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ON THE NUMBER OF SEMISTAR OPERATIONS OF SOME CLASSES OF PRÜFER DOMAINS

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Abstract. The purpose of this paper is to compute the number of semistar operations of certain classes of finite dimensional Prüfer domains. We prove that $|SStar(R)| = |Star(R)| + |Spec(R)| + |Idem(R)|$ where $Idem(R)$ is the set of all nonzero idempotent prime ideals of R if and only if R is a Prüfer domain with Y-graph spectrum, that is, R is a Prüfer domain with exactly two maximal ideals M and N and $Spec(R) = \{(0) \subsetneq P_1 \subsetneq \cdots \subsetneq P_{n-1} \subsetneq M, N \mid P_{n-1} \subsetneq N\}$. We also characterize non-local Prüfer domains R such that $|SStar(R)| = 7$, respectively $|SStar(R)| = 14$.

1. Introduction

Let R be an integral domain with quotient field K, $\mathcal{F}(R)$ the set of nonzero fractional ideals of R, and $\bar{\mathcal{F}}(R)$ the set of nonzero R-submodules of K.

A mapping $* : \bar{\mathcal{F}}(R) \to \bar{\mathcal{F}}(R)$, $E \mapsto E^*$, is called a *semistar operation* on R if the following conditions hold for all $a \in K \setminus \{0\}$ and $E, F \in \bar{\mathcal{F}}(R)$:

(I) $(aE)^* = aE^*$;
(II) $E \subseteq E^*$; if $E \subseteq F$, then $E^* \subseteq F^*$; and
(III) $(E^*)^* = E^*$.

In case where $R^* = R$, the restriction $*_{|\mathcal{F}(R)}$ is a star operation. The simplest semistar operations are the d-operation defined by $E^d = E$ for every $E \in \bar{\mathcal{F}}(R)$, the e-operation defined by $E^e = K$ for every $E \in \bar{\mathcal{F}}(R)$; and the v-operation defined by $E^v = (R : (R : E))$ for every $E \in \bar{\mathcal{F}}(R)$. The notion of semistar operations was introduced by Okabe and Matsuda in [37] as a generalization of star operations introduced by Krull in [18, Section 6.43] and developed in Gilmer's book [9]. Since then, many investigations of semistar operations have been done and tens of papers were published. Two well-studied problems in the literature of semistar operations are: (1) Compute the cardinality of the

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set $SStar(R)$ of all semistar operations on an integral domain R (see [19–32, 34–36]).

(2) Study ring-theoretic properties of integral domains subject to some specific conditions on the lattice of their semistar operations, see for instance [1, 2, 5–8, 10, 33, 38].

The author of this paper, together with E. Houston and M. H. Park, has studied some ring-theoretic properties of integral domains having only finitely many star operations in different contexts of integral domains, see [12–16]. The purpose of this paper is to continue the investigation of both the cardinality of the set of all semistar operations and the ring-theoretic properties of certain classes of Prüfer domains. Firstly, we prove that for a non-local Prüfer domain R, $|SStar(R)| \geq |Star(R)| + |Spec(R)| + |Idem(R)|$, where $Star(R)$ is the set of all star operations on R and $Idem(R)$ is the set of all nonzero idempotent prime ideals of R, and the equality holds if and only if R has a Y-graph spectrum, that is, R has exactly two maximal ideals M and N and $Spec(R) = \{(0) \subsetneq P_1 \subsetneq \cdots \subsetneq P_{n-1} \subsetneq M, N \mid P_{n-1} \subsetneq N\}$ (Theorem 1). In particular if R is an n-dimensional non-local Prüfer domain with $n \geq 2$ and finite prime spectrum, then $|SStar(R)| = |Star(R)| + |Spec(R)|$ if and only if R is a strongly discrete Prüfer domain with two maximal ideals and Y-graph spectrum (Corollary 3). Secondly, we deal with ring-theoretic properties of some classes of Prüfer domains R such that $|SStar(R)| = 7$ respectively $|SStar(R)| = 14$. First, notice that in [32], Matsuda proved that if R is a one-dimensional Prüfer domain with exactly two maximal ideals M and N, then $|SStar(R)| \in \{7, 14, 30\}$ depending on whether both maximal ideals are divisorial ($|SStar(R)| = 7$), or one maximal ideal is divisorial and the other one is non-divisorial ($|SStar(R)| = 14$) or both maximal ideals are not divisorial ($|SStar(R)| = 30$). Also notice that Elliot proved (separately) that if R is a Dedekind domain with exactly two maximal ideals, then $|SStar(R)| = 7$, [4, Table 1, page 238]. In this vein, our objective is to seek for possible characterizations of Prüfer domains R such that $|SStar(R)| \in \{7, 14, 30\}$. First, we prove that for a Prüfer domain R, $|SStar(R)| = 7$ if and only if R is a Dedekind domain with exactly two maximal ideals (equivalently, R is a one-dimensional Prüfer domain with exactly two maximal ideals and both are divisorial) (Theorem 4). Second, we characterize non-local Prüfer domains R such that $|SStar(R)| = 14$. It turns out that a non-local Prüfer domain R has exactly 14 semistar operations if and only if one of the following conditions holds: (1) R is a one-dimensional Prüfer domain with exactly two maximal ideals M and N, M is invertible and N is idempotent. (2) R is a two-dimensional strongly discrete Prüfer domain with exactly two maximal ideals and $Spec(R) = \{(0) \subsetneq P \subsetneq M, N, P \not\subseteq N\}$. (3) R has exactly two maximal ideals and Y-graph spectrum, $7 \leq |Spec(R)| \leq 10$, $0 \leq |Idem(R)| \leq 3$ and $|Spec(R)| + |Idem(R)| = 10$. However, the case of a Prüfer domain with $|SStar(R)| = 30$ seems more difficult to characterize and left open in this paper. It is worth to mention that recently, D. Spirito has developed the study of semilocal Prüfer domains with finitely many semistar operations by linking it to the concept of Jaffard family, see [38].

