

UNICITY OF MERMORPHIC FUNCTIONS CONCERNING SHARED FUNCTIONS WITH THEIR DIFFERENCE

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ABSTRACT. In this paper, we investigate the uniqueness of meromorphic functions of finite order concerning sharing small functions and prove that if $f(z)$ and $\Delta_c f(z)$ share $a(z), b(z), \infty$ CM, where $a(z), b(z) (\neq \infty)$ are two distinct small functions of $f(z)$, then $f(z) \equiv \Delta_c f(z)$. The result improves the results due to Li et al. ([9]), Cui et al. ([1]) and Lü et al. ([12]).

1. Introduction

Throughout this paper, a meromorphic function always means a function which is meromorphic in the whole complex plane \mathbb{C} . We assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna theory (see [7, 15, 16]).

In addition, we denote by $S(r, f)$ any quantity satisfying $S(r, f) = o(T(r, f))$ as $r \rightarrow \infty$ possibly outside of a set E with finite linear or logarithmic measure, not necessarily the same at each occurrence. We say that $a(z)$ is a small function of $f(z)$ if $T(r, a) = S(r, f)$.

We use $\rho(f)$ to denote the order of f . We say that two meromorphic functions f and g share a IM (ignoring multiplicities) if $f - a$ and $g - a$ have the same zeros. If $f - a$ and $g - a$ have the same zeros with the same multiplicities, then we say that they share a CM (counting multiplicities), where a is a small function of f and g .

For a meromorphic function $f(z)$, we define its shift by $f_c(z) = f(z + c)$ and its difference operator by $\Delta_c f(z) = f(z + c) - f(z)$.

In 1929, Nevanlinna [13] proved the following famous five-value theorem.

Theorem 1.1. *Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions, and let a_j ($j = 1, 2, 3, 4, 5$) be five distinct values in the extended complex plane. If $f(z)$ and $g(z)$ share a_j ($j = 1, 2, 3, 4, 5$) IM, then $f(z) \equiv g(z)$.*

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In 2000, Li and Qiao [10] proved that Theorem 1.1 is still valid for five small functions, they proved:

Theorem 1.2. *Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions, and let $a_j(z)$ ($j = 1, 2, 3, 4, 5$) (one of them can be ∞) be five distinct small functions of $f(z)$ and $g(z)$. If $f(z)$ and $g(z)$ share $a_j(z)$ ($j = 1, 2, 3, 4, 5$) IM, then $f(z) \equiv g(z)$.*

Recently, value distribution in difference analogue of meromorphic functions has become a subject of some interests, see ([1–6, 8, 17]).

In 2014, Zhang and Liao [17], Liu et al. [11] proved the following result independently.

Theorem 1.3. *Let $f(z)$ be a transcendental entire function of finite order, and let a, b be two distinct constants. If $f(z)$ and $\Delta_c f(z)$ share a, b CM, then $f(z) \equiv \Delta_c f(z)$.*

In fact, Liu et al. [11] proved the following more general case.

Theorem 1.4. *Let $f(z)$ be a transcendental entire function of finite order, and let $a(z) (\neq 0), b(z) (\neq 0)$ be two distinct small functions of $f(z)$. If $f(z)$ and $\Delta_c f(z)$ share $a(z), b(z)$ CM, then $f(z) \equiv \Delta_c f(z)$.*

More recently, Li et al. [9], Cui et al. [1], Lü et al. [12], proved that Theorem 1.3 still holds for meromorphic functions of finite order if $f(z)$ and $\Delta_c f(z)$ sharing ∞ CM. They proved:

Theorem 1.5. *Let $f(z)$ be a transcendental meromorphic function of finite order, and let a, b be two distinct constants. If $f(z)$ and $\Delta_c f(z)$ share a, b, ∞ CM, then $f(z) \equiv \Delta_c f(z)$.*

A nature problem arise: Does Theorem 1.5 still hold if $f(z)$ and $\Delta_c f(z)$ share $a(z), b(z)$ and ∞ CM, where $a(z), b(z)$ are two distinct small functions of $f(z)$?

In this paper, we study the problem and give a positive answer to the question.

Theorem 1.6. *Let $f(z)$ be a transcendental meromorphic function of finite order, and let $a(z) (\neq \infty), b(z) (\neq \infty)$ be two distinct small functions of $f(z)$. If $f(z)$ and $\Delta_c f(z)$ share $a(z), b(z), \infty$ CM, then $f(z) \equiv \Delta_c f(z)$.*

Example 1.7. Let $f(z) = \frac{e^z}{e^{2z}+1}$, $c = \pi i$, $a(z) \equiv 0$. Then $\Delta_c f(z) = f(z+c) - f(z) = -2\frac{e^z}{e^{2z}+1}$. Obviously, $\Delta_c f(z)$ and $f(z)$ share $a(z), \infty$, but $\Delta_c f(z) \not\equiv f(z)$. This example shows that the number of shared functions can not be reduce to two.

2. Some lemmas

Lemma 2.1. *Let $A (\neq -1)$ be a nonzero constant. Suppose that $f(z) \not\equiv 0$ is a meromorphic solution of finite order to the following difference equation*

$$(1) \quad Af(z) + f(z+c) \equiv 0.$$

Then there exists a real number $B > 0$ such that $T(r, f) \geq Br$.

Proof. We consider three cases.

