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A MULTIPLE RECURRENCE THEOREM FOR COUNTABLE DIRECTED PARTIAL SEMIGROUP ACTIONS

Tiaoying Zeng

Abstract. We show a multiple recurrence theorem for topological dynamical systems with countable directed partial semigroup actions, which generalizes the well-know IP-version of multiple recurrence theorem proved by Furstenberg and Weiss.

1. Introduction

Recurrence is one of the central topics in the study of topological dynamics. A classical result by Birkhoff states that if T is a continuous map from a compact metric space X to itself then there exists $x \in X$ and an increasing sequence $\{n_k\}_{k=1}^{\infty}$ of positive integers such that $\lim_{k\to\infty} T^{n_k}x = x$.

In the seminal paper [5], Furstenberg and Weiss showed that the well-known van der Waerden theorem is equivalent to the multiple recurrence theorem in topological dynamics. They also proved the following IP-version of multiple recurrence theorem (see Theorem 8.19 of [4] for this version).

Theorem 1.1. *Let $\{S_1^\alpha\}$, $\{S_2^\alpha\}$, $\ldots$, $\{S_\ell^\alpha\}$ be ℓ IP-systems of maps of a compact metric space X, all contained in a commutative group of homeomorphisms of X. Then there exists a point $x \in X$ and a homomorphism $\phi\colon \mathcal{F} \to \mathcal{F}$ such that*

$$S_i^{\phi(\alpha)}x \to x, \ i = 1, 2, \ldots, \ell.$$

Recently, the authors in [1] studied topological dynamical systems indexed by directed partial semigroups. Examples of directed partial semigroup actions are IP-systems in [5] and dynamics systems indexed by words in [2]. Under some conditions, the authors in [1] obtained the following multiple recurrence theorem for directed partial semigroup actions.

Theorem 1.2. *Let $(\Lambda, \prec, *)$ be a directed partial semigroup and $\mathcal{B}$ a suitable coideal basis for $(\Lambda, \prec, *)$ with the (D)-property. Let X be a compact metric space and G an abelian subgroup of homeomorphisms of X. Assume that $\ell \geq 1$*

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and $\{T_i^\lambda\}_{\lambda\in\Lambda}$ *are* Λ*-systems on* G *for* $i = 1,\dots,\ell$*. If the systems* $\{T_i^\lambda\}_{\lambda\in\Lambda}$ *and* $\{(T_i^\lambda)^{-1}\}_{\lambda\in\Lambda}$ *are equicontinuous, then for every* $B \in \mathcal{B}$*, there exist* $A \in \mathcal{B}$ *with* $A \subset B$ *and* $x \in X$ *such that*

$$\lim_{\lambda\in A} T_i^\lambda x = x,\ i = 1,\dots,\ell.$$

It should be noticed that Theorem 1.2 requires equicontinuity of the action, which is a strong condition. The main idea of this paper is to prove the following reasonable multiple recurrence theorem for directed partial semigroup actions. We do not require the condition of equicontinuity, but the convergence that we obtain is slightly weaker.

Theorem 1.3. *Let* $(\Lambda, \prec, *)$ *be a directed partial semigroup and* $\mathcal{B}$ *a suitable coideal basis for* $(\Lambda, \prec, *)$ *with the* (D)*-property. Let* X *be a compact metric space and* G *an abelian subgroup of homeomorphisms of* X*. Assume that* $\ell \geq 1$ *and* $\{T_i^\lambda\}_{\lambda\in\Lambda}$ *are* Λ*-systems in* G *for* $i = 1,\dots,\ell$*. Then for every* $B \in \mathcal{B}$*, there exist* $x \in X$ *and a sequence* $\{\lambda_n\}_{n=1}^\infty$ *in* B *tending to infinity, such that*

$$\lim_{n\to\infty} T_i^{\lambda_n} x = x,\ i = 1,\dots,\ell.$$

The organization of this paper is as follows. In Section 2, we introduce some notions and results which will be used later. In Section 3, after some preparation we prove the main result Theorem 1.3.

2. Preliminaries

Let $\mathbb{N}$ denote the set of all positive integers.

2.1. Directed set and coideal basis

Definition 2.1. Let Λ be a non-empty countable set and $\prec$ a relation on Λ. If the relation $\prec$ satisfies the following conditions:

(1) For every $\lambda_1, \lambda_2 \in \Lambda$ with $\lambda_1 \prec \lambda_2$, $\lambda_1 \neq \lambda_2$.
(2) For every $\lambda_1, \lambda_2, \lambda_3 \in \Lambda$ with $\lambda_1 \prec \lambda_2$ and $\lambda_2 \prec \lambda_3$, $\lambda_1 \prec \lambda_3$.
(3) For every $\lambda_1, \lambda_2 \in \Lambda$, there exists $\lambda_3 \in \Lambda$ such that $\lambda_1 \prec \lambda_3$ and $\lambda_2 \prec \lambda_3$.

Then we say that $(\Lambda, \prec)$ is a *directed set*.

Remark 2.2. The definition of directed partial semigroup in this paper requires the set Λ is a non-empty countable set, while the one in reference [1] only requires the set Λ is a non-empty infinite set.

