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MILD SOLUTIONS FOR THE RELATIVISTIC VLASOV-KLEIN-GORDON SYSTEM

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Abstract. In this paper, the relativistic Vlasov-Klein-Gordon system in one dimension is investigated. This non-linear dynamics system consists of a transport equation for the distribution function combined with Klein-Gordon equation. Without any assumption of continuity or compact support of any initial particle density f_0, we prove the existence and uniqueness of the mild solution via the iteration method.

1. Introduction

The relativistic Vlasov-Klein-Gordon system describes the dynamics of an ensemble of relativistic charged particles coupled to a self-consistent Klein-Gordon field. In the present paper, we consider the one-dimensional relativistic Vlasov-Klein-Gordon system:

$$\partial_t f + \widehat{v} \cdot \partial_x f - \partial_x u \cdot \partial_v f = 0, \tag{1}$$

$$\partial_t^2 u - \partial_x^2 u + u = -\rho(t,x). \tag{2}$$

In this system, the unknown $f = f(t,x,v) \geq 0$ denotes the particle density in phase space depending on time $t \in \mathbb{R}$, position $x \in \mathbb{R}$ and momentum $v \in \mathbb{R}$, $u = u(t,x)$ is a scalar Klein-Gordon field, $\rho(t,x) = \int_{\mathbb{R}} f(t,x,v)dv$ is the density in space and $\widehat{v} = \frac{v}{\sqrt{1+|v|^2}}$ is the relativistic velocity with momentum v. Initial data are given by

$$f(0,x,v) = f_0(x,v), \quad u(0,x) = u_0(x), \quad \partial_t u(0,x) = u_1(x). \tag{3}$$

In [13], the locally classical solution for general initial data to the relativistic Vlasov-Klein-Gordon system in three dimensions was proved. Moveover, a continuation criterion (i.e., a local solution can be extended as long as the particle momenta is under control) was established. Based on the continuation criterion, global existence and uniqueness for initial data $f_0(x,v) \in C_c^1(\mathbb{R}^2)$, $u_0(x) \in C_b^3(\mathbb{R})$, $u_1(x) \in C_b^2(\mathbb{R})$ in one-dimensional version of the system was

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obtained. By the adaptation of the way in [2], Ha and Lee [9] investigated global classical solutions for a damped Vlasov-Klein-Gordon system with small data. The global existence of classical solutions for general initial data is still an open problem. Nevertheless, in order to get global solution, one method is to weaken the solution concept. Applying momentum averaging [6–8] and compactness method, [14, 21] proved the existence of global weak solutions for the relativistic Vlasov-Klein-Gordon system with initial data satisfying a size restriction in three or two dimensions. Furthermore, global existence of weak solutions for general initial data is obtained in [20].

The purpose of this paper is to prove a more general existence and uniqueness result for the Cauchy problem (1)-(3) under less restrictive hypotheses. In this paper, we suppose initial Klein-Gordon field $u_0(x) \in W^{2,\infty}(\mathbb{R})$, $u_1(x) \in W^{1,\infty}(\mathbb{R})$ and initial density f_0 is uniformly bounded with respect to x by some function $h_0(p)$ (which will be specified in Theorem 1.1). Adopting the iteration method, we construct a unique global mild solution (f, u), i.e., the particle density is solution by characteristics and the Klein-Gordon field is Lipschitz continuous. This method is inspired by [4, 5].

We observe that if we replace the Klein-Gordon field with electrostatic field, (1)-(3) will become the classical Vlasov-Poisson system, whose main properties, such as existence, uniqueness, propagation of moments, and asymptotic behavior of compactly supported classical solutions, etc., have already been achieved (see, e.g., [1, 3, 7, 10–12, 15–19] and the references therein).

Throughout the paper, $T > 0$ is an arbitrary positive real number. c and C denote generic positive constants but C depends on T and the initial data, however, c and C may change from line to line. Based on these notations, we are in a position to state the precise definition of mild solutions to the system (1)-(3) and the main results of this paper.