Case 1. $f(z) \neq 0, \infty$. Then $f(z) = e^{p(z)}$, where $p(z)$ is a polynomial. It follows from (1) that $p(z)$ is a nonconstant polynomial. Let $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$, where $a_n \neq 0$, $n \geq 1$. Hence there exists $B > 0$ such that

$$T(r, f) \geq \frac{1}{2} T(r, e^{a_n z^n}) \geq \frac{T(r, e^z)}{2} \geq Br.$$

Case 2. There exists z_0 such that $f(z_0) = 0$. Without loss of generality, let $z_0 = 0$. Then it follows from (1) that nc ($n = 0, 1, 2, \dots$) are also zeros of $f(z)$. For $|2nc| \leq r < |(2n+1)c|$, we have

$$\begin{aligned} T(r, f) &\geq N(r, \frac{1}{f}) + O(1) = \int_0^r \frac{n(t, \frac{1}{f}) - n(0, \frac{1}{f})}{t} dt + n(0, \frac{1}{f}) \log r + O(1) \\ &\geq \sum_{j=1}^{2n-1} j \int_{|jc|}^{|(j+1)c|} \frac{1}{t} dt + n(0, \frac{1}{f}) \log r + O(1) \\ &\geq \log \frac{(2n)^{2n-1}}{(2n-1)!} + n(0, \frac{1}{f}) \log r + O(1) \\ &\geq n \log 2 + n(0, \frac{1}{f}) \log r + O(1) \\ &\geq \frac{r + |c|}{2|c|} \log 2 + n(0, \frac{1}{f}) \log r + O(1) \\ &\geq \frac{r}{4|c|}. \end{aligned}$$

Case 3. There exists z_1 such that $f(z_1) = \infty$. Without loss of generality, let $z_1 = 0$. Then it follows from (1) that nc ($n = 0, 1, 2, \dots$) are also poles of $f(z)$. For $|2nc| \leq r < |(2n+1)c|$, we have

$$\begin{aligned} T(r, f) &\geq N(r, f) = \int_0^r \frac{n(t, f) - n(0, f)}{t} dt + n(0, f) \log r \\ &\geq \sum_{j=1}^{2n-1} j \int_{jc}^{(j+1)c} \frac{1}{t} dt + n(0, f) \log r \\ &\geq \log \frac{(2n)^{2n-1}}{(2n-1)!} + n(0, f) \log r \\ &\geq n \log 2 + n(0, f) \log r + O(1) \\ &\geq \frac{r + |c|}{2|c|} \log 2 + n(0, f) \log r + O(1) \\ &\geq \frac{r}{4|c|}. \end{aligned}$$

This completes the proof of Lemma 2.1. \square

Lemma 2.2 ([4, 5]). *Let $f(z)$ be a meromorphic function of finite order, and let c be a nonzero complex constant. Then*

$$T(r, f(z+c)) = T(r, f) + S(r, f).$$

Lemma 2.3 ([4, 5]). *Let $c \in \mathbb{C}$, let k be a positive integer, and let $f(z)$ be a meromorphic function of finite order. Then*

$$m\left(r, \frac{\Delta_c^k f(z)}{f(z)}\right) = S(r, f).$$

Lemma 2.4 (see [15] Theorem 1.51). *Suppose that $f_i(z)$ ($i = 1, 2, \dots, n$) and $g_i(z)$ ($i = 1, 2, \dots, n$) ($n \geq 2$) are entire functions satisfying*

- (i) $\sum_{i=1}^n f_i(z)e^{g_i(z)} \equiv 0$.
- (ii) $g_j(z) - g_k(z)$ are not constants for $1 \leq j < k < n$.
- (iii) For $1 \leq i \leq n$, $1 \leq k < l \leq n$,

$$T(r, f_i) = o\{T(r, e^{g_k - g_l})\} \quad (r \rightarrow \infty, r \notin E).$$

Then $f_i(z) \equiv 0$ ($i = 1, 2, \dots, n$).

Lemma 2.5 ([14]). *Suppose that $f(z)$ is a meromorphic function in the complex plane, and that $F(z) = a_n(z)f^n(z) + a_{n-1}(z)f^{n-1}(z) + \dots + a_0(z)$, where $a_0(z), a_1(z), \dots, a_n(\neq 0)$ are small functions of $f(z)$. Then*

$$(2) \quad T(r, F) = nT(r, f) + S(r, f).$$

3. Proof of Theorem 1.6

Proof. Since $f(z)$ and $\Delta_c f(z)$ share $a(z), b(z)$ and ∞ CM, and f is a transcendental meromorphic function with finite order, combing that with Lemma 2.3, we have

$$(3) \quad \frac{\Delta_c f(z) - a(z)}{f(z) - a(z)} = e^{\alpha(z)}, \quad \frac{\Delta_c f(z) - b(z)}{f(z) - b(z)} = e^{\beta(z)},$$

where $\alpha(z)$ and $\beta(z)$ are two polynomials such that

$$(4) \quad \max\{\deg \alpha(z), \deg \beta(z)\} \leq \rho(f).$$

It follows from (3) that

$$(5) \quad (e^{\alpha(z)} - e^{\beta(z)})f(z) = a(z)e^{\alpha(z)} - b(z)e^{\beta(z)} - [a(z) - b(z)].$$

If $e^{\alpha(z)} \equiv e^{\beta(z)}$, then from (5) we obtain

$$(a(z) - b(z))(e^{\alpha(z)} - 1) = 0.$$

Since $a(z) \not\equiv b(z)$, we get $e^{\alpha(z)} \equiv 1$, hence $f(z) \equiv \Delta_c f(z)$.