Definition 2.3. Let $(\Lambda, \prec)$ be a directed set. A collection $\mathcal{B}$ of subsets of Λ is a *coideal basis* on $(\Lambda, \prec)$ if it satisfies the following conditions:

(1) For every $A \in \mathcal{B}$ and $\lambda_1 \in \Lambda$ there exists $\lambda_2 \in A$ such that $\lambda_1 \prec \lambda_2$.
(2) For every $A \in \mathcal{B}$ and $A = A_1 \cup A_2$ there exists $B \in \mathcal{B}$ such that either $B \subset A_1$ or $B \subset A_2$.

Let $\mathcal{B}$ be a coideal basis on $(\Lambda, \prec)$. It is clear that for every $A \in \mathcal{B}$, $(A, \prec)$ is also a directed set.

Definition 2.4. Let $(\Lambda, \prec)$ be a directed set. If a sequence $\{\lambda_n\}_{n=1}^{\infty}$ in Λ satisfies the following two conditions:

(1) $\lambda_n \prec \lambda_{n+1}$, $\forall n \in \mathbb{N}$.
(2) for every $\lambda \in \Lambda$, there exists $n \in \mathbb{N}$ such that $\lambda \prec \lambda_n$.

Then we say that the sequence $\{\lambda_n\}_{n=1}^{\infty}$ *tends to infinity*, denoted by $\lambda_n \to \infty$ as $n \to \infty$.

Definition 2.5. Let X be a non-empty set and $(\Lambda, \prec)$ be a directed set. We say that a sequence $\{x_\lambda\}_{\lambda \in \Lambda}$ in X indexed by Λ is a Λ-*sequence*.

Lemma 2.6. *Let $(\Lambda, \prec)$ be a directed set and $\mathcal{B}$ be a coideal basis on $(\Lambda, \prec)$. Let X be a compact metric space and $\{x_\lambda\}_{\lambda \in \Lambda}$ be a Λ-sequence in X. Then for every $B \in \mathcal{B}$, there exist a point $x \in X$ and a sequence $\{\lambda_n\}_{n=1}^{\infty}$ in B tending to infinity such that $\lim_{n\to\infty} x_{\lambda_n} = x$.*

Proof. Fix a $B \in \mathcal{B}$. Let d be a compatible metric on X. For every $x \in X$ and $\varepsilon > 0$, we set $U(x, \varepsilon) = \{y \in X : d(x, y) \leq \varepsilon\}$. Since X is compact, we have that $X = \bigcup_{i=1}^{m_1} U(x_i^1, \frac{1}{2})$ for some $x_1^1, \ldots, x_{m_1}^1 \in X$.

Let $B_1 = B$. Then $B_1 = \bigcup_{i=1}^{m_1} C_i$, where $C_i = \{\lambda \in B_1 : x_\lambda \in U(x_i^1, \frac{1}{2})$. As $\mathcal{B}$ is a coideal basis, there exist $B_2 \in \mathcal{B}$ with $B_2 \subseteq B_1$ and $1 \leq i_1 \leq m_1$ such that $B_2 \subseteq C_{i_1}$ and consequently $\{x_\lambda : \lambda \in B_2\} \subseteq U(x_i^1, \frac{1}{2})$. We continue analogously. Since X is compact, there exist $x_1^2, \ldots, x_{m_2}^2 \in X$ such that $U(x_{i_1}^1, \frac{1}{2}) \subseteq \bigcup_{i=1}^{m_1} U(x_i^2, \frac{1}{4})$, and consequently there exist $B_3 \in \mathcal{B}$ with $B_3 \subseteq A_2$ and $1 \leq i_2 \leq m_2$ such that $\{x_\lambda : \lambda \in B_3\} \subseteq U(x_{i_1}^1, \frac{1}{2}) \cap U(x_{i_2}^2, \frac{1}{4})$.

Inductively, we construct a sequence $\{B_n\}_{n=1}^{\infty}$ with $B_n \in \mathcal{B}$ and $B_1 \supseteq B_2 \supseteq \cdots$, and also a sequence $\{U(x_{i_n}^n, \frac{1}{2^n})\}_{n=1}^{\infty}$ such that

$$\{x_\lambda : \lambda \in B_{n+1}\} \subseteq \bigcap_{j=1}^{n} U(x_{i_j}^j, \tfrac{1}{2^j}) \text{ for every } n \in \mathbb{N}.$$

As X is compact, without loss of generality, assume that $x_{i_n}^n \to x_0 \in X$ as $n \to \infty$. As Λ is countable, let $\{\alpha_1, \alpha_2, \ldots, \alpha_n, \alpha_{n+1}, \ldots\}$ be an enumeration of Λ. Since $\mathcal{B}$ is a coideal basis on $(\Lambda, \prec)$ and $B_n \in \mathcal{B}$ for each $n \in \mathbb{N}$, then we can choose a sequence $\{\lambda_n\}_{n=1}^{\infty}$ inductively such that $\lambda_n \in B_n$, $\lambda_n \prec \lambda_{n+1}$ and $\alpha_n \prec \lambda_{n+1}$ for all $n \in \mathbb{N}$. Then $\{\lambda_n\}_{n=1}^{\infty}$ tends to infinity and $\lim_{n\to\infty} x_{\lambda_n} = x_0$. □