Finally, notice that each star operation $*$ on R can be extended (but not in a unique way) to a semistar operation $\bar{*}$ on R by setting $E^{\bar{*}} = E^*$ if $E \in \mathcal{F}(R)$ and $E^{\bar{*}} = K$ if $E \in \bar{\mathcal{F}}(R) \setminus \mathcal{F}(R)$. Also if T is a proper overring of R, then T induced a semistar operation on R denoted by $*_T$ and defined by $E^{*_T} = ET$ for every $E \in \bar{F}(R)$. In particular, if P is a nonzero prime ideal of R, $*_P$ (or $*_{R_P}$) will denote the semistar operation induced by the overring R_P. Moreover, if $*$ is a semistar operation of T, then $*$ induces a semistar operation $\widetilde{*}$ on R defined by $E^{\widetilde{*}} = (ET)^*$ for every $E \in \bar{F}(R)$. We denote by:

(1) $\overline{Star(R)} = \{\bar{*} \,|\, * \in Star(R)\}$.

(2) If T is a proper overring of R, $\overline{Star(T)} = \{\bar{*} \,|\, * \in Star(T)\}$, $\widetilde{SStar(T)} = \{\widetilde{*} \,|\, * \in SStar(T)\}$; and $\widetilde{\overline{Star(T)}} = \{\widetilde{\bar{*}} \,|\, * \in Star(T)\}$.

2. Main result

Our next Theorem characterizes Prüfer domains R such that $|SStar(R)| = |Star(R)| + |Spec(R)| + |Idem(R)|$, where $Idem(R)$ is the set of all non-zero prime idempotent ideals of R. Recall that a domain R is conducive if $(R:T) \neq 0$ for every overring $T \neq K$ of R (equivalently, $(R:V) \neq 0$ for some valuation overring V of R, [3, Theorem 3.2]). In this case, $\bar{\mathcal{F}}(R) = \mathcal{F}(R) \cup \{K\}$.

Theorem 1. *Let R be an n-dimensional non-local Prüfer domain with $n \geq 2$ and with finite prime spectrum; and let $Idem(R)$ be the set of all nonzero idempotent prime ideals of R. Then $|SStar(R)| \geq |Star(R)| + |Spec(R)| + |Idem(R)|$; and the equality holds if and only if R has exactly two maximal ideals M and N and Y-graph spectrum, that is, $Spec(R) = \{(0) = P_0 \subset P_1 \subset \cdots \subset P_{n-1}, M, N\}$ with $P_{n-1} \subseteq M \cap N$.*

Proof. It is clear that for every $* \in Star(R)$, $Q \in Spec(R)$ and $P \in Idem(R)$, $R^{\bar{*}} = R \subset R_Q = R^{*_Q}$; $R^{\bar{*}} = R \subset R_P = R^{\tilde{v_P}}$; $P^{\tilde{v_P}} = (PR_P)_{v_P} = R_P$. If $P \nsubseteq Q$, then $P^{*_Q} = PR_Q = R_Q \neq R_P = P^{\tilde{v_P}}$. If $P \subseteq Q$, then $P^{*_Q} = PR_Q = PR_P \subsetneq R_P = P^{\tilde{v_P}}$. Thus $\bar{*} \neq *_Q$, $\bar{*} \neq \tilde{v_P}$ and $*_Q \neq \tilde{v_P}$. Hence $\overline{Star(R)} \dot{\cup} \{*_Q | Q \in Spec(R)\} \dot{\cup} \{\tilde{v_P} | P \in Idem(R)\} \subseteq SStar(R)$ and therefore $|SStar(R)| \geq |Star(R)| + |Spec(R)| + |Idem(R)|$.

$\Rightarrow$) Assume that $|SStar(R)| = |Star(R)| + |Spec(R)| + |Idem(R)|$, and suppose that R has at least three maximal ideals M_1, M_2 and M_3. Set $T_1 = R_{M_2} \cap R_{M_3}$, $T_2 = R_{M_1} \cap R_{M_3}$ and $T_3 = R_{M_1} \cap R_{M_2}$; and let $*_i = *_{T_i}$ be the semistar operation induced by T_i for $i = 1, 2, 3$. Since $R^{*_i} = T_i \neq R_Q = R^{*_Q}$, $*_i \neq *_Q$ for every $Q \in Spec(R)$ and since $*_{i|F(R)} \notin Star(R)$, $*_i \neq \bar{*}$ for every $* \in Star(R)$. Also since $R^{*_i} = T_i \neq R_P = R^{\tilde{v_P}}$, $*_i \neq \tilde{v_P}$ for every $P \in Idem(R)$. Thus $\overline{Star(R)}\} \dot{\cup} \{*_Q | Q \in Spec(R)\} \dot{\cup} \{\tilde{v_P} | P \in Idem(R) \dot{\cup} \{*_1, *_2, *_3\} \subseteq SStar(R)$. Hence $|SStar(R)| \geq |Star(R)| + |Spec(R)| + |Idem(R)| + 3$, which is absurd. Thus R must have at most two maximal ideal and since R is not local, then R

has exactly two maximal ideals M and N. Similarly if R has two non-maximal non-comparable prime ideals P_1 and Q_1, set $T = R_{P_1} \cap R_{Q_1}$ and let $*_T$ be the semistar operation induced by T. Then

$$\overline{Star(R)}\}\dot{\cup}\{*_Q \mid Q \in Spec(R)\}\dot{\cup}\{\tilde{v_P} \mid P \in Idem(R)\}\dot{\cup}\{*_T\} \subseteq SStar(R);$$

and so $|SStar(R)| \geq |Star(R)| + |Spec(R)| + |Idem(R)| + 1$, which is again a contradiction. Thus every non-maximal prime ideals of R are comparable and therefore R has Y-graph spectrum.