Next, we consider the case of $e^{\alpha(z)} \not\equiv e^{\beta(z)}$.

It follows from (5) and the first equation in (3) that

$$(6) \quad f(z) = \frac{a(z)e^{\alpha(z)} - b(z)e^{\beta(z)} - [a(z) - b(z)]}{e^{\alpha(z)} - e^{\beta(z)}},$$

$$(7) \quad \Delta_c f(z) = \frac{e^{\alpha(z)} \{a(z)e^{\alpha(z)} - b(z)e^{\beta(z)} - [a(z) - b(z)]\}}{e^{\alpha(z)} - e^{\beta(z)}} - a(z)[e^{\alpha(z)} - 1].$$

It follows from (6) that

$$(8) \quad \begin{aligned} \Delta_c f(z) &= f(z+c) - f(z) \\ &= \frac{a(z+c)e^{\alpha(z+c)} - b(z+c)e^{\beta(z+c)} - [a(z+c) - b(z+c)]}{e^{\alpha(z+c)} - e^{\beta(z+c)}} \\ &\quad - \frac{a(z)e^{\alpha(z)} - b(z)e^{\beta(z)} - [a(z) - b(z)]}{e^{\alpha(z)} - e^{\beta(z)}}. \end{aligned}$$

By (7) and (8), we obtain

$$(9) \quad \begin{aligned} &[a(z) - b(z)]e^{\alpha(z)+\alpha(z+c)+\beta(z)} + [a(z) - a(z+c) + b(z)]e^{\alpha(z)+\alpha(z+c)} \\ &- [a(z) - b(z)]e^{\alpha(z)+\beta(z)+\beta(z+c)} - [a(z) + b(z) - b(z+c)]e^{\alpha(z)+\beta(z+c)} \\ &- [a(z) - a(z+c) + b(z)]e^{\alpha(z+c)+\beta(z)} \\ &+ [a(z) + b(z) - b(z+c)]e^{\beta(z)+\beta(z+c)} \\ &+ [a(z+c) - b(z+c)]e^{\alpha(z)} - [a(z+c) - b(z+c)]e^{\beta(z)} \\ &- [a(z) - b(z)]e^{\alpha(z+c)} + [a(z) - b(z)]e^{\beta(z+c)} \equiv 0. \end{aligned}$$

It follows from (6) and $T(r, a) + T(r, b) = S(r, f)$ that

$$\begin{aligned} T(r, f(z)) &= T(r, \frac{a(z)e^{\alpha(z)} - b(z)e^{\beta(z)} - (a(z) - b(z))}{e^{\alpha(z)} - e^{\beta(z)}}) \\ &\leq 2(T(r, e^\alpha) + T(r, e^\beta)) + S(r, f). \end{aligned}$$

Hence, we have

$$(10) \quad T(r, a) + T(r, b) = o(T(r, e^\alpha) + T(r, e^\beta)).$$

By (10) and Lemma 2.2, we also obtain

$$(11) \quad T(r, a(z+c)) + T(r, b(z+c)) = o(T(r, e^\alpha) + T(r, e^\beta)).$$

Next, we consider three cases.

Case 1. $\deg \alpha(z) > \deg \beta(z)$. Then (9) can be rewritten as

$$(12) \quad H_2(z)e^{2\alpha(z)} + H_1(z)e^{\alpha(z)} + H_0(z) \equiv 0,$$

where

$$(13) \quad \begin{aligned} H_2(z) &= [a(z) - b(z)]e^{\Delta_c \alpha(z)+\beta(z)} + [a(z) - a(z+c) + b(z)]e^{\Delta_c \alpha(z)}, \\ H_1(z) &= -[a(z) - b(z)]e^{\beta(z)+\beta(z+c)} - [a(z) + b(z) - b(z+c)]e^{\beta(z+c)} \\ &\quad + [a(z+c) - b(z+c)] - [a(z) - a(z+c) + b(z)]e^{\Delta_c \alpha(z)+\beta(z)} \\ &\quad - [a(z) - b(z)]e^{\Delta_c \alpha(z)}, \\ H_0(z) &= [a(z) + b(z) - b(z+c)]e^{\beta(z)+\beta(z+c)} - [a(z+c) - b(z+c)]e^{\beta(z)} \\ &\quad + [a(z) - b(z)]e^{\beta(z+c)}. \end{aligned}$$

It follows from (10), (11) and $\deg \alpha(z) > \deg \beta(z)$ that $T(H_i(z)) = S(r, e^{\alpha(z)})$ ($i = 0, 1, 2$). If $H_2(z) \not\equiv 0$, then by Lemma 2.5, we have

$$\begin{aligned} 2T(r, e^\alpha) + S(r, e^\alpha) &= T(r, H_2 e^{2\alpha}) + S(r, e^\alpha) \\ &= T(r, -H_1 e^\alpha - H_0) + S(r, e^\alpha) \\ &\leq T(r, e^\alpha) + S(r, e^\alpha). \end{aligned}$$

Hence, we get $T(r, e^\alpha) = S(r, e^\alpha)$. This implies e^α is a constant, but it contradicts $\deg \alpha(z) > \deg \beta(z) \geq 0$.

By the same way, we deduce that $H_0(z) \equiv H_1(z) \equiv H_2(z) \equiv 0$.