Definition 2.7 ([1])**.** Let $(\Lambda, \prec)$ be a directed set and $\mathcal{B}$ be a coideal basis on $(\Lambda, \prec)$. We say that $\mathcal{B}$ has the (D)-*property* if for every sequence $\{A_n\}_{n=1}^{\infty}$ in $\mathcal{B}$ with $A_n \supset A_{n+1}$, there exists $A \in \mathcal{B}$ such that for every $n \in \mathbb{N}$, there exists $k_n \in \mathbb{N}$ satisfying

$$\max\{k \in \mathbb{N} \colon \exists \lambda_1, \ldots, \lambda_k \in A \setminus A_n \text{ with } \lambda_1 \prec \lambda_2 \prec \cdots \prec \lambda_k\} \leq k_n.$$

Definition 2.8. Assume that X is a topological space and $\{x_\lambda\}_{\lambda\in\Lambda}$ is a Λ-sequence in X. For $x \in X$, if for any neighborhood V of x there exists $\lambda_0 = \lambda_0(V) \in \Lambda$ such that $x_\lambda \in X$ for every $\lambda \in \Lambda$ with $\lambda_0 \prec \lambda$, then we say that *the* Λ*-sequence* $\{x_\lambda\}_{\lambda\in\Lambda}$ *converges to* x, or x *is the limit point of the* Λ*-sequence* $\{x_\lambda\}_{\lambda\in\Lambda}$, which is denoted by $\lim_{\lambda\in\Lambda} x_\lambda = x$. It is clear that if X is Hausdorff, then the limit of any Λ-sequence is unique.

The following result is Theorem 2.10 in [1].

Theorem 2.9. *Let* $(\Lambda, \prec)$ *be a directed set and* $\mathcal{B}$ *be a coideal basis on* $(\Lambda, \prec)$ *with the* (D)*-property. Assume that* X *is a compact metric space and* $\{x_\lambda\}_{\lambda\in\Lambda}$ *is a* Λ*-sequence in* X*. Then for every* $B \in \mathcal{B}$ *there exists* $A \in \mathcal{B}$ *with* $A \subset B$ *such that* $\{x_\lambda\}_{\lambda\in A}$ *is convergent in* X*.*

Remark 2.10. It should be noticed that if $\{x_\lambda\}_{\lambda\in A}$ converges to x as an A-sequence then there exists a sequence $\{\lambda_n\}_{n=1}^{\infty}$ in A tending to infinity, such that $\lim_{n\to\infty} x_{\lambda_n} = x$. So the conclusion of Theorem 2.9 is stronger than the one of Lemma 2.6, but in Lemma 2.6, we do not require the coideal basis to have the (D)-property.

2.2. *G*-system

Definition 2.11. Let X be a compact metric space. Denote

$$\mathrm{End}(X) = \{T\colon X \to X \mid T \text{ is continuous}\} \text{ and}$$
$$\mathrm{Aut}(X) = \{T\colon X \to X \mid T \text{ is a homeomorphsim}\}.$$

$\mathrm{End}(X)$ and $\mathrm{Aut}(X)$ are semigroups under the composition operator of maps.

Let $(G, \cdot)$ be a countable semigroup. If there exists a semigroup homomorphism $\phi\colon G \to \mathrm{End}(X)$, then we say that (X, G) is a *G-system.* In fact, we also identify the element $g \in G$ and its images $\phi(g) \in \mathrm{End}(X)$, that is, g is regarded as a continuous self-map on X.

Definition 2.12. Let (X, G) is a G-system and $Y \subset X$. If for every $g \in G$, $g(Y) \subset Y$, then we say that Y is *G-invariant.* If there is no non-empty proper closed G-invariant subset of X, then we say that (X, G) is *minimal.*

The following three lemmas are well-known result for G-systems, we refer the reader to Section 1.4 of [4].

Lemma 2.13. *For every* G*-system* (X, G)*, there exists a non-empty closed* G*-invariant subset* Y *of* X *such that* (Y, G) *is minimal.*

Lemma 2.14. *Let* (X, G) *be a* G*-system. The following statements are equivalent*:

(1) (X, G) *is minimal.*
(2) *For each* $x \in X$*, the set* $\{gx : g \in G\}$ *is dense in* X*.*

(3) *For every open subset U of X, there exists finitely many elements $g_1, \dots,$ $g_n \in G$ such that*

$$\bigcup_{i=1}^{n} g_i^{-1} U = X.$$

Lemma 2.15. *If a G-system (X, G) is minimal, then for every $\varepsilon > 0$ there exists a finite subset G_0 of G such that for any $x, y \in X$ there exists $g \in G_0$ with $d(gx, y) < \varepsilon$.*

2.3. Directed partial semigroup and its action

Definition 2.16 ([1])**.** Let $(\Lambda, \prec)$ be a directed set. If an operator $*$ on Λ satisfies the following conditions: for every $\lambda_1, \lambda_2, \lambda_3 \in \Lambda$ with $\lambda_1 \prec \lambda_2 \prec \lambda_3$, one has $\lambda_1 \prec \lambda_2 * \lambda_3$, $\lambda_1 * \lambda_2 \prec \lambda_3$ and $(\lambda_1 * \lambda_2) * \lambda_3 = \lambda_1 * (\lambda_2 * \lambda_3)$, then we say that $(\Lambda, \prec, *)$ is a *directed partial semigroup.*

