k \to A_k \vee (B_k\times B_k)$ be the inclusion for $k = 1,2$. Define $\lambda_1 = (1_{A_1}\vee m_1)i_1(\phi_1 h_1^{-1}\vee 1_{B_1})\psi_1$ and $\lambda_2 = (1_{A_2}\vee m_2)i_2(\phi_2 h_2^{-1}\vee 1_{B_2})\psi_2$, where (h_1^{-1},h_2^{-1}) is the inverse morphism of (h_1,h_2). Then, we have $j\lambda_1 \simeq (h_1\times(f_1+g_1))\Delta$ and $j\lambda_2 \simeq (h_2 \times (f_2+g_2))\Delta$. This conclusion follows from the following diagram:



for $k = 1,2$, where the two triangular diagrams on the left are commutative up to homotopy and the other diagrams are commutative. Furthermore, since

$$(j\vee 1_{B_k})\circ(\phi_k h_k^{-1}\vee 1_{B_k}) = j\phi_k h_k^{-1}\vee 1_{B_k} \simeq (h_k\times f_k)\Delta h_k^{-1}\vee 1_{B_k},$$

we have $j\lambda_k \simeq (1_{A_k}\times m_k)\circ((h_k\times f_k)\Delta h_k^{-1}\times 1_{B_k} = (1_{A_k}\times m_k)\circ((h_k\times f_k)\Delta h_k^{-1}h_k \times g_k)\Delta = (1_{A_k}\times m_k)(h_k\times f_k)\Delta\times g_k)\Delta = (h_k\times m_k(f_k\times g_k))\Delta = (h_k\times(f_k+g_k))\Delta$ for $i = 1,2$.

Hence, to show that (λ_1, λ_2) is an affiliated map of $[f_1 + g_1, f_2 + g_2]$ with respect to (h_1, h_2), it is sufficient to show that (λ_1, λ_2) is a morphism from α to $\alpha \vee \beta$. In fact,

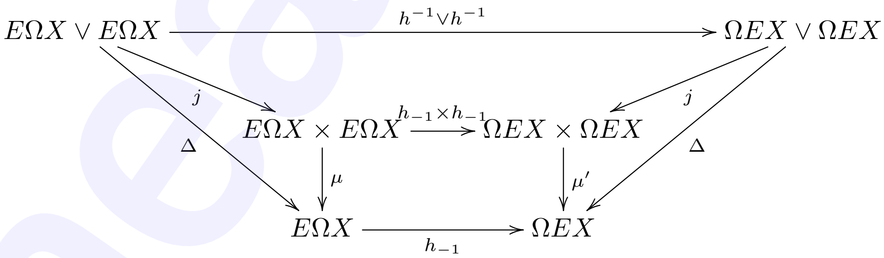
$$\begin{aligned}
\lambda_2 \circ \alpha &= [(1_{A_2} \vee m_2) i_2 (\phi_2 h_2^{-1} \vee 1_{B_2}) \psi_2] \circ \alpha \\
&= (1_{A_2} \vee m_2) i_2 (\phi_2 h_2^{-1} \vee 1_{B_2})(\alpha \vee \beta) \psi_1 \\
&= (1_{A_2} \vee m_2) i_2 (\phi_2 h_2^{-1} \alpha \vee \beta) \psi_1 \\
&= (1_{A_2} \vee m_2) i_2 ((\alpha \vee \beta) \phi_1 h_1^{-1} \vee \beta) \psi_1 \\
&= (1_{A_2} \vee m_2) i_2 ((\alpha \vee \beta) \vee \beta) \circ (\phi_1 h_1^{-1} \vee 1_{B_1}) \psi_1 \\
&= (1_{A_2} \vee m_2)(\alpha \vee (\beta \times \beta)) \circ i_1 (\phi_1 h_1^{-1} \vee 1_{B_1}) \psi_1 \\
&= (\alpha \vee m_2(\beta \times \beta)) i_1 (\phi_1 h_1^{-1} \vee 1_{B_1}) \psi_1 \\
&= (\alpha \vee \beta m_1) i_1 (\phi_1 h_1^{-1} \vee 1_{B_1}) \psi_1 \\
&= (\alpha \vee \beta)(1_{A_1} \vee m_1) i_1 (\phi_1 h_1^{-1} \vee 1_{B_1}) \psi_1 \\
&= (\alpha \vee \beta) \lambda_1.
\end{aligned}$$