Definition. The pair of $(f(t,x,v), u(t,x))$ is said to be a mild solution on $[0,T)$ to the system (1)-(3), if $u \in C([0,T]; C^1(\mathbb{R}))$, $\partial_t u, \partial_x u \in L^\infty((0,T); W^{1,\infty}(\mathbb{R}))$ and the solution by characteristic of equation (1) with initial datum $f_0 \in L^1(\mathbb{R}^2)$ is given by

$$f(t,x,v) = f_0(X(0,t,x,v), V(0,t,x,v)), \qquad (t,x,v) \in [0,T] \times \mathbb{R}^2.$$

Then, the main result of this paper is represented by:

Theorem 1.1. *Suppose that initial data $u_0(x) \in W^{2,\infty}(\mathbb{R})$, $u_1(x) \in W^{1,\infty}(\mathbb{R})$, $f_0(x,v) \in L^1(\mathbb{R}^2)$ and that there exists some function $h_0 : \mathbb{R} \to [0,\infty)$ nondecreasing on $\mathbb{R}^-$ and nonincreasing on $\mathbb{R}^+$, $h_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $\int_{\mathbb{R}} |v|^2 h_0(v)\, dv < +\infty$ such that $0 \le f_0(x,v) \le h_0(v)$ for any $(x,v) \in \mathbb{R}^2$. Then, for any $T > 0$, there exists a unique mild solution*

$$(f,u) \in L^\infty((0,T); L^1(\mathbb{R}^2)) \times W^{2,\infty}((0,T) \times \mathbb{R})$$

to the system (1)-(3). *Moreover,*

$$\int_{\mathbb{R}} (1+|v|^2) f dv \in L^\infty((0,T)\times\mathbb{R}) \quad \textit{for any} \;\; T>0.$$

To prove Theorem 1.1, we make use of the iteration method. Namely, we first construct the iteration scheme $(f^{(n)}, u^{(n)})$. Then we establish the uniform bounds on $u^{(n)}$ and its first and second order derivatives, by which we can deduce compactness of $u^{(n)}$ in $C^1([0,T]\times\mathbb{R})$. Finally, by the definition of the mild solutions of the system (1)-(3), we prove that there exists a unique mild solution with desired properties described in Theorem 1.1.

2. Preliminaries

Assume that $u \in C^1([0,T]\times\mathbb{R})$ and $\partial_t u, \partial_x u \in L^\infty((0,T); W^{1,\infty}(\mathbb{R}))$. Then for any $(t,x,v) \in [0,T]\times\mathbb{R}^2$, the characteristic equations

$$\left\{ \begin{array}{ll} \frac{dX(s,t,x,v)}{ds} = \widehat{V}(s,t,x,v), & X(t,t,x,v) = x, \\ \frac{dV(s,t,x,v)}{ds} = -\partial_x u(s, X(s,t,x,v)), & V(t,t,x,v) = v \end{array} \right. \tag{4}$$

of the equation (1) has a unique solution $(X(s),V(s)) = (X(s,t,x,v), V(s,t,x,v))$ defined on $[0,T]$ such that $(X(s),V(s)) \in C^1([0,T]\times[0,T]\times\mathbb{R}^2;\mathbb{R}^2)$. Furthermore, for any fixed $s,t \in [0,T]$, there is a measure preserving homomorphism from $\mathbb{R}^2$ onto $\mathbb{R}^2$.

Lemma 2.1. *Assume that* $u^{(k)} \in C^1([0,T]\times\mathbb{R}) \cap L^\infty((0,T)\times\mathbb{R})$, *and* $\partial_t u^{(k)}$, $\partial_x u^{(k)} \in L^\infty((0,T);\, W^{1,\infty}(\mathbb{R}))$ *with* $k \in \{1,2\}$. *For any* $(x,v) \in \mathbb{R}^2$ *denote* $(X^{(k)}(t),\, V^{(k)}(t)) = (X^{(k)}\ (t,0,x,v),\, V^{(k)}(t,0,x,v))$, *then we get for any* $0 \le s \le t < T$

$$\begin{aligned} &|X^{(1)}(s) - X^{(2)}(s)| + |V^{(1)}(s) - V^{(2)}(s)| \\ \le\ & e^{Qs} \int_0^s \|\partial_x u^{(1)}(\tau) - \partial_x u^{(2)}(\tau)\|_{L^\infty(\mathbb{R})} d\tau, \end{aligned}$$

where Q *is a positive constant depending on* $\max\limits_{k=\{1,2\}} \|\partial_x^2 u^{(k)}(s)\|_{L^\infty(\mathbb{R})}$.