$\Leftarrow$) First notice that R must be conducive. Indeed, let P be a nonmaximal prime ideal of R. Then $P^{-1} = (P : P) = R_P$ ([17, Theorem 3.8]). Hence $(R : R_P) = P_v = P \neq (0)$ ([17, Proposition 3.10]) and so R is conducive. Since R_P is a valuation domain $Star(R_P) = \{d_P\}$ (the trivial star operation of R_P) if P is not idempotent and $Star(R_P) = \{d_P, v_P\}$ (v_P is the v-operation on R_P) if P is idempotent. Now let $*$ be any semistar operation on R and set $T = R^*$. If $T = R$, then $*_{|F(R)}$ is a star operation on R and so $* = \overline{*_{|F(R)}}$ (the extension of $*_{|F(R)}$ as R is conducive). So we may assume that $R \subset T \subset K = qf(R)$ (notice that if $T = K$, then $* = e = *_{(0)}$). Since R has Y-graph spectrum, $T = R_P$ for some nonzero prime ideal P of R. In this case $*_{|F(R_P)}$ is a star operation on R_P and so it is either equal to d_P or equal v_P (depending on whether P is idempotent or not). Thus $* = *_P$ or $* = \tilde{v_P}$. Hence $|SStar(R)| = |Star(R)| + |Spec(R)| + |Idem(R)|$. □

Example 2. ([14, Example 5.5]) Let R be a Prüfer domain with exactly two maximal ideals M, N, and exactly one nonzero prime ideal P with $P \subset M \cap N$.

(1) If M and N are invertible, then $|\mathrm{Star}(R)| = 4$,
$|SStar(R)| = 8$ if P is not idempotent and $|SStar(R)| = 9$ if P is idempotent.

(2) If M is not invertible but N is invertible, then $|\mathrm{Star}(R)| = 10$,
$|SStar(R)| = 15$ if P is not idempotent and $|SStar(R)| = 16$ if P is idempotent.

(3) If neither M nor N is invertible, then $|\mathrm{Star}(R)| = 25$,
$|SStar(R)| = 31$ if P is not idempotent and $|SStar(R)| = 32$ if P is idempotent.

Recall that a Prüfer domain R is said to be strongly discrete if R has no nonzero idempotent prime ideal, equivalently, $Idem(R) = \phi$. Our next corollary characterizes Prüfer domains R such that $|SStar(R)| = |Star(R)| + |Spec(R)|$.

Corollary 3. *Let R be an n-dimensional non-local Prüfer domain with $n \geq 2$ and finite prime spectrum. Then $|SStar(R)| = |Star(R)| + |Spec(R)|$ if and only if R is a strongly discrete Prüfer domain with two maximal ideals and Y-graph spectrum.*

Next, we consider a one-dimensional Prüfer domain R with Y-graph spectrum, that is R is a one-dimensional Prüfer domain with exactly two maximal ideals M and N. In [32], Matsuda proved that $|SStar(R)| = 7$ if both M

and N are divisorial, equivalently, R is a Dedekind domain with exactly two maximal ideals. The same result was obtained separately by Elliot in [4, Table 1, page 238]. Our first result shows that this is in fact a characterization of such Prüfer domains.

Theorem 4. *Let R be a non-local integrally closed domain of finite dimension. Then $|SStar(R)| = 7$ (if and) only if R is a Dedekind domain with exactly two maximal ideals.*

Proof. By [14, Theorem 3.1], R is a Prüfer domain. If R has more than three maximal ideals, say M_1, M_2 and M_3, then set $T_1 = R_{M_2} \cap R_{M_3}$, $T_2 = R_{M_1} \cap R_{M_3}$ and $T_3 = R_{M_1} \cap R_{M_2}$. So $\{e, d, *_{M_1}, *_{M_2}, *_{M_3}, *_{T_1}, *_{T_2}, *_{T_3}\} \subseteq SStar(R)$, which is absurd. Hence R has exactly two maximal ideals, say M and N. We claim that $\dim R = 1$. By way of contradiction, suppose that $htM = \dim R \geq 2$. Let P be a height-one prime ideal contained in M. Two cases are then possible.

Case 1. $Spec(R)$ contains a nonzero prime ideal Q that is not in $\{P, M, N\}$. Suppose that $P \not\subseteq N$ and set $T = R_P \cap R_N$. If $Q \not\subseteq M$, set $S = R_Q \cap R_M$. Then $\{e, d, *_P, *_Q, *_M, *_N, *_T, *_S\} \subseteq SStar(R)$, which is absurd. Hence $Q \subsetneq M$ and since $htP = 1$, $P \subsetneq Q$. But since $P \not\subseteq N$, $Q \not\subseteq N$. Again set $S = R_Q \cap R_N$. Then $\{e, d, *_P, *_Q, *_M, *_N, *_T, *_S\} \subseteq SStar(R)$, which is a contradiction too. Hence $P \subsetneq N$ and so $P \subseteq M \cap N$. Thus $PR_P = P$ and by [11, Theorem 5.1], R is not divisorial. Thus, $SStar(R) = \{d, e, v, *_P, *_Q, *_M, *_N\}$, which implies that $Q \subseteq M \cap N$. Hence $Spec(R)$ must of the form $\{(0) \subsetneq P \subsetneq Q \subsetneq M \cap N\}$. By [16, Theorem 4.3], $|Star(R)| \geq 4$ and by Theorem 1, $7 = |SStar(R)| = |Star(R)| + |Spec(R)| + |Idem(R)| \geq 4 + 5 + |idem(R)|$ which is a contradiction.