Let $\mathcal{B}$ be a coideal basis on $(\Lambda, \prec)$. If for every $B \in \mathcal{B}$ and $\lambda_1, \lambda_2 \in B$ with $\lambda_1 \prec \lambda_2$, $\lambda_1 * \lambda_2 \in B$, then we called that $\mathcal{B}$ is *suitable.*

Example 2.17. Let $n \in \mathbb{N}$. For $\mathbf{x} = (x_1, x_2, \dots, x_n)$ and $\mathbf{y} = (y_1, y_2, \dots, y_n) \in \mathbb{N}^n$, if $\max_{1 \le i \le n} x_i < \min_{1 \le i \le n} y_i$, then we say that $\mathbf{x} \prec \mathbf{y}$. We define an operator $+$ on $\mathbb{N}^n$ as $\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)$. Then $(\mathbb{N}^n, \prec, +)$ is a directed partial semigroup. Let

$$\mathcal{B} = \big\{ B = \{\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_k, \dots\} \subset \mathbb{N}^n \colon \mathbf{b}_k \prec \mathbf{b}_{k+1}, \forall k \in \mathbb{N} \big\}.$$

Then $\mathcal{B}$ is a coideal basis with the (D)-property for $(\mathbb{N}^n, \prec, +)$, but it is not suitable.

Example 2.18. Let $\mathcal{F}$ be the collection of all finite non-empty subsets of $\mathbb{N}$. For $\alpha, \beta \in \mathcal{F}$, if $\max \alpha < \min \beta$ then we say that $\alpha \prec \beta$. Then $(\mathcal{F}, \prec, \cup)$ is a directed partial semigroup.

For a sequence $\{\alpha_n\}_{n=1}^{\infty}$ in $\mathcal{F}$, we set

$$\mathrm{FU}(\{\alpha_n\}_{n=1}^{\infty}) = \Big\{ \bigcup_{i \in \beta} \alpha_i \colon \beta \in \mathcal{F} \Big\}.$$

By the well-known Hindman theorem (see [6]), it is not hard to see that the collection

$$\mathcal{B} = \{\mathrm{FU}(\{\alpha_n\}_{n=1}^{\infty}) \colon \{\alpha_n\}_{n=1}^{\infty} \text{ is a sequence in } \mathcal{F} \text{ and } \alpha_1 \prec \alpha_2 \prec \cdots\}$$

is a suitable coideal basis with the (D)-property for $(\mathcal{F}, \prec, \cup)$.

Definition 2.19. Let $(\Lambda, \prec, *)$ be a directed partial semigroup and $(G, \cdot)$ be a semigroup. If a Λ-sequence $\{x_\lambda\}_{\lambda \in \Lambda}$ in G satisfies the following property: for every $\lambda_1, \lambda_2 \in \Lambda$ with $\lambda_1 \prec \lambda_2$, $x_{\lambda_1 * \lambda_2} = x_{\lambda_1} \cdot x_{\lambda_2}$, then we say that $\{x_\lambda\}_{\lambda \in \Lambda}$ is a Λ-*system* in G.

Definition 2.20. Let $(\Lambda, \prec, *)$ be a directed partial semigroup and X be a compact metric space. If $\{T^\lambda\}_{\lambda \in \Lambda}$ be a Λ-system on $\mathrm{End}(X)$, then we say that $(X, \{T^\lambda\}_{\lambda \in \Lambda})$ is a Λ-*topological dynamical system.*

Definition 2.21. Let $(X, \{T^\lambda\}_{\lambda\in\Lambda})$ be a Λ-topological dynamical system and $x \in X$.

(1) If there exists a sequence $\{\lambda_n\}_{n=1}^\infty$ in Λ tending to infinity such that $\lim_{n\to\infty} T^{\lambda_n}x = x$, then we say that x is *recurrent.*
(2) Let $\mathcal{B}$ be a coideal basis for $(\Lambda, \prec, *)$ and $B \in \mathcal{B}$. If there exists $A \in \mathcal{B}$ with $A \subset B$ such that $\lim_{\lambda\in A} T^\lambda x = x$, then we say that x is B-*recurrent.*

Theorem 2.22 ([1, Theorem 3.6])**.** *If a directed partial semigroup* $(\Lambda, \prec, *)$ *admits a suitable coideal basis* $\mathcal{B}$ *with the* (D)*-property, then for every* Λ-*topological dynamical system* $(X, \{T^\lambda\}_{\lambda\in\Lambda})$ *and* $B \in \mathcal{B}$, X *contains some* B-*recurrent point.*

Remark 2.23. By Remark 2.10, if x is B-recurrent, then it is also recurrent. By Theorem 2.22 if a directed partial semigroup $(\Lambda, \prec, *)$ admits a suitable coideal basis $\mathcal{B}$ with the (D)-property, then every Λ-topological dynamical system contains some recurrent point. But the following question is still open.