Hence, by (13) and $H_2(z) \equiv H_0(z) \equiv 0$, we obtain

$$(14) \quad [a(z) - b(z)]e^{\beta(z)} + [a(z) - a(z+c) + b(z)] \equiv 0,$$

$$(15) \quad [a(z) + b(z) - b(z+c)]e^{\beta(z+c)} - [a(z+c) - b(z+c)] \\ + [a(z) - b(z)]e^{\beta(z+c)-\beta(z)} \equiv 0.$$

It follows from $H_1(z) \equiv 0$ and (14)-(15) that

$$(16) \quad e^{\beta(z)+\beta(z+c)} - e^{\beta(z+c)-\beta(z)} - [e^{2\beta(z)} - 1]e^{\Delta_c \alpha(z)} \equiv 0.$$

Next, we consider two subcases.

Case 1.1. $\deg \Delta_c \alpha(z) > \deg \beta(z)$. Then it is easy to deduce that $e^{2\beta(z)} = 1$, which implies that $\beta(z)$ is a constant satisfying $e^\beta = 1$, or $e^\beta = -1$.

If $e^\beta = 1$, then by the second equation of (3), we obtain $f(z) \equiv \Delta_c f(z)$.

If $e^\beta = -1$, then it follows from (14)-(15) that

$$2b(z) - a(z+c) \equiv 0,$$

$$2b(z) - b(z+c) \equiv 0.$$

Hence, we get $a(z) \equiv b(z)$, a contradiction.

Case 1.2. $\deg \Delta_c \alpha(z) = \deg \beta(z) \geq 1$.

Case 1.2.1. $\deg[2\beta(z) - \Delta_c \alpha(z)] = \deg \beta(z)$ and $\deg[2\beta(z) + \Delta_c \alpha(z)] = \deg \beta(z)$. Then (16) can be rewritten as follows.

$$(17) \quad \sum_{i=1}^4 L_i(z) e^{g_i(z)} \equiv 0,$$

where

$$\begin{aligned} L_1(z) &= e^{\Delta_c \beta(z)}, & g_1(z) &= 2\beta(z), \\ L_2(z) &= -1, & g_2(z) &= \Delta_c \beta(z), \\ L_3(z) &= -1, & g_3(z) &= 2\beta(z) + \Delta_c \alpha(z), \\ L_4(z) &= 1, & g_4(z) &= \Delta_c \alpha(z). \end{aligned}$$

Obviously, for any $1 \leq i < j \leq 4$, $n = 1, 2, 3, 4$, we have

$$T(r, L_n) = o\{T(r, e^{g_i - g_j})\}.$$

Hence, it follows from (17) and Lemma 2.4 that $L_1(z) \equiv L_2(z) \equiv L_3(z) \equiv L_4(z) \equiv 0$. But $L_1(z) = e^{\Delta_c \beta(z)} (\neq 0)$, $L_2(z) = -1 (\neq 0)$, $L_3(z) = -1 (\neq 0)$, $L_4(z) = 1 (\neq 0)$, we get a contradiction.

Case 1.2.2. $\deg[2\beta(z) - \Delta_c \alpha(z)] < \deg \beta(z)$. Let $2\beta(z) - \Delta_c \alpha(z) = -p_1(z)$, then $\Delta_c \alpha(z) = 2\beta(z) + p_1(z)$. So (16) can be rewritten as follows.

$$(18) \quad \sum_{i=1}^3 M_i(z) e^{g_i(z)} \equiv 0,$$

where

$$\begin{aligned} M_1(z) &= e^{\Delta_c \beta(z)} + e^{p_1(z)}, & g_1(z) &= 2\beta(z), \\ M_2(z) &= -1, & g_2(z) &= \Delta_c \beta(z), \\ M_3(z) &= -e^{p_1(z)}, & g_3(z) &= 4\beta(z). \end{aligned}$$

Obviously, for any $1 \leq i < j \leq 3$, $n = 1, 2, 3$, we have

$$T(r, M_n) = o\{T(r, e^{g_i - g_j})\}.$$

Hence, it follows from (18) and Lemma 2.4 that $M_1(z) \equiv M_2(z) \equiv M_3(z) \equiv 0$. But $M_2(z) = -1 (\neq 0)$, $M_3(z) = -e^{p_1(z)} (\neq 0)$, we get a contradiction.

Case 1.2.3. $\deg[2\beta(z) + \Delta_c \alpha(z)] < \deg \beta(z)$. Let $2\beta(z) + \Delta_c \alpha(z) = p_2(z)$, then $\Delta_c \alpha(z) = -2\beta(z) + p_2(z)$. So (16) can be rewritten as follows.

$$(19) \quad \sum_{i=1}^3 M_i(z) e^{g_i(z)} \equiv 0,$$

where

$$\begin{aligned} M_1(z) &= e^{\Delta_c \beta(z)}, & g_1(z) &= 2\beta(z), \\ M_2(z) &= -e^{\Delta_c \beta(z)} - e^{p_2(z)}, & g_2(z) &= 0, \\ M_3(z) &= e^{p_2(z)}, & g_3(z) &= -2\beta(z). \end{aligned}$$

Obviously, for any $1 \leq i < j \leq 3$, $n = 1, 2, 3$, we have

$$T(r, M_n) = o\{T(r, e^{g_i - g_j})\}.$$

Hence, it follows from (19) and Lemma 2.4 that $M_1(z) \equiv M_2(z) \equiv M_3(z) \equiv 0$. But $M_1(z) = e^{\Delta_c \beta(z)} (\neq 0)$, we get a contradiction.