Case 2. $Spec(R) = \{(0) \subsetneq P \subsetneq M, N\}$. If $P \not\subseteq N$, then R is not conducive and so $d \neq \bar{d}$. Since R_P, R_M and R_N are valuation domains, $\bar{\mathcal{F}}(R_P) = \mathcal{F}(R_P) \cup \{K\}$, $\bar{\mathcal{F}}(R_M) = \mathcal{F}(R_M) \cup \{K\}$ and $\bar{\mathcal{F}}(R_N) = \mathcal{F}(R_N) \cup \{K\}$. Also notice that since the largest prime ideal contained in M and N is the zero prime ideal, $R_P R_N = R_M R_N = K$. Now define $*_1$, $*_2$ and $*_3$ by $E^{*_1} = E$ if $ER_P \in F(R_P)$ and $E^{*_1} = K$ if $ER_R = K$; $E^{*_2} = E$ if $ER_M \in F(R_M)$ and $E^{*_2} = K$ if $ER_M = K$; and $E^{*_3} = E$ if $ER_N \in F(R_N)$ and $E^{*_3} = K$ if $ER_N = K$. Then it is easy to check that $*_1$, $*_2$ and $*_3$ are semistar operations on R; $*_3 \neq *_i$, $i = 1, 2$ and so $\{d, \bar{d}, e, *_P, *_M, *_N, *_1, *_2, *_3\} \subseteq SStar(R)$, which is absurd. Thus $P \subset N$ and so R has a Y-graph spectrum of the form $Spec(R) = \{(0) \subsetneq P \subsetneq M \cap N\}$. By [16, Theorem 4.3] $|Star(R)| \geq 4$, and Theorem 1, $7 = |SStar(R)| = |Star(R)| + |Spec(R)| + |Idem(R)| \geq 4 + 4 + |idem(R)|$ which is a contradiction.

It follows that $\dim R = 1$ and $Spec(R) = \{(0), M, N\}$. Now, by [32, Theorem 1], M and N must be invertible and therefore R is a Dedekind domain. □

Next, we deal with the case where R is a non-local Prüfer domain with two maximal ideals M and N and $|SStar(R)| = 14$. We continue to use d for the trivial semistar operation, v the v-operation on R, $\bar{v}$ it extension to a semistar operation, and if Q is a nonzero prime ideal of R, we use d_Q to denote the d-(semi)star operation on R_Q and v_Q the v-operation on R_Q. Also recall that if Q is a prime ideal of a Prüfer domain R, then $Star(R_Q) = \{d_Q\}$ if Q is not idempotent and $Star(R_Q) = \{d_Q, v_Q\}$ if Q is idempotent. Finally $\tilde{v_Q}$ will denote the extension of v_Q to a semistar operation on R, that is, $E^{\tilde{v_Q}} = (ER_Q)^{v_Q}$; and since $K^* = K$ for any semistar operation $*$ on R, we will always assume that $E \subsetneq K$ whenever E is in $\bar{F}(R)$. Also notice that if $E \in \bar{F}(R) - F(R)$, then either $ER_M = K$ or $ER_N = K$. But since $E = ER_M \cap ER_N$, either $E = ER_N$ or $E = ER_M$.

Lemma 5. *Let R be a two-dimensional strongly discrete Prüfer domain with exactly two maximal ideals M and N and $Spec(R) = \{(0) \subsetneq P \subsetneq M, N \mid P \not\subseteq N\}$. Then $|SStar(R)| = 14$.*

Proof. Clearly R is divisorial ([11, Theorem 5.1]) but not conducive. Moreover, it is easy to check that the following operations are semistar operations on R.

(1) $\star_1$ defined by $E^{\star_1} = E$ if $ER_M \in \mathcal{F}(R_M)$, and $E^{\star_1} = K$ if $ER_M = K$.
(2) $\star_2$ defined by $E^{\star_2} = E$ if $ER_N \in \mathcal{F}(R_N)$, and $E^{\star_2} = K$ if $ER_N = K$.
(3) $\star_3$ defined by $\begin{cases} E^{\star_3} = E & \text{if } E \in \mathcal{F}(R), \\ E^{\star_3} = E & \text{if } E \in \bar{\mathcal{F}}(R) \setminus \mathcal{F}(R),\ ER_P = K \text{ and } \\ & ER_N \subset K, \\ E^{\star_3} = ER_P & \text{if } E \in \bar{\mathcal{F}}(R) \setminus \mathcal{F}(R),\ ER_P \subset K \text{ and } \\ & ER_N = K. \end{cases}$
(4) $\star_4$ defined by $\begin{cases} E^{\star_4} = E & \text{if } E \in \mathcal{F}(R), \\ E^{\star_4} = K & \text{if } E \in \bar{\mathcal{F}}(R) \setminus \mathcal{F}(R),\ ER_M = K, \\ E^{\star_4} = ER_P & \text{if } E \in \bar{\mathcal{F}}(R) \setminus \mathcal{F}(R),\ ER_M \subset K. \end{cases}$