Proof. Due to (4), we have

$$\frac{d}{ds}|X^{(1)}(s) - X^{(2)}(s)| \le |\widehat{V}^{(1)}(s) - \widehat{V}^{(2)}(s)| \le |V^{(1)}(s) - V^{(2)}(s)|$$

and

$$\begin{aligned} &\frac{d}{ds}|V^{(1)}(s) - V^{(2)}(s)| \\ \le\ & \Big|\partial_x u^{(1)}(s, X^1(s)) - \partial_x u^{(2)}(s, X^2(s))\Big| \\ \le\ & Q|X^{(1)}(s) - X^{(2)}(s)| + \|\partial_x u^{(1)}(s) - \partial_x u^{(2)}(s)\|_{L^\infty(\mathbb{R})}, \end{aligned}$$

where Q is a positive constant depending on $\max_{k=\{1,2\}} \|\partial_x^2 u^{(k)}(s)\|_{L^\infty(\mathbb{R})}$. Without loss of generality we may assume that $Q \geq 1$. Integrating against s in the above two inequalities, we obtain for $0 \leq s \leq t \leq T$

$$\begin{aligned} &|X^{(1)}(s) - X^{(2)}(s)| + |V^{(1)}(s) - V^{(2)}(s)| \\ \leq\ & Q \int_0^s \Big(|X^{(1)}(\tau) - X^{(2)}(\tau)| + |V^{(1)}(\tau) - V^{(2)}(\tau)|\Big) d\tau \\ &+ \int_0^s \|\partial_x u^{(1)}(\tau) - \partial_x u^{(2)}(\tau)\|_{L^\infty(\mathbb{R})} d\tau. \end{aligned}$$

Hence, by Gronwall's lemma, we deduce the desired inequality. □

3. Proof of Theorem 1.1

In this section, we first define the following iteration scheme. For $n \geq 0$, assume $u^{(n-1)} \in C^1([0,T] \times \mathbb{R})$ and $\partial_t u^{(n-1)}, \partial_x u^{(n-1)} \in L^\infty((0,T); W^{1,\infty}(\mathbb{R}))$, then the equations

$$\begin{cases} \frac{dX^{(n)}(s,t,x,v)}{ds} = \widehat{V}^{(n)}(s,t,x,v), & X^{(n)}(t,t,x,v) = x, \\ \frac{dV^{(n)}(s,t,x,v)}{ds} = -\partial_x u^{(n-1)}(s, X^{(n)}(s,t,x,v)), & V^{(n)}(t,t,x,v) = v \end{cases} \tag{5}$$

have a unique solution and the Cauchy problem

$$\begin{cases} \partial_t f^{(n)} + \widehat{v} \cdot \partial_x f^{(n)} - \partial_x u^{(n-1)} \cdot \partial_v f^{(n)} = 0, \\ f^{(n)}(0,x,v) = f_0(x,v) \end{cases} \tag{6}$$

has a characteristic solution $f^{(n)}(t,x,v)$. Define $\rho^{(n)}(t,x) = \int_{\mathbb{R}} f^{(n)}(t,x,v)dv$ and let $u^{(n)}$ be the solution of

$$\begin{cases} \partial_t^2 u^{(n)} - \partial_x^2 u^{(n)} + u^{(n)} = -\rho^{(n)}(t,x), \\ u^{(n)}(0,x) = u_0(x), \quad \partial_t u^{(n)}(0,x) = u_1(x). \end{cases} \tag{7}$$