Case 1.3. $\deg \Delta_c \alpha(z) = \deg \beta(z) = 0$. Then $\beta(z)$ and $\Delta_c \alpha(z)$ are two constants, and $\alpha(z) = Az + B$, ($A \neq 0$). So, by (14)-(15), we obtain

$$[2b(z) - b(z+c)](e^\beta - 1) \equiv 0.$$

This implies that $e^\beta = 1$, or $2b(z) - b(z+c) \equiv 0$.

If $e^\beta = 1$, then it follows from the second equation of (3) that $f(z) \equiv \Delta_c f(z)$.

Next, we consider the case $2b(z) - b(z+c) \equiv 0$. In this case, we divide it into two subcases.

Case 1.3.1. $b(z) \not\equiv 0$. Then by Lemma 2.1, there exists $D_1 > 0$ such that $T(r, b(z)) \geq D_1 r$. On the other hand, by $\alpha(z) = Az + B$, there exists

$D_2 > 0$ such that $T(r, e^{\alpha(z)}) = T(r, e^{Az+B}) \leq D_2 r$. Hence, we have $T(r, b(z)) \geq \frac{D_1}{D_2} T(r, e^{\alpha(z)})$, but it contradicts with (10).

Case 1.3.2. $b(z) \equiv 0$. Then it follows from (14) that $(e^\beta + 1)a(z) - a(z+c) \equiv 0$, obviously, $e^\beta + 1 \neq 1$. Since $a(z) \not\equiv b(z)$, we have $a(z) \not\equiv 0$. Using the same argument as case 1.3.1, we get a contradiction.

Case 2. $\deg \alpha(z) < \deg \beta(z)$. Then (9) can be rewritten as

$$(20) \quad K_2(z)e^{2\beta(z)} + K_1(z)e^{\beta(z)} + K_0(z) \equiv 0,$$

where

$$(21) \quad \begin{aligned} K_2(z) &= -[a(z) - b(z)]e^{\alpha(z) + \Delta_c \beta(z)} + [a(z) + b(z) - b(z+c)]e^{\Delta_c \beta(z)}, \\ K_1(z) &= [a(z) - b(z)]e^{\alpha(z) + \alpha(z+c)} - [a(z) + b(z) - b(z+c)]e^{\alpha(z) + \Delta_c \beta(z)} \\ &\quad - [a(z) - a(z+c) + b(z)]e^{\alpha(z+c)} - [a(z+c) - b(z+c)] \\ &\quad + [a(z) - b(z)]e^{\Delta_c \beta(z)}, \\ K_0(z) &= [a(z) - a(z+c) + b(z)]e^{\alpha(z) + \alpha(z+c)} + [a(z+c) - b(z+c)]e^{\alpha(z)} \\ &\quad - [a(z) - b(z)]e^{\alpha(z+c)}. \end{aligned}$$

It follows from (10)-(11) and $\deg \alpha(z) < \deg \beta(z)$ that $T(r, K_i(z)) = S(r, e^{\beta(z)})$ ($i = 0, 1, 2$). Using the same argument as case 1, we obtain $K_0(z) \equiv K_1(z) \equiv K_2(z) \equiv 0$.

Hence, by (21) and $K_2(z) \equiv K_0(z) \equiv 0$, we obtain

$$(22) \quad [a(z) - b(z)]e^{\alpha(z)} - [a(z) + b(z) - b(z+c)] \equiv 0,$$

$$(23) \quad [a(z) - a(z+c) + b(z)]e^{\alpha(z+c)} + [a(z+c) - b(z+c)] \\ - [a(z) - b(z)]e^{\alpha(z+c) - \alpha(z)} \equiv 0.$$

It follows from $K_1(z) \equiv 0$ and (22)-(23) that

$$(24) \quad e^{\alpha(z) + \alpha(z+c)} - e^{\alpha(z+c) - \alpha(z)} - [e^{2\alpha(z)} - 1]e^{\Delta_c \beta(z)} \equiv 0.$$

Next, we consider two subcases.

Case 2.1. $\deg \Delta_c \beta(z) > \deg \alpha(z)$. Then it is easy to deduce that $e^{2\alpha(z)} = 1$, which implies that $\alpha(z)$ is a constant satisfying $e^\alpha = 1$, or $e^\alpha = -1$.

If $e^\alpha = 1$, then by the first equation of (3), we obtain $f(z) \equiv \Delta_c f(z)$.

If $e^\alpha = -1$, then it follows from (22)-(23) that

$$2a(z) - b(z+c) \equiv 0,$$

$$2a(z) - a(z+c) \equiv 0.$$

Hence, we get $a(z) \equiv b(z)$, a contradiction.

Case 2.2. $\deg \Delta_c \beta(z) = \deg \alpha(z) \geq 1$.