Since $(R_M)^{\star_1} = R_M$, $(R_M)^{\star_2} = K$, $(R_M)^{\star_3} = R_P$, $(R_M)^{\star_4} = R_P$, $(R_N)^{\star_3} = R_N$ and $(R_N)^{\star_4} = K$, $\star_1, \star_2, \star_3$ and $\star_4$ are distinct semistar operations of R. Let $T = R_P \cap R_N$. Since T is a Dedekind domain with exactly two maximal ideals, $|SStar(T)| = 7$. Now, let $* \in SStar(R)$. If $R^* = K$, then $* = e$. If $R^* \in \{T, R_N, R_P\}$, then $* \in \widetilde{SStar(T)}$. Indeed, if $R^* = T$, then $*|\bar{\mathcal{F}}(T) \in SStar(T)$ (for if $E \in \bar{\mathcal{F}}(T)$, then $E^*T = E^*R^* \subseteq (E^*R^*)^* = (ER)^* = E^*$ and so $E^* \in \bar{\mathcal{F}}(T)$). Moreover, for every $E \in \bar{\mathcal{F}}(R)$, $E^* = (ER)^* = (ER^*)^* = (ET)^* = (ET)^{*|\bar{\mathcal{F}}(T)}$ and so $* = \widetilde{*|\bar{\mathcal{F}}(T)}$. Assume that $R^* = R_Q$, $Q = P, N$. Then $*|\mathcal{F}(R_Q) \in Star(R_Q) = \{d_Q\}$ since R_Q is a strongly discrete valuation domain. Now Define $\star$ on T by $E^\star = ER_Q$ for every $E \in \bar{\mathcal{F}}(T)$. Then let $E \in \bar{\mathcal{F}}(R)$. If $ER_Q \neq K$, then $E^* = (ER)^* = (ER^*)^* = (ER_Q)^* = ER_Q = (ET)R_Q = (ET)^\star = E^{\widetilde{\star}}$. If $ER_Q = K$, then $E^* = (ER)^* = (ER^*)^* = (ER_Q)^* = K^* = K$ and $E^{\widetilde{\star}} = (ET)R_Q = ER_Q = K$. Thus $* = \widetilde{\star}$, as desired. If $R^* = R_M$, then $* = *_{R_M}$. Thus we may assume that $R^* = R$. Then $*|_{F(R)} \in Star(R)$. But since R is divisorial $*|_{F(R)} = d$. Thus for every $E \in F(R)$, $E^* = E$. Since $[R_M, K] = \{R_M, R_P, K\}$ and $[R_N, K] = \{R_N, K\}$, there are six possibilities:
Case 1. $(R_M)^* = R_M$ and $(R_N)^* = R_N$. Then $*|_{F(R_M)} = d_{R_M}$ and $*|_{F(R_N)} = d_{R_N}$. In this case $* = d$. Indeed, let $E \in \bar{F}(R) - F(R)$. If $E = ER_M$, then

$E^* = (ER_M)^* = ER_M = E$ and similarly if $E = ER_N$, then $E^* = (ER_N)^* = ER_N = E$.
Case 2. $(R_M)^* = R_M$ and $(R_N)^* = K$. Then $*|_{F(R_M)} = d_{R_M}$. Let $E \in \bar{F}(R) - F(R)$. If $E = ER_M$, then $E^* = (ER_M)^* = ER_M = E$. If $ER_M = K$, then $E = ER_N$ and so $E^* = (ER_N)^* = (E(R_N)^*)^* = (EK)^* = K$. Hence $* = \star_1$.
Case 3. $(R_M)^* = K$ and $(R_N)^* = R_N$. Then $*|_{F(R_N)} = d_{R_N}$. Let $E \in \bar{F}(R) - F(R)$. If $E = ER_N$, then $E^* = (ER_N)^* = ER_N = E$. If $ER_N = K$, then $E = ER_M$ and so $E^* = (ER_M)^* = (E(R_M)^*)^* = (EK)^* = K$. Hence $* = \star_2$.
Case 4. $(R_M)^* = K$, $(R_N)^* = K$. Let $E \in \bar{F}(R) - F(R)$. If $E = ER_M$, then $E^* = (ER_M)^* = (E(R_M)^*)^* = (EK)^* = K$; and if $E = ER_N$, then $E^* = (ER_N)^* = (E(R_N)^*)^* = (EK)^* = K$. Thus $* = \overline{d}$.
Case 5. $(R_M)^* = R_P$, $(R_N)^* = R_N$. Then $(R_P)^* = (R_M)^{**} = (R_M)^* = R_P$ and so $*|_{F(R_P)} = d_{R_P}$. Let $E \in \bar{F}(R) - F(R)$. If $ER_P = K$ and $ER_N \subsetneq R_N$, necessarily $ER_M = K$ and so $E = ER_N$. Thus $E^* = (ER_N)^* = ER_N = E$. If $ER_P \subsetneq K$ and $ER_N = K$, necessarily $E = ER_M$. Then $E^* = (ER_M)^* = (E(R_M)^*)^* = (ER_P)^* = ER_P$. Thus $* = \star_3$.
Case 6. $(R_M)^* = R_P$ and $(R_N)^* = K$. Let $E \in \bar{F}(R) - F(R)$. If $ER_P \subsetneq K$, then $ER_M \subsetneq K$ and so $E = ER_M$. Thus $E^* = (ER_M)^* = (E(R_M)^*)^* = (ER_P)^* = ER_P$. Assume that $ER_P = K$. If $E = ER_M$, then $E^* = (ER_M)^* = (E(R_M)^*)^* = (ER_P)^* = K^* = K$; and if $E = ER_N$, then $E^* = (ER_N)^* = (E(R_N)^*)^* = (EK)^* = K^* = K$. $* = \star_4$.