Question 2.24. *Does every* Λ-*topological dynamical system* $(X, \{T^\lambda\}_{\lambda\in\Lambda})$ *contain some recurrent point?*

Definition 2.25. Let $(\Lambda, \prec, *)$ be a directed partial semigroup and $\{\lambda_n\}_{n=1}^\infty$ be a sequence in Λ. We define the *finite product* of $\{\lambda_n\}_{n=1}^\infty$ as

$$FP(\{\lambda_n\}_{n=1}^\infty) = \{\lambda_{i_1} * \lambda_{i_2} * \cdots * \lambda_{i_k} : 1 \le i_1 < i_2 < \cdots < i_k, k \in \mathbb{N}\}.$$

Proposition 2.26. *Let* $(X, \{T^\lambda\}_{\lambda\in\Lambda})$ *be a* Λ-*topological dynamical system. If a point* $x \in X$ *is recurrent, then there exists a sequence* $\{\lambda_n\}_{n=1}^\infty$ *in* λ *such that for any open neighborhood* U *of* x, *there exists* $k \in \mathbb{N}$ *such that for any* $\lambda \in FP(\{\lambda_n\}_{n=k}^\infty)$, $T^\lambda x \in U$.

Proof. As Λ is countable, enumerate it as $\{\alpha_1, \alpha_2, \dots, \alpha_n, \alpha_{n+1}, \dots\}$. Let d be a compatible metric on X. Fix an open neighborhood V_1 of x with diameter less than 1. As x is recurrent, there exists $\lambda_1 \in \Lambda$ with $\alpha_1 \prec \lambda_1$ such that $T^{\lambda_1}x \in V_1$. Then $V_1 \cap (T^{\lambda_1})^{-1}(V_1)$ is also an open neighborhood of x. Pick an open neighborhood V_2 of x with diameter less than $\frac{1}{2}$ and $V_2 \subset V_1 \cap (T^{\lambda_1})^{-1}(V_1)$. There exists $\lambda_2 \in \Lambda$ with $\alpha_2 \prec \lambda_2$ and $\lambda_1 \prec \lambda_2$ such that $T^{\lambda_2}x \in V_2$. Then $T^{\lambda_1 * \lambda_2}x \in T^{\lambda_1}V_2 \subset V_1$. Assume that $\lambda_1, \lambda_2, \dots, \lambda_n$ and $V_1, V_2, \dots, V_n$ have been found that

(1) $\lambda_i \prec \lambda_{i+1}$ for $i = 1, 2, \dots, n-1$;
(2) $\alpha_i \prec \lambda_i$ for $i = 1, 2, \dots, n$;
(3) $V_{i+1} \subset V_i \cap (T^{\lambda_i})^{-1}V_i$ for $i = 1, 2, \dots, n-1$;
(4) V_i is an open neighborhood of x with diameter less than $\frac{1}{i}$ for $i = 1, 2, \dots, n$.

Pick an open neighborhood V_{n+1} of x with diameter less than $\frac{1}{n+1}$ and $V_{n+1} \subset V_n \cap (T^{\lambda_n})^{-1}(V_n)$. There exists $\lambda_{n+1} \in \Lambda$ with $\alpha_{n+1} \prec \lambda_{n+1}$ and $\lambda_n \prec \lambda_{n+1}$ such that $T^{\lambda_{n+1}}x \in V_{n+1}$. By induction, we get a sequence $\{\lambda_n\}_{n=1}^\infty$ in Λ and a sequence $\{V_n\}_{n=1}^{\infty}$ of neighborhoods of x satisfying the previous properties. It is clear that λ_n tends to infinity and the diameter of V_n tends to 0 as $n \to \infty$. For every $k \in \mathbb{N}$ and any $\lambda \in FP(\{\lambda_n\}_{n=k}^{\infty})$, $T^{\lambda}x \in V_k$. Then the sequence $\{\lambda_n\}_{n=1}^{\infty}$ is as required. □

3. Proof of the main result

In this section, we give the proof of the main result in this article. Before doing this, we need some preparation.

Definition 3.1. Let $(X, \{T^{\lambda}\}_{\lambda \in \Lambda})$ be a Λ-topological dynamical system and Y be a closed subset of X. If there exists a semigroup G of $\mathrm{End}(X)$ such that for any $g \in G$ and $\lambda \in \Lambda$, $g \circ T^{\lambda} = T^{\lambda} \circ g$, Y is G-invariant and (Y, G) is minimal, then we say that Y is *homogeneous* in $(X, \{T^{\lambda}\}_{\lambda \in \Lambda})$.

Lemma 3.2. *Let $(\Lambda, \prec, *)$ be a directed partial semigroup and $\mathcal{B}$ is a coideal basis for $(\Lambda, \prec, *)$. Assume that $(X, \{T^{\lambda}\}_{\lambda \in \Lambda})$ is a Λ-topological dynamical system, Y is a homogeneous subset of X and $B \in \mathcal{B}$. Consider the following statements.*

1. *For every $\varepsilon > 0$ and $\lambda_0 \in \Lambda$, there exist $x, y \in Y$ and $\lambda \in B$ with $\lambda_0 \prec \lambda$ and $d(T^{\lambda}y, x) < \varepsilon$;*
2. *For every $\varepsilon > 0$, $x \in Y$ and $\lambda_0 \in \Lambda$, there exist $y \in Y$ and $\lambda \in B$ with $\lambda_0 \prec \lambda$ such that $d(T^{\lambda}y, x) < \varepsilon$;*
3. *For every $\varepsilon > 0$ and $\lambda_0 \in \Lambda$, there exist $z \in Y$ and $\lambda \in B$ with $\lambda_0 \prec \lambda$ such that $d(T^{\lambda}z, z) < \varepsilon$.*

Then, we have (1) $\Rightarrow$ (2) $\Rightarrow$ (3).