Case 2.2.1. $\deg[2\alpha(z) - \Delta_c\beta(z)] = \deg \alpha(z)$ and $\deg[2\alpha(z) + \Delta_c\beta(z)] = \deg \alpha(z)$. Then (24) can be rewritten as follows.

$$(25) \quad \sum_{i=1}^4 J_i(z)e^{g_i(z)} \equiv 0,$$

where

$$\begin{aligned} J_1(z) &= e^{\Delta_c\alpha(z)}, & g_1(z) &= 2\alpha(z), \\ J_2(z) &= -1, & g_2(z) &= \Delta_c\alpha(z), \\ J_3(z) &= -1, & g_3(z) &= 2\alpha(z) + \Delta_c\beta(z), \\ J_4(z) &= 1, & g_4(z) &= \Delta_c\beta(z). \end{aligned}$$

Obviously, for any $1 \leq i < j \leq 4$, $n = 1, 2, 3, 4$, we have

$$T(r, J_n) = o\{T(r, e^{g_i - g_j})\}.$$

Hence, it follows from (25) and Lemma 2.4 that $J_1(z) \equiv J_2(z) \equiv J_3(z) \equiv J_4(z) \equiv 0$. But $J_1(z) = e^{\Delta_c\alpha(z)}$, $J_2(z) = -1$, $J_3(z) = -1$, $J_4(z) = 1$, we get a contradiction.

Case 2.2.2. $\deg[2\alpha(z) - \Delta_c\beta(z)] < \deg \alpha(z)$. Let $2\alpha(z) - \Delta_c\beta(z) = -p_3(z)$, then $\Delta_c\beta(z) = 2\alpha(z) + p_3(z)$. So (24) can be rewritten as follows.

$$(26) \quad \sum_{i=1}^3 N_i(z)e^{g_i(z)} \equiv 0,$$

where

$$\begin{aligned} N_1(z) &= e^{\Delta_c\alpha(z)} + e^{p_3(z)}, & g_1(z) &= 2\alpha(z), \\ N_2(z) &= -1, & g_2(z) &= \Delta_c\alpha(z), \\ N_3(z) &= -e^{p_3(z)}, & g_3(z) &= 4\alpha(z). \end{aligned}$$

Obviously, for any $1 \leq i < j \leq 3$, $n = 1, 2, 3$, we have

$$T(r, N_n) = o\{T(r, e^{g_i - g_j})\}.$$

Hence, it follows from (26) and Lemma 2.4 that $N_1(z) \equiv N_2(z) \equiv N_3(z) \equiv 0$. But $N_1(z) = e^{\Delta_c\alpha(z)} + e^{p_3(z)}$, $N_2(z) = -1$, $N_3(z) = -e^{p_3(z)}$, we get a contradiction.

Case 2.2.3. $\deg[2\alpha(z) + \Delta_c\beta(z)] < \deg \alpha(z)$. Let $2\alpha(z) + \Delta_c\beta(z) = p_4(z)$, then $\Delta_c\beta(z) = -2\alpha(z) + p_4(z)$. So (24) can be rewritten as follows.

$$(27) \quad \sum_{i=1}^3 N_i(z)e^{g_i(z)} \equiv 0,$$

where

$$\begin{aligned} N_1(z) &= e^{\Delta_c \alpha(z)}, & g_1(z) &= 2\alpha(z), \\ N_2(z) &= -e^{\Delta_c \alpha(z)} - e^{p_4(z)}, & g_2(z) &= 0, \\ N_3(z) &= e^{p_4(z)}, & g_3(z) &= -2\alpha(z). \end{aligned}$$

Obviously, for any $1 \leq i < j \leq 3$, $n = 1, 2, 3$, we have

$$T(r, N_n) = o\{T(r, e^{g_i - g_j})\}.$$

Hence, it follows from (27) and Lemma 2.4 that $N_1(z) \equiv N_2(z) \equiv N_3(z) \equiv 0$. But $N_1(z) = e^{\Delta_c \alpha(z)}$, we get a contradiction.

Case 2.3. $\deg \Delta_c \beta(z) = \deg \alpha(z) = 0$. Then $\alpha(z)$ and $\Delta_c \beta(z)$ are two constants, and $\beta(z) = A_2 z + B_2$, ($A_2 \neq 0$). So, by (22)-(23), we obtain

$$[2a(z) - a(z+c)](e^\alpha - 1) \equiv 0.$$

This implies that $e^\alpha = 1$, or $2a(z) - a(z+c) \equiv 0$.

If $e^\alpha = 1$, then it follows from the first equation of (3) that $f(z) \equiv \Delta_c f(z)$.

Next, we consider the case $2a(z) - a(z+c) \equiv 0$. In this case, we divide it into two subcases.

Case 2.3.1. $a(z) \not\equiv 0$. Then by Lemma 2.1, there exists $D_1 > 0$ such that $T(r, a(z)) \geq D_1 r$. On the other hand, by $\beta(z) = Az + B$, there exists $D_2 > 0$ such that $T(r, e^{\beta(z)}) = T(r, e^{Az+B}) \leq D_2 r$. Hence, we have $T(r, a(z)) \geq \frac{D_1}{D_2} T(r, e^{\beta(z)})$, but it contradicts with (10).

Case 2.3.2. $a(z) \equiv 0$. Then it follows from (22) that $(e^\alpha + 1)b(z) - b(z+c) \equiv 0$, obviously, $e^\alpha + 1 \neq 1$. Since $a(z) \not\equiv b(z)$, we have $b(z) \not\equiv 0$. Using the same argument as case 2.3.1, we get a contradiction.