A direct calculation shows

$$\begin{aligned} & u^{(n)}(t,x) \\ =\ & \frac{1}{2}(u_0(x+t) + u_0(x-t)) - \frac{t}{2}\int_{x-t}^{x+t} \frac{J_1(\sqrt{t^2 - |x-y|^2})}{\sqrt{t^2 - |x-y|^2}} u_0(y) dy \\ &+ \frac{1}{2}\int_{x-t}^{x+t} u_1(y) J_0(\sqrt{t^2 - |x-y|^2}) dy \\ &- \frac{1}{2}\int_0^t \int_{x-(t-s)}^{x+t-s} \rho^{(n)}(s,y) J_0(\sqrt{(t-s)^2 - |x-y|^2}) dy ds, \end{aligned} \tag{8}$$

where

$$J_\alpha(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+\alpha}}{2^{2m+\alpha} m! (m+\alpha)!}$$

is a Bessel function of the first kind of order α.

Next, we devote to proving Theorem 1.1 and carry out the proof in several steps.

Step 1. Uniform bounds. Due to (8) and the uniform boundedness of $J_0(\cdot)$ and $\frac{J_1(\cdot)}{(\cdot)}$, we get

$$\|u^{(n)}(t)\|_{L^\infty(\mathbb{R})} \leq (1+ct^2)\|u_0\|_{L^\infty(\mathbb{R})} + ct\|u_1\|_{L^\infty(\mathbb{R})} + ct\|f_0\|_{L^1(\mathbb{R})}$$

for any $t \in [0,T]$. Meanwhile, the expression of $u^{(n)}$ in (8) reveals

$$\begin{aligned} &\partial_x u^{(n)}(t,x) \qquad (9)\\ = &-\frac{t}{4}(u_0(x+t)-u_0(x-t)) + \frac{1}{2}(u_0'(x+t)+u_0'(x-t))\\ &+\frac{1}{2}(u_1(x+t)-u_1(x-t))\\ &-\frac{t}{2}\int_{x-t}^{x+t}\frac{J_2(\sqrt{t^2-|x-y|^2})}{t^2-|x-y|^2}(x-y)u_0(y)dy\\ &+\frac{1}{2}\int_{x-t}^{x+t}\frac{J_1(\sqrt{t^2-|x-y|^2})}{\sqrt{t^2-|x-y|^2}}(x-y)u_1(y)dy\\ &-\frac{1}{2}\int_0^t[\rho^{(n)}(s,x+(t-s))-\rho^{(n)}(s,x-(t-s))]ds\\ &-\frac{1}{2}\int_0^t\int_{x-(t-s)}^{x+t-s}\rho^{(n)}(s,y)\frac{J_1(\sqrt{(t-s)^2-|x-y|^2})}{\sqrt{(t-s)^2-|x-y|^2}}(x-y)dyds \end{aligned}$$

and

$$\begin{aligned} &\partial_t u^{(n)}(t,x) \qquad (10)\\ = &-\frac{t}{4}(u_0(x+t)+u_0(x-t)) + \frac{1}{2}(u_0'(x+t)-u_0'(x-t))\\ &+\frac{1}{2}(u_1(x+t)+u_1(x-t))\\ &-\frac{1}{2}\int_{x-t}^{x+t}\frac{J_1(\sqrt{t^2-|x-y|^2})}{\sqrt{t^2-|x-y|^2}}u_0(y)dy\\ &+\frac{t^2}{2}\int_{x-t}^{x+t}\frac{J_2(\sqrt{t^2-|x-y|^2})}{t^2-|x-y|^2}u_0(y)dy\\ &-\frac{t}{2}\int_{x-t}^{x+t}\frac{J_1(\sqrt{t^2-|x-y|^2})}{\sqrt{t^2-|x-y|^2}}u_1(y)dy\\ &-\frac{1}{2}\int_0^t[\rho^{(n)}(s,x+(t-s))+\rho^{(n)}(s,x-(t-s))]ds\\ &+\frac{1}{2}\int_0^t\int_{x-(t-s)}^{x+t-s}\rho^{(n)}(s,y)\frac{J_1(\sqrt{(t-s)^2-|x-y|^2})}{\sqrt{(t-s)^2-|x-y|^2}}(t-s)dyds. \end{aligned}$$