Proof. (1) $\Rightarrow$ (2) Since Y is a homogeneous subset of X, there exists a semigroup G of $\mathrm{End}(X)$ such that for any $g \in G$ and $\lambda \in \Lambda$, $g \circ T^{\lambda} = T^{\lambda} \circ g$ and (Y, G) is minimal. Fix $\varepsilon > 0$, $x \in Y$ and $\lambda_0 \in \Lambda$. By Lemma 2.15 there exists a finite subset G_0 of G such that for every $u, v \in Y$

$$\min_{g \in G_0} d(gu, v) < \frac{\varepsilon}{2}.$$

There exists $\delta > 0$ such that if $u, v \in X$ and $d(u, v) < \delta$, then $d(gu, gv) < \frac{\varepsilon}{2}$ for every $g \in G_0$. According to the hypothesis, there exist $x_0, y_0 \in Y$ and $\lambda \in B$ with $\lambda_0 \prec \lambda$ and $d(T^{\lambda}y_0, x_0) < \delta$. Then for every $g \in G_0$,

$$d(T^{\lambda}(gy_0), gx_0) = d(g(T^{\lambda}y_0), gx_0) < \frac{\varepsilon}{2}.$$

Since there exists $g \in G_0$ such that $d(gx_0, x) < \frac{\varepsilon}{2}$, we have that

$$d(T^{\lambda}(gy_0), x) \leq d(T^{\lambda}(gy_0), gx_0) + d(gx_0, x) < \varepsilon,$$

which ends the proof.

(2) $\Rightarrow$ (3) We fix $\varepsilon > 0$ and $\lambda_0 \in \Lambda$. Let $z_0 \in Y$. By our hypothesis, there exists $\lambda_1 \in B$ with $\lambda_0 \prec \lambda_1$ and $z_1 \in Y$ such that

$$d(T^{\lambda_1}z_1, z_0) < \frac{\varepsilon}{2}.$$

Let $\eta_2 = \frac{\varepsilon}{2} - d(T^{\lambda_1} z_1, z_0) > 0$. Then there exists $0 < \delta_2 < \frac{\varepsilon}{2}$ such that $d(T^{\lambda_1}u, T^{\lambda_1}v) < \eta_2$ for any $u, v \in X$ with $d(u, v) < \delta_2$. Again by our hypothesis, there exist $z_2 \in Y$ and $\lambda_2 \in B$ with $\lambda_1 \prec \lambda_2$ such that

$$d(T^{\lambda_2} z_2, z_1) < \delta_2 < \frac{\varepsilon}{2}.$$

It follows that

$$d(T^{\lambda_1 * \lambda_2}(z_2), z_0) \le d(T^{\lambda_1 * \lambda_2}(z_2), T^{\lambda_1}(z_1)) + d(T^{\lambda_1} z_1, z_0) < \frac{\varepsilon}{2}.$$

Assume that there exist $z_1, \dots, z_k \in Y$ and $\lambda_1, \dots, \lambda_k \in B$ satisfying

$$d(T^{\lambda_{i+1} * \cdots * \lambda_j} z_j, z_i) < \frac{\varepsilon}{2} \quad \text{for each } 0 \le i < j \le k.$$

Let

$$\eta_{k+1} = \min\Big\{\frac{\varepsilon}{2} - d(T^{\lambda_{i+1} * \cdots * \lambda_j} z_j, z_i) \colon 0 \le i < j \le k\Big\} > 0.$$

There exists $0 < \delta_{k+1}$ such that for every $u, v \in X$, if $d(u, v) < \delta_{k+1}$ then

$$d(T^{\lambda}u, T^{\lambda}v) < \eta_{k+1} \quad \text{for all } \lambda = \lambda_{i+1} * \cdots * \lambda_j, 0 \le i < j \le k.$$

By our hypothesis, there exist $z_{k+1} \in B$ and $\lambda_{n_{k+1}} \in B$ such that

$$d(T^{\lambda_{k+1}} z_{k+1}, z_k) < \delta_{k+1} < \frac{\varepsilon}{2}.$$

Then for every $i = 1, 2, \dots, k$,

$$\begin{aligned} d(T^{\lambda_{i+1} * \cdots * \lambda_k * \lambda_{k+1}} z_{k+1}, z_i) &\le d(T^{\lambda_{i+1} * \cdots * \lambda_k * \lambda_{k+1}} z_{k+1}, T^{\lambda_{i+1} * \cdots * \lambda_k} z_k) \\ &\quad + d(T^{\lambda_{i+1} * \cdots * \lambda_k} z_k, z_i) \\ &< \eta_{k+1} + d(T^{\lambda_{i+1} * \cdots * \lambda_k} z_k, z_i) < \frac{\varepsilon}{2}. \end{aligned}$$

Inductively, we can construct a sequence $\{z_k\}_{k=1}^{\infty}$ and $\{\lambda_k\}_{k=1}^{\infty}$ such that for each $0 \le i < j$,

$$d(T^{\lambda_{i+1} * \cdots * \lambda_j} z_j, z_i) < \frac{\varepsilon}{2}.$$

Since X is compact, there exists $0 \le i < j$ such that $d(z_i, z_j) < \frac{\varepsilon}{2}$. Let $z = z_j$ and $\lambda = \lambda_{i+1} * \cdots * \lambda_j$. Since $\mathcal{B}$ is suitable, $\lambda \in B$. Then we obtain that

$$d(T^{\lambda} z, z) \le d(T^{\lambda_{i+1} * \cdots * \lambda_j} z_j, z_i) + d(z_i, z_j) < \varepsilon.$$

This ends the proof. □

Remark 3.3. In proving (2) ⇒ (3), we only use the assumption that Y is closed rather than Y is homogeneous.