Case 3. $\deg \alpha(z) = \deg \beta(z)$. Then (9) can be rewritten as follows.

$$(28) \quad \sum_{i=1}^7 W_i(z) e^{g_i(z)} \equiv 0,$$

where

$$\begin{aligned} W_1(z) &= [a(z) - b(z)]e^{\Delta_c \alpha(z)}, & g_1(z) &= 2\alpha(z) + \beta(z), \\ W_2(z) &= [a(z) - a(z+c) + b(z)]e^{\Delta_c \alpha(z)}, & g_2(z) &= 2\alpha(z), \\ W_3(z) &= -[a(z) - b(z)]e^{\Delta_c \beta(z)}, & g_3(z) &= \alpha(z) + 2\beta(z), \\ W_4(z) &= [a(z) + b(z) - b(z+c)]e^{\Delta_c \beta(z)}, & g_4(z) &= 2\beta(z), \\ W_5(z) &= -[a(z) - a(z+c) + b(z)]e^{\Delta_c \alpha(z)} \\ &\quad - [a(z) + b(z) - b(z+c)]e^{\Delta_c \beta(z)}, & g_5(z) &= \alpha(z) + \beta(z), \\ W_6(z) &= [a(z+c) - b(z+c)] - [a(z) - b(z)]e^{\Delta_c \alpha(z)}, & g_6(z) &= \alpha(z), \\ W_7(z) &= [a(z) - b(z)]e^{\Delta_c \beta(z)} - [a(z+c) - b(z+c)], & g_7(z) &= \beta(z). \end{aligned}$$

If $\deg[\alpha(z) - \beta(z)]$, $\deg[\alpha(z) + \beta(z)]$, $\deg[2\alpha(z) - \beta(z)]$, $\deg[2\beta(z) - \alpha(z)]$ are all equal to $\deg \alpha(z)$. Then, for any $1 \leq i < j \leq 7$, $n = 1, 2, \dots, 7$, we have $T(r, W_n) = o\{T(r, e^{g_i - g_j})\}$. Hence, it follows from (28) and Lemma 2.4 that $W_i(z) \equiv 0$ ($i = 1, 2, \dots, 7$). By $W_1(z) = (a(z) - b(z))e^{\Delta_c \alpha(z)} \equiv 0$, we get $a(z) \equiv b(z)$, a contradiction.

Next, we only need to discuss the cases that some of $\deg[\alpha(z) - \beta(z)]$, $\deg[\alpha(z) + \beta(z)]$, $\deg[2\alpha(z) - \beta(z)]$, $\deg[2\beta(z) - \alpha(z)]$ are less than $\deg \alpha(z)$.

Case 3.1. $\deg[\alpha(z) - \beta(z)] < \deg \alpha(z)$. Let $\alpha(z) - \beta(z) = -p_4(z)$, then $\beta(z) = \alpha(z) + p_4(z)$ and (28) can be rewritten as follows.

$$(29) \quad F_3(z)e^{2\alpha(z)} + F_2(z)e^{\alpha(z)} + F_1(z) \equiv 0,$$

where

$$\begin{aligned} F_3(z) &= [a(z) - b(z)]e^{\Delta_c \alpha(z) + p_4(z)} - [a(z) - b(z)]e^{\Delta_c \beta(z) + 2p_4(z)}, \\ F_2(z) &= [a(z) - a(z+c) + b(z)]e^{\Delta_c \alpha(z)} + [a(z) + b(z) - b(z+c)]e^{\Delta_c \beta(z) + 2p_4(z)} \\ &\quad - \left\{ [a(z) - a(z+c) + b(z)]e^{\Delta_c \alpha(z)} \right. \\ &\quad \left. + [a(z) + b(z) - b(z+c)]e^{\Delta_c \beta(z)} \right\} e^{p_4(z)}, \\ F_1(z) &= a(z+c) - b(z+c) - [a(z) - b(z)]e^{\Delta_c \alpha(z)} \\ &\quad - \left\{ [a(z+c) - b(z+c)] - [a(z) - b(z)]e^{\Delta_c \beta(z)} \right\} e^{p_4(z)}. \end{aligned}$$

Obviously, for any $1 \leq i < j \leq 3$, $n = 1, 2, 3$, we have

$$T(r, F_n) = o\{T(r, e^\alpha)\}.$$

Hence, it follows from (29) and Lemma 2.4 that $F_i(z) \equiv 0$ ($i = 1, 2, 3$).

By $F_3 \equiv 0$, we get

$$(30) \quad e^{\Delta_c \alpha(z)} - e^{\Delta_c \beta(z) + p_4(z)} \equiv 0.$$

It follows from (30) and $F_1(z) \equiv 0$ that

$$[a(z+c) - b(z+c)][1 - e^{p_4(z)}] \equiv 0.$$

Combing this with $a(z) \not\equiv b(z)$, we obtain that $e^{p_4(z)} = 1$, this implies that $e^{\alpha(z)} \equiv e^{\beta(z)}$, which contradicts with our assumption.