Thus, $[f_1+g_1, f_2+g_2] \in DG^{(h_1,h_2)}(\alpha, \beta)$, so that $DG^{(h_1,h_2)}(\alpha, \beta)$ is closed under the operation of addition. Hence, $DG^{(h_1,h_2)}(\alpha, \beta)$ is a subgroup of $\Pi(\alpha, \beta)$. This completes the proof. ☐

Definition 3.5. A morphism $(h_1, h_2) : \alpha \to \alpha$ is called a *structural morphism* if $DG^{(h_1,h_2)}(\alpha, \beta)$ is a subgroup, where $\beta : B_1 \to B_2$ is an H-object.

Example 4. Let $\alpha : A_1 \to A_2$ be an object and $\beta : B_1 \to B_2$ be an H-object. Then, the identity morphism $(1_{A_1}, 1_{A_2})$ and the constant morphism (c_{A1}, c_{A_2}) are structural morphisms. Furthermore, by Theorem 3.8, a homotopy equivalent morphism $(h_1, h_2) : \alpha \to \alpha$ is a structural morphism.

Let EX and ΩX be the path and loop space of a space X, respectively. Then, the map $p : E\Omega X \to \Omega X$ given by $p(\alpha) = \alpha(0)$ is an H-group. Let $h : \Omega EX \to E\Omega X$ be a homeomorphism given by $h(\alpha)(s)(t) = \alpha(t)(s)$. Define $\mu : E\Omega X \times E\Omega X \to E\Omega X$ by $h\mu'(h^{-1} \times h^{-1})$, where μ' is the H-group structure of ΩEX. Then, μ is an H-group structure of $E\Omega X$. In fact, the following diagram is commutative:



Moreover, the inverse structure of $E\Omega X$ is given by $h\nu h^{-1}$, where ν is the inverse structure of ΩEX. Thus, to show that p is an H-structure object, it is sufficient to show that the following diagram is commutative:

$$\begin{array}{ccc} E\Omega X \times E\Omega X & \xrightarrow{\mu_1} & E\Omega X \\ \downarrow{\scriptstyle p\times p} & & \downarrow{\scriptstyle p} \\ \Omega X \times \Omega X & \xrightarrow[\mu_2]{} & \Omega X \end{array}$$

where μ_1 and μ_2 are the H-group structures of $E\Omega X$ and ΩX, respectively. From the definition of μ_1 and μ_2, it is easy to show that $p \circ \mu_1 = \mu_2 \circ (p \times p)$. Moreover, $p\nu_1(\alpha)(t) = \nu p(\alpha)(t)$, where $\nu : \Omega X \to \Omega X$ is the inverse structure.

Corollary 3.9. *Let $\alpha : A_1 \to A_2$ be an object and $p : E\Omega X \to \Omega X$ be the H-structure. Then, $DG^{(h_1,h_2)}(\alpha, p)$ is a subgroup of $\Pi(\alpha, p)$ if $(h_1, h_2) : \alpha \to \alpha$ is a structural morphism.*

In general, the map $p_n : E\Omega^{n-1}X \to \Omega^{n-1}X$ is an H-group object. Thus, $DG^{(h_1,h_2)}(\alpha, p_n)$ is a subgroup of $\Pi(\alpha, p_n) = \Pi_n(\alpha, X)$ for a structural morphism $(h_1, h_2) : \alpha \to \alpha$. Denote $DG^{(h_1,h_2)}(\alpha, p_n)$ by $DG_n^{(h_1,h_2)}(\alpha, X)$ for $n > 1$.