Proposition 3.4. *Let $(\Lambda, \prec, *)$ be a directed partial semigroup and $\mathcal{B}$ is a coideal basis for $(\Lambda, \prec, *)$. Let X be a compact metric space and G is an abelian subgroup of $\mathrm{Aut}(X)$ such that (X, G) is minimal. Assume that $\ell \ge 1$ and $\{T_i^{\lambda}\}_{\lambda \in \Lambda}$ are Λ-systems on G for $i = 1, \dots, \ell$. Then the following four statements are equivalent:*

(1) *For every non-empty open subset U of X, $B \in \mathcal{B}$ and $\lambda_0 \in B$ there exists $\lambda \in B$ with $\lambda_0 \prec \lambda$ such that*

$$U \cap \bigcap_{i=1}^{\ell} (T_i^{\lambda})^{-1} U \neq \emptyset.$$

(2) *For every $\varepsilon > 0$, for every $B \in \mathcal{B}$ and $\lambda_0 \in \Lambda$ there exist $x \in X$ and $\lambda \in B$ with $\lambda_0 \prec \lambda$ such that $d(T_i^{\lambda} x, x) < \varepsilon$ for $i = 1, 2, \ldots, \ell$.*

(3) *For every $B \in \mathcal{B}$ there exists a sequence $\{\lambda_n\}_{n=1}^{\infty}$ in B tending to infinity and $x \in X$ such that*

$$\lim_{n\to\infty} T_i^{\lambda_n} x = x, \ i = 1, \ldots, \ell.$$

(4) *For every $B \in \mathcal{B}$ there exists a dense G_δ subset A of X such that for any $x \in A$ there exists a sequence $\{\lambda_n\}_{n=1}^{\infty}$ in B tending to infinity satisfying*

$$\lim_{n\to\infty} T_i^{\lambda_n} x = x, \ i = 1, \ldots, \ell.$$

Proof. (2) $\Rightarrow$ (1). Let U be non-empty open subset of X, $B \in \mathcal{B}$ and $\lambda_0 \in B$. Since (X, G) is minimal, there exists a finite subset G_0 of G such that $X = \bigcup_{g \in G_0} g^{-1} U$. Let ε be a Lebesgue number of the open cover $\{g^{-1}U \colon g \in G_0\}$. There exist $x \in X$ and $\lambda \in B$ with $\lambda_0 \prec \lambda$ such that $d(T_i^{\lambda} x, x) < \varepsilon$ for $i = 1, 2, \ldots, \ell$. Choose $g \in G_0$ such that $x, T_i^{\lambda} x \in g^{-1}U$ for every $1 \le i \le \ell$. Hence

$$gx \in U \cap \bigcap_{i=1}^{\ell} (T_i^{\lambda})^{-1}(U).$$

(3) $\Rightarrow$ (2) and (4) $\Rightarrow$ (3) are obvious.

(1) $\Rightarrow$ (4). Fix $B \in \mathcal{B}$. For every $m \in \mathbb{N}$ and $\lambda \in B$, we set

$$A(m, \lambda) = \Big\{ x \in X : d(x, T_i^{\lambda} x) < \frac{1}{m}, \ i = 1, 2, \ldots, \ell \Big\}.$$

It is clear that $A(m, \lambda)$ is an open subset of X. We first show the following claim.

Claim: For every $\lambda_0 \in B$, the set $\bigcup_{\lambda \in B, \lambda_0 \prec \lambda} A(m, \lambda)$ is dense in X.

Indeed, let U be a non-empty open subset of X. Choose a non-empty open subset V of U such that the diameter of V is less than $\frac{1}{m}$. There exists $\lambda_1 \in B$ with $\lambda_0 \prec \lambda_1$ such that

$$V \cap \bigcap_{i=1}^{\ell} (T_i^{\lambda})^{-1} V \neq \emptyset.$$

Pick a point $x \in V \cap \bigcap_{i=1}^{\ell} (T_i^{\lambda_1})^{-1} V$. Then $x, T_i^{\lambda_1} x \in V$ for $i = 1, 2, \ldots, \ell$. As the diameter of V is less than $\frac{1}{m}$, $d(x, T_i^{\lambda_1} x) < \frac{1}{m}$ for $i = 1, 2, \ldots, \ell$. Then $x \in U \cap A(m, \lambda)_1$.