Case 3.2. $\deg[\alpha(z) + \beta(z)] < \deg \alpha(z)$. Let $\alpha(z) + \beta(z) = p_5(z)$, then $\beta(z) = -\alpha(z) + p_5(z)$ and (28) can be rewritten as follows.

$$(31) \quad G_2(z)e^{2\alpha(z)} + G_1(z)e^{\alpha(z)} + G_0(z) + G_{-1}e^{-\alpha(z)} + G_{-2}e^{-2\alpha(z)} \equiv 0,$$

where

$$\begin{aligned} G_2(z) &= [a(z) - a(z+c) + b(z)]e^{\Delta_c \alpha(z)}, \\ G_1(z) &= [a(z) - b(z)]e^{\Delta_c \alpha(z) + p_5(z)} + [a(z+c) - b(z+c)] \\ &\quad - [a(z) - b(z)]e^{\Delta_c \alpha(z)}, \end{aligned}$$

$$\begin{aligned}
G_0(z) &= - \left\{ [a(z) - a(z+c) + b(z)]e^{\Delta_c \alpha(z)} \right. \\
&\quad \left. + [a(z) + b(z) - b(z+c)]e^{\Delta_c \beta(z)} \right\} e^{p_5(z)}, \\
G_{-1}(z) &= - [a(z) - b(z)]e^{\Delta_c \beta(z) + 2p_5(z)} + \{ [a(z) - b(z)]e^{\Delta_c \beta(z)} \\
&\quad - [a(z+c) - b(z+c)] \} e^{p_5(z)}, \\
G_{-2}(z) &= [a(z) + b(z) - b(z+c)]e^{\Delta_c \beta(z) + 2p_5(z)}.
\end{aligned}$$

Obviously, for any $-2 \leq i < j \leq 2$, $n = -2, -1, \dots, 2$, we have

$$T(r, G_n) = o\{T(r, e^\alpha)\}.$$

Hence, it follows from (31) and Lemma 2.4 that $G_i(z) \equiv 0$ ($i = -2, -1, \dots, 2$). By $G_2 \equiv 0$ and $G_{-2} \equiv 0$, we get $a(z) - a(z+c) + b(z) \equiv 0$ and $a(z) + b(z) - b(z+c) \equiv 0$, which implies $a(z) \equiv b(z)$, a contradiction.

Case 3.3. $\deg[2\alpha(z) - \beta(z)] < \deg \alpha(z)$. Let $2\alpha(z) - \beta(z) = -p_6(z)$, then $\beta(z) = 2\alpha(z) + p_6(z)$ and (28) can be rewritten as follows.

$$(32) \quad \sum_{i=1}^5 D_i(z) e^{i\alpha(z)} \equiv 0,$$

where

$$\begin{aligned}
D_1(z) &= [a(z+c) - b(z+c)] - [a(z) - b(z)]e^{\Delta_c \alpha(z)}, \\
D_2(z) &= [a(z) - a(z+c) + b(z)]e^{\Delta_c \alpha(z)} \\
&\quad + \left\{ [a(z) - b(z)]e^{\Delta_c \beta(z)} - [a(z+c) - b(z+c)] \right\} e^{p_6(z)}, \\
D_3(z) &= - \left\{ [a(z) - a(z+c) + b(z)]e^{\Delta_c \alpha(z)} \right. \\
&\quad \left. + [a(z) + b(z) - b(z+c)]e^{\Delta_c \beta(z)} \right\} e^{p_6(z)}, \\
D_4(z) &= [a(z) - b(z)]e^{\Delta_c \alpha(z) + p_6(z)} + [a(z) + b(z) - b(z+c)]e^{\Delta_c \beta(z) + 2p_6(z)}, \\
D_5(z) &= - [a(z) - b(z)]e^{\Delta_c \beta(z) + 2p_6(z)}.
\end{aligned}$$

Obviously, for any $1 \leq i < j \leq 5$, $n = 1, 2, \dots, 5$, we have

$$T(r, D_n) = o\{T(r, e^\alpha)\}.$$

Hence, it follows from (32) and Lemma 2.4 that $D_i(z) \equiv 0$ ($i = 1, 2, \dots, 5$). By $D_5 \equiv 0$, we get $a(z) \equiv b(z)$, a contradiction.

Case 3.4. $\deg[2\beta(z) - \alpha(z)] < \deg \alpha(z)$. Let $2\beta(z) - \alpha(z) = -p_7(z)$, then $\alpha(z) = 2\beta(z) + p_7(z)$ and (28) can be rewritten as follows.

$$(33) \quad \sum_{i=1}^5 X_i(z) e^{i\beta(z)} \equiv 0,$$

where

$$X_1(z) = [a(z) - b(z)]e^{\Delta_c \beta(z)} - [a(z+c) - b(z+c)],$$

$$\begin{aligned}
X_2(z) &= [a(z) + b(z) - b(z+c)]e^{\Delta_c \beta(z)} \\
&\quad + \left\{ [a(z+c) - b(z+c)] - [a(z) - b(z)]e^{\Delta_c \alpha(z)} \right\} e^{p_7(z)}, \\
X_3(z) &= - \left\{ [a(z) - a(z+c) + b(z)]e^{\Delta_c \alpha(z)} \right. \\
&\quad \left. + [a(z) + b(z) - b(z+c)]e^{\Delta_c \beta(z)} \right\} e^{p_7(z)}, \\
X_4(z) &= [a(z) - a(z+c) + b(z)]e^{\Delta_c \alpha(z)+2p_7(z)} - [a(z) - b(z)]e^{\Delta_c \beta(z)+p_7(z)}, \\
X_5(z) &= [a(z) - b(z)]e^{\Delta_c \alpha(z)+2p_7(z)}.
\end{aligned}$$

Obviously, for any $1 \leq i < j \leq 5$, $n = 1, 2, \dots, 5$, we have

$$T(r, X_n) = o \{T(r, e^\beta)\}.$$

Hence, it follows from (33) and Lemma 2.4 that $X_i(z) \equiv 0$ ($i = 1, 2, \dots, 5$). By $X_5 \equiv 0$, we get $a(z) \equiv b(z)$, a contradiction.

Thus, Theorem 1.6 is proved. \square

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