It follows that $SStar(R) = \widetilde{SStar(T)} \dot{\bigcup} \{d, \overline{d}, *_{R_M}, \star_1, \star_2, \star_3, \star_4\}$. Therefore $|SStar(R)| = 14$, as desired. □

Theorem 6. *Let R be a non-local Prüfer domain. Then $|SStar(R)| = 14$ if and only if one of the following conditions holds*:

(1) *R is a one-dimensional Prüfer domain with exactly two maximal ideals M and N, M is invertible and N is idempotent.*

(2) *R is a two-dimensional strongly discrete Prüfer domain with exactly two maximal ideals and $Spec(R) = \{(0) \subsetneq P \subsetneq M, N \mid P \nsubseteq N\}$.*

(3) *R has exactly two maximal ideals and Y-graph spectrum, $7 \le |Spec(R)| \le 10$, $0 \le |Idem(R)| \le 3$ and $|Spec(R)| + |Idem(R)| = 10$.*

Proof. Assume that $|SStar(R)| = 14$. If $|\mathrm{Max}(R)| \ge 4$, then let M_1, M_2, M_3 and M_4 be maximal ideals of R. For $i, j \in \{1, 2, 3, 4\}$ with $i \ne j$, set $T_{ij} = R_{M_i} \cap R_{M_j}$ and for each $k = 1, 2, 3, 4$, set $S_k = \bigcap_{i=1, i \ne k}^{i=4} R_{M_i}$. Then

$$\{*_{R_{M_i}}, *_{T_{ij}}, *_{S_k}, e, d\} \subseteq SStar(R)$$

and so $16 = 4+6+4+2 \le |SStar(R)|$, a contradiction. Hence $2 \le |\mathrm{Max}(R)| \le 3$. Suppose that $|\mathrm{Max}(R)| = 3$ and set $\mathrm{Max}(R) = \{M_1, M_2, M_3\}$. Suppose that $\dim R = 1$. If all M_i are invertible, R is a Dedekind domain with exactly three maximal ideals and by [4, Theorem 1.2], $|SStar(R)|$=61, a contradiction.

Hence at least one maximal ideal M_i of R is not invertible, say M_1. Set $T = R_{M_1} \cap R_{M_2}$. Then T is a one-dimensional Prüfer domain with exactly two maximal ideals such that at least one maximal ideal is non-invertible (so non-divisorial). By [32], $|SStar(T)| \in \{14, 30\}$. Since $\widetilde{SStar(T)}\dot{\bigcup}\{*_{R_{M_3}}\} \subseteq SStar(R)$, $15 \leq |SStar(R)|$, a contradiction. It follows that $\dim R \geq 2$.

If the Jacobson radical $J(R)$ contains a nonzero prime ideal P of R, then for each $i = 1, 2, 3$, let $P_i \subseteq M_i$ be a prime ideal of R containing P such that $ht(P_i/P) = 1$ and set $T = R_{P_1} \cap R_{P_2} \cap R_{P_3}$. By [16, Theorem 3.5], $|SStarR| \geq |\widetilde{Star(T)}| = |Star(T)| = 45$, a contradiction. Hence $J(R)$ does not contain any nonzero prime ideal. Now for every $i \neq j$, let $P_{i,j} = M_i \wedge M_j$ be the largest prime ideal of R contained in $M_i \cap M_j$. If for each $i \neq j$, $P_{i,j} = (0)$, then let P_i be a height-one prime ideal of R contained in M_i and set $T = R_{P_1} \cap R_{P_2} \cap R_{P_3}$. Then T is a one-dimensional Prüfer domain with exactly three maximal ideals and so $|SStar(R)| \geq |SStar(T)| \geq 15$, a contradiction. Hence there is $i \neq j$ such that $P_{ij} \neq (0)$. Without loss of generality, we may assume that $P = P_{1,2} = M_1 \wedge M_2 \neq (0)$. Necessarily $P \not\subseteq M_3$ and clearly R is not divisorial and R is not conducive (so $d \neq \bar{d}$). Set $T = R_{M_1} \cap R_{M_2}$. Then T is a Prüfer domain with exactly two maximal ideals and $P = PT \subseteq J(T)$, by [16, Theorem 4.3], $|Star(T)| \geq 4$. Since

$$\widetilde{Star(T)}\dot{\bigcup}\{*_{R_P}, *_{R_{M_1}}, *_{R_{M_2}}, *_{R_{M_3}}, *_{R_P \cap R_{M_3}}, *_{R_{M_1} \cap R_{M_3}}, *_{R_{M_2} \cap R_{M_3}}, e, d, \bar{d}, v\},$$

$15 \leq |SStar(R)|$, which absurd. It follows that $|\mathrm{Max}(R)| = 2$.

Now, assume that $\mathrm{Max}(R) = \{M, N\}$. If $\dim R = 1$, then (1) of the theorem is satisfied by [32]. Assume that $\dim R \geq 2$ and without loss of generality, we may assume that $htM = \dim R = n \geq 2$. If $M \wedge N = (0)$, let P and Q be height-one prime ideals such that $P \subsetneq M$ and $Q \subseteq N$ and set $T = R_P \cap R_Q$. Since T is a one-dimensional Prüfer domain with exactly two maximal ideals, by [32], $SStar(T) \in \{7, 14, 30\}$. If $|SStar(T)| \geq 14$, then $|SStar(R)| \geq 15$ since $\widetilde{SStar(T)}\dot{\cup}\{*_{R_M}\} \subseteq SStar(R)$, which is absurd. Hence T must be a Dedekind domain, $|SStar(T)| = 7$ and P and Q are strongly discrete. If $\dim(R) = htM \geq 3$, then let $(0) \subsetneq P \subsetneq P_2 \subsetneq M$ with $htP_2 = 2$ and set $S = R_{P_2} \cap R_Q$. Then S is a two-dimensional Prüfer domain with exactly two maximal ideals and $Spec(S) = \{(0) \subsetneq PS \subsetneq P_2S, QS \,|\, PS \not\subseteq QS\}$. If P_2S is not divisorial, then by Lemma 5, $|SStar(S)| \geq 14$ ($|SStar(S)| = 14$ if P_2S is strongly discrete and $|SStar(S)| \geq 14$ if P_2S is not divisorial in S). Since $\widetilde{SStar(S)}\dot{\cup}\{*_{R_M}\} \subseteq SStar(R)$, $|SStar(R)| \geq 15$, which is absurd. Hence $\dim R = 2$. If $htN \geq 2$, let $Q_2 \subseteq N$ such that $htQ_2 = 2$ and set $S' = R_P \cap R_{Q_2}$ again by Lemma 5, $|SStar(S')| \geq 14$ and since $\widetilde{SStar(S')}\dot{\cup}\{*_{R_M}\} \subseteq SStar(R)$, $|SStar(R)| \geq 15$, which is absurd. Hence $htN = 1$ and so $N = Q$. Therefore $Spec(R) = \{(0) \subsetneq P \subsetneq M, N \,|\, P \not\subseteq N\}$. Finally, if M is idempotent, then M is not divisorial and so $\widetilde{SStar(T)}\dot{\bigcup}\{d, \overline{d}, *_{R_M}, *_1, *_2, *_3, *_4, v_{R_M}\} \subseteq SStar(R)$