To finish the proof of the statement, let

$$A = \cap_{m=1}^{\infty} \cap_{\lambda_0 \in B} \cup_{\lambda \in B, \lambda_0 \prec \lambda} A(m, \lambda).$$

As B is countable, by the Baire's theorem, A is dense G_δ subset of X. It is easy to see that $z \in X$ if and only if there exists a sequence $\{\lambda_n\}_{n=1}^\infty$ in B tending to infinity such that

$$\lim_{n\to\infty} T_i^{\lambda_n} z = z \text{ for } i = 1, \ldots, \ell,$$

which completes the proof of the claim. □

Remark 3.5. The "countability" assumption is needed in the proof of Proposition 3.4 (1) ⇒ (4) in order to use Baire's theorem.

Now we give the proof of the main result.

Proof of Theorem 1.3. Without loss of generality, we assume that (X, G) is minimal, otherwise we can replace X by a G-minimal subset of X. We proceed by induction on ℓ. For $\ell = 1$ the theorem is valid by Theorem 2.22 and Remark 2.23. Assume that the theorem holds for $\ell - 1$. Let $B \in \mathcal{B}$ and $\{T_1^\lambda\}_{\lambda\in\Lambda}, \ldots, \{T_\ell^\lambda\}_\lambda \in \Lambda$ be ℓ Λ-topological dynamical systems satisfying the hypothesis of the theorem. Set

$$\Delta_\ell = \{(x, x, \ldots, x) \in X^\ell : x \in X\},$$
$$\hat{T}^\lambda = T_1^\lambda \times \ldots \times T_\ell^\lambda,$$
$$g(x_1, \ldots, x_\ell) = (gx_1, \ldots, gx_\ell).$$

If we set $\pi : \Delta_\ell \to X, (x, x, \ldots, x) \mapsto x$, then π is a topological conjugacy between (Δ_ℓ, G) and (X, G). It follows that (Δ_ℓ, G) is also a minimal system. Then Δ_ℓ is homogeneous in $(X^\ell, \{\hat{T}^\lambda\}_{\lambda\in\Lambda})$. We first show the following claim.

Claim: For every $\varepsilon > 0$ and $\lambda_0 \in \Lambda$, there exist $x^*, y^* \in \Delta_\ell$ and $\lambda \in B$ with $\lambda_0 \prec \lambda$ such that

$$d(T^\lambda y^*, x^*) < \varepsilon.$$

Indeed, for $1 \le i \le \ell - 1$, set

$$R_i^\lambda = T_i^\lambda \circ (T_\ell^\lambda)^{-1}.$$

Applying the induction hypothesis, we have the existence of $x \in X$ and a sequence $\{\lambda_n\}_{n=1}^\infty$ in B tending to infinity such that

$$\lim_{n\to\infty} R_i^{\lambda_n} x = x \quad \text{for } i = 1, \ldots, \ell - 1.$$

Let

$$x^* = (x, x, \ldots, x) \text{ and } y^* = ((T_\ell^{\lambda_n})^{-1}x, (T_\ell^{\lambda_n})^{-1}x, \ldots, (T_\ell^{\lambda_n})^{-1}x).$$

Then we have that

$$\begin{aligned} d(\hat{T}^{\lambda_n} y^*, x^*) &= d(T_1^{\lambda_n} \times T_2^{\lambda_n} \times \ldots T_\ell^{\lambda_n} y^*, x^*) \\ &= d((T_1^{\lambda_n} \circ (T_\ell^{\lambda_n})^{-1}x, \ldots, T_\ell^{\lambda_n} \circ (T_\ell^{\lambda_n})^{-1}x, x), (x, x, \ldots, x)) \\ &= d((R_1^{\lambda_n} x, \ldots, R_{\ell-1}^{\lambda_n} x, x), (x, x, \cdots, x)). \end{aligned}$$

For every $\varepsilon > 0$ and $\lambda_0 \in \Lambda$, we can select an large enough n such that $\lambda_n \in B$ with $\lambda_n \succ \lambda_0$ and satisfying

$$d(\hat{T}^{\lambda_n} y^*, x^*) < \varepsilon.$$

To end the proof, by Lemma 3.2, with the above claim we deduced that for every $\varepsilon > 0$ and $\lambda_0 \in \Lambda$ there exist $x \in X$ and $\lambda \in B$ with $\lambda_0 \prec \lambda$ such that $d(T_i^{\lambda} x, x) < \varepsilon$ for $i = 1, 2, \ldots, \ell$. Then the result follows from Proposition 3.4(2) $\Rightarrow$ (3). □

Remark 3.6. According to Example 2.18 and Proposition 2.26, our Theorem 1.3 is a generalization of Furstenberg-Weiss's multiple recurrence theorem (Theorem 1.1).

Remark 3.7. A semigroup with a digital representation was studied in [3]. The authors in [1] showed that a semigroup with digital representation in a proper relation has s suitable coideal basis satisfying the (D)-property. Theorem 1.3 can be applied to this setting too.

Remark 3.8. In the proof of Theorem 1.3, we only require the (D)-property of the suitable coideal basis $\mathcal{B}$ in Theorem 2.22. If Question 2.24 has a positive answer or holds under some weak conditions, then Theorem 1.3 also holds under those conditions.

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Tiaoying Zeng
Department of Mathematics
Shantou University
Shantou, Guangdong 515063, P. R. China
and
School of Mathematics
Jiaying University
Meizhou, Guangdong 514015, P. R. China
Email address: zengtiaoying@126.com