Theorem 3.10. *Let $(f_1, f_2) : \alpha \to \beta$ be a morphism. Then f_1 is a cocyclic map with respect to h_1 with ϕ_1 as an affiliated map and f_2 is a cocyclic map with respect to h_2 with ϕ_2 as an affiliated map. If α is a cofibration and $\phi_2\alpha$ is homotopic to $(\alpha \vee \beta)\phi_1$, then there exists a coaffiliated map ϕ' with respect to h_2 such that (ϕ_1, ϕ_2') is a morphism from α to $\alpha \vee \beta$ such that (f_1, f_2) is a cocyclic morphism with respect to (h_1, h_2).*

Proof. Let $\alpha : A_1 \to A_2$ and $\beta : B_1 \to B_2$ be objects. According to the hypothesis, the following diagram commutes up to homotopy:

$$\begin{array}{ccc} A_1 \times A_1 & \xrightarrow{h_1 \times f_1} & A_1 \times B_1 \\ \uparrow{\scriptstyle \Delta} & & \uparrow{\scriptstyle j} \\ A_1 & \xrightarrow{\phi_1} & A_1 \vee B_1 \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha\vee\beta} \\ A_2 & \xrightarrow[\phi_2]{} & A_2 \vee B_2 \\ \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle j} \\ A_2 \times A_2 & \xrightarrow[h_2 \times f_2]{} & A_2 \times B_2 \end{array}$$

Since $\phi_2\alpha \simeq (\alpha \vee \beta)\phi_1$, there exists a homotopy $H : A_1 \times I \to A_2 \vee B_2$ such that $H|_{A_1\times 1} = (\alpha \vee \beta)\phi_1$. Since α is a cofibration, there exists a homotopy $H' : A_2 \times I \to A_2 \vee B_2$ such that $H'(\alpha \times 1_I) = H$. Define $\phi_2' = H'|_{A_2\times 1}$. Then, we have

$$\phi_2'\alpha = H'|_{A_2\times 1}\alpha = H'(\alpha\times 1)|_{A_1\times 1} = H|_{A_1\times 1} = (\alpha\vee\beta)\phi_1.$$

Thus, (ϕ_1, ϕ_2') is a morphism from α to $\alpha\vee\beta$. However, $\phi_2 \simeq \phi_2'$ by the homotopy H. Therefore, (ϕ_1, ϕ_2') is a coaffiliated morphism of (f_1, f_2) with respect to (h_1, h_2). □

References

[1] D. H. Gottlieb, *A certain subgroup of the fundamental group*, Amer. J. Math. **87** (1965), 840–856. https://doi.org/10.2307/2373248

[2] ______, *Evaluation subgroups of homotopy groups*, Amer. J. Math. **91** (1969), 729–756. https://doi.org/10.2307/2373349

[3] P. Hilton, *Homotopy Theory and Duality*, Gordon and Breach Science Publishers, New York, 1965.

[4] J. Kim and K. Y. Lee, *Gottlieb subsets with respect to a morphism in the category of pairs*, Bull. Korean Math. Soc. **47** (2010), no. 6, 1311–1327. https://doi.org/10.4134/BKMS.2010.47.6.1311

[5] K. Y. Lee and M. H. Woo, *Cyclic morphisms in the category of pairs and generalized G-sequences*, J. Math. Kyoto Univ. **38** (1998), no. 2, 271–285. https://doi.org/10.1215/kjm/1250518118

[6] ______, *Cocyclic morphisms and dual G-sequences*, Topology Appl. **116** (2001), no. 1, 123–136. https://doi.org/10.1016/S0166-8641(00)00081-X

[7] K. L. Lim, *Cocyclic maps and coevaluation subgroups*, Canad. Math. Bull. **30** (1987), no. 1, 63–71. https://doi.org/10.4153/CMB-1987-009-1

[8] J. Z. Pan and M. H. Woo, *Exactness of G-sequences and monomorphisms*, Topology Appl. **109** (2001), no. 3, 315–320. https://doi.org/10.1016/S0166-8641(99)00178-9

[9] K. Varadarajan, *Generalised Gottlieb groups*, J. Indian Math. Soc. (N.S.) **33** (1969), 141–164 (1970).

[10] M. H. Woo and J.-R. Kim, *Certain subgroups of homotopy groups*, J. Korean Math. Soc. **21** (1984), no. 2, 109–120.

[11] M. H. Woo and K. Y. Lee, *On the relative evaluation subgroups of a CW-pair*, J. Korean Math. Soc. **25** (1988), no. 1, 149–160.

[12] ______, *Exact G-sequences and relative G-pairs*, J. Korean Math. Soc. **27** (1990), no. 2, 177–184.

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