where $T = R_P \cap R_N$ and v_{R_M} is the v-operation on R_M, which is absurd. Hence M is strongly discrete and therefore (2) of the theorem holds.

Assume that $M \wedge N = P \neq (0)$. Then R is not divisorial and R is conducive. Let m_1, n_1 (respectively, m_2, n_2) be the numbers of non-idempotent (respectively idempotent) prime ideals strictly between P and M (respectively between P and N). If M or N is not divisorial, for instance M is not divisorial, by [16, Theorem 4.3], $|Star(R)| \geq 10$. Since $\overline{Star(R)} \dot{\bigcup} \{*_{R_M}, *_{R_N}, *_{R_P}, e, v_{R_M}\} \subseteq SStar(R)$, $15 \leq |SStar(R)|$, which is a contradiction. Hence M and N are divisorial. Now, suppose that $n_1 \geq 1$ or $n_2 \geq 1$, for instance, $n_1 \geq 1$. Let P_1 be an idempotent prime ideal strictly between P and M. By [16, Theorem 4.3 (1)], $|Star(R)| \geq 8$, and so $\overline{Star(R)} \dot{\bigcup} \{*_{R_M}, *_{R_N}, *_{R_P}, *_{R_{P_1}}, *_{R_{P_1} \cap R_N}, e, v_{R_{P_1}}\} \subseteq SStar(R)$. Hence $15 \leq |SStar(R)|$, which is a contradiction. Thus $n_1 = n_2 = 0$, equivalently there are no idempotent primes strictly between P and M and strictly between P and N. Again suppose that $m_1 \geq 1$ or $m_2 \geq 1$, for instance, $m_1 \geq 1$. Let Q be a non-idempotent prime strictly between P and M and set $T = R_Q \cap R_N$. By [16, Theorem 4.3], $|Star(R)| \geq 6$ and $|Star(T)| \geq 4$. But since $\overline{Star(R)} \dot{\bigcup} \overline{Star(T)} \dot{\bigcup} \{*_{R_M}, *_{R_N}, *_{R_P}, *_{R_Q}, e\} \subseteq SStar(R)$, $15 \leq |SStar(R)|$, which is a absurd. Thus $m_1 = m_2 = 0$, and therefore there are no primes strictly between P and M and strictly between P and N, equivalently, $ht(M/P) = ht(N/P) = 1$. By [16, Theorem 4.3(1)], $|Star(R)| = 4$ and by Theorem 1, $14 = |SStar(R)| = |Star(R)| + |Spec(R)| + |Idem(R)| = 4 + |Spec(R)| + |Idem(R)|$. Thus $|Spec(R)| + |Idem(R)| = 10$. Since M and N are divisorial, then M and N are not idempotent. Thus $|Idem(R)| \leq |Spec(R)| - 3$. Thus $|Spec(R)| \geq 7$ and $|Idem(R)| \leq 3$, with $|Spec(R)| + |Idem(R)| = 10$, as desired.

Conversely, if (1) is satisfied, $|SStar(R)| = 14$ by [32]. If (2) holds, then $|SStar(R)| = 14$ by Lemma 5. Assume that (3) is satisfied. Then by [16, Theorem 4.3 (1)], $|Star(R)| = 4$ and by Theorem 1, $|SStar(R)| = |Star(R)| + |Spec(R)| + |Idem(R)| = 4 + 10 = 14$. ☐

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References

[1] G. W. Chang and M. Fontana, *Uppers to zero and semistar operations in polynomial rings*, J. Algebra **318** (2007), no. 1, 484–493. https://doi.org/10.1016/j.jalgebra.2007.06.010

[2] G. W. Chang, M. Fontana, and M. H. Park, *Polynomial extensions of semistar operations*, J. Algebra **390** (2013), 250–263. https://doi.org/10.1016/j.jalgebra.2013.05.020

[3] D. E. Dobbs and R. Fedder, *Conducive integral domains*, J. Algebra **86** (1984), no. 2, 494–510. https://doi.org/10.1016/0021-8693(84)90044-9

[4] J. Elliott, *Semistar operations on Dedekind domains*, Comm. Algebra **43** (2015), no. 1, 236–248. https://doi.org/10.1080/00927872.2014.897571

[5] C. A. Finocchiaro, M. Fontana, and D. Spirito, *Spectral spaces of semistar operations*, J. Pure Appl. Algebra **220** (2016), no. 8, 2897–2913. https://doi.org/10.1016/j.jpaa.2016.01.008
[6] C. A. Finocchiaro and D. Spirito, *Some topological considerations on semistar operations*, J. Algebra **409** (2014), 199–218. https://doi.org/10.1016/j.jalgebra.2014.04.002
[7] M. Fontana, P. Jara, and E. Santos, *Local-global properties for semistar operations*, Comm. Algebra **32** (2004), no. 8, 3111–3137. https://doi.org/10.1081/AGB-120039282
[8] G. Fusacchia, *Injective modules and semistar operations*, J. Pure Appl. Algebra **216** (2012), no. 1, 77–90. https://doi.org/10.1016/j.jpaa.2011.05.004
[9] R. Gilmer, *Multiplicative Ideal Theory*, Marcel Dekker, Inc., New York, 1972.
[10] F. Halter-Koch, *Localizing systems, module systems, and semistar operations*, J. Algebra **238** (2001), no. 2, 723–761. https://doi.org/10.1006/jabr.2000.8671
[11] W. Heinzer, *Integral domains in which each non-zero ideal is divisorial*, Mathematika **15** (1968), 164–170. https://doi.org/10.1112/S0025579300002527
[12] E. Houston, A. Mimouni, and M. H. Park, *Integral domains which admit at most two star operations*, Comm. Algebra **39** (2011), no. 5, 1907–1921. https://doi.org/10.1080/00927872.2010.480956
[13] ———, *Noetherian domains which admit only finitely many star operations*, J. Algebra **366** (2012), 78–93. https://doi.org/10.1016/j.jalgebra.2012.05.015
[14] ———, *Integrally closed domains with only finitely many star operations*, Comm. Algebra **42** (2014), no. 12, 5264–5286. https://doi.org/10.1080/00927872.2013.837477
[15] ———, *Star operations on overrings of Noetherian domains*, J. Pure Appl. Algebra **220** (2016), no. 2, 810–821. https://doi.org/10.1016/j.jpaa.2015.07.018
[16] ———, *Star operations on overrings of Prüfer domains*, Comm. Algebra **45** (2017), no. 8, 3297–3309. https://doi.org/10.1080/00927872.2016.1236199
[17] J. A. Huckaba and I. J. Papick, *When the dual of an ideal is a ring*, Manuscripta Math. **37** (1982), no. 1, 67–85. https://doi.org/10.1007/BF01239947
[18] W. Krull, *Idealtheorie*, Zweite, ergänzte Auflage. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 46, Springer-Verlag, Berlin, 1968.
[19] R. Matsuda, *Note on the number of semistar-operations*, Math. J. Ibaraki Univ. **31** (1999), 47–53. https://doi.org/10.5036/mjiu.31.47
[20] ———, *A note on the number of semistar-operations. II*, Far East J. Math. Sci. (FJMS) **2** (2000), no. 1, 159–172.
[21] ———, *On the number of semistar-operations*, in Proceedings of the 5th Symposium on Algebra, Languages and Computation (Matsue, 2001), 69–73, Shimane Univ., Matsue, 2002.
[22] ———, *Note on the number of semistar-operations. III*, in Commutative rings, 77–81, Nova Sci. Publ., Hauppauge, NY, 2002.
[23] ———, *Note on the number of semistar-operations. IV*, Sci. Math. Jpn. **55** (2002), no. 2, 345–347.
[24] ———, *Note on the number of semistar-operations. V*, Sci. Math. Jpn. **57** (2003), no. 1, 57–62.
[25] ———, *Note on the number of semistar operations. VIII*, Math. J. Ibaraki Univ. **37** (2005), 53–79. https://doi.org/10.5036/mjiu.37.53
[26] ———, *Note on the number of semistar operations. XIII*, Adv. Algebra Anal. **1** (2006), no. 3, 147–158.
[27] ———, *Note on the number of semistar operations. X*, Math. J. Ibaraki Univ. **38** (2006), 1–19. https://doi.org/10.5036/mjiu.38.1
[28] ———, *Note on the number of semistar operations. VI*, in Focus on commutative rings research, 187–192, Nova Sci. Publ., New York, 2006.

[29] ______, *Note on the number of semistar operations. XI*, Math. J. Ibaraki Univ. **39** (2007), 11–22. https://doi.org/10.5036/mjiu.39.11

[30] ______, *Note on the number of semistar operations. XIV*, Math. J. Ibaraki Univ. **40** (2008), 11–17. https://doi.org/10.5036/mjiu.40.11

[31] ______, *The semistar operations on certain Prüfer domain*, Math. J. Ibaraki Univ. **43** (2011), 1–12. https://doi.org/10.5036/mjiu.43.1

[32] ______, *The semistar operations on certain Prüfer domain, II*, Math. J. Ibaraki Univ. **46** (2014), 1–8. https://doi.org/10.5036/mjiu.46.

[33] ______, *The construction of all the star operations and all the semistar operations on 1-dimensional Prüfer domains*, Math. J. Ibaraki Univ. **47** (2015), 19–37. https://doi.org/10.5036/mjiu.47.19

[34] R. Matsuda and T. Sugatani, *Semistar-operations on integral domains. II*, Math. J. Toyama Univ. **18** (1995), 155–161.

[35] A. Mimouni, *Krull dimension, overrings and semistar operations of an integral domain*, J. Algebra **321** (2009), no. 5, 1497–1509. https://doi.org/10.1016/j.jalgebra.2008.11.028

[36] A. Mimouni and M. Samman, *On the cardinality of semistar operations on integral domains*, Comm. Algebra **33** (2005), no. 9, 3311–3321. https://doi.org/10.1081/AGB-200058203

[37] A. Okabe and R. Matsuda, *Semistar-operations on integral domains*, Math. J. Toyama Univ. **17** (1994), 1–21.

[38] D. Spirito, *Towards a classification of stable semistar operations on a Prüfer domain*, Comm. Algebra **46** (2018), no. 4, 1831–1842. https://doi.org/10.1080/00927872.2017.1360329

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