To estimate $\partial_x u^{(n)}$ and $\partial_t u^{(n)}$, the estimate of $\rho^{(n)}$ is needed. In fact, by (5), it comes

$$|V^{(n)}(0,t,x,v)-v| \le \int_0^t \|\partial_x u^{(n-1)}(s)\|_{L^\infty(\mathbb{R})} ds =: R(t).$$

If $v > R(t)$, then $V^{(n)}(0,t,x,v) \ge v - R(t) > 0$. Thus, $h_0(V^{(n)}(0,t,x,v)) \le h_0(v-R(t))$. If $v < -R(t)$, then $V^{(n)}(0,t,x,v) \le v + R(t) < 0$, we have $h_0(V^{(n)}(0,t,x,v)) \le h_0(v+R(t))$. In addition, if $|v| \le R(t)$, then $h_0(V^{(n)}(0,t,x,v)) \le h_0(0)$. Taken together, we have

$$\begin{aligned}
&\rho^{(n)}(t,x)\\
&= \int_{\mathbb{R}} f_0(X^{(n)}(0,t,x,v), V^{(n)}(0,t,x,v))dv\\
&\le \int_{\mathbb{R}} h_0(V^{(n)}(0,t,x,v))dv\\
&= \int_{-\infty}^{-R(t)} h_0(v+R(t))dv + \int_{R(t)}^{\infty} h_0(v-R(t))dv + \int_{-R(t)}^{R(t)} h_0 dv\\
&\le \|h_0\|_{L^1(\mathbb{R})} + 2\|h_0\|_{L^\infty(\mathbb{R})} \int_0^t \|\partial_x u^{(n-1)}(s)\|_{L^\infty(\mathbb{R})} ds.
\end{aligned} \tag{11}$$

Substituting (11) into (9), since $\frac{J_1(\cdot)}{(\cdot)}$ and $\frac{J_2(\cdot)}{(\cdot)^2}$ are uniformly bounded we get for any $0 \le t \le T$

$$\begin{aligned}
&\|\partial_x u^{(n)}(t)\|_{L^\infty(\mathbb{R})}\\
&\le \frac{t}{2}\|u_0\|_{L^\infty(\mathbb{R})} + \|u_0'\|_{L^\infty(\mathbb{R})} + \|u_1\|_{L^\infty(\mathbb{R})}\\
&\quad + \frac{t}{2}\|u_0\|_{L^\infty(\mathbb{R})} \int_{x-t}^{x+t} \Big|\frac{J_2(\sqrt{t^2-|x-y|^2})}{t^2-|x-y|^2}\Big| |x-y| dy\\
&\quad + \frac{1}{2}\|u_1\|_{L^\infty(\mathbb{R})} \int_{x-t}^{x+t} \Big|\frac{J_1(\sqrt{t^2-|x-y|^2})}{\sqrt{t^2-|x-y|^2}}\Big| |x-y| dy\\
&\quad - \frac{1}{2}\int_0^t \int_{x-(t-s)}^{x+t-s} |\rho^{(n)}(s,y)| \Big|\frac{J_1(\sqrt{t^2-|x-y|^2})}{\sqrt{t^2-|x-y|^2}}\Big| |x-y| dyds\\
&\quad + \int_0^t \|\rho^{(n)}(s)\|_{L^\infty(\mathbb{R})} ds\\
&\le (\frac{t}{2} + ct^3)\|u_0\|_{L^\infty(\mathbb{R})} + \|u_0'\|_{L^\infty(\mathbb{R})} + (1+ct^2)\|u_1\|_{L^\infty(\mathbb{R})} + \frac{ct^2}{2}\|f_0\|_{L^1(\mathbb{R})}\\
&\quad + \int_0^t \Big(\|h_0\|_{L^1(\mathbb{R})} + 2\|h_0\|_{L^\infty(\mathbb{R})} \int_0^s \|\partial_x u^{(n-1)}(\tau)\|_{L^\infty(\mathbb{R})} d\tau\Big) ds\\
&\le C + 2\|h_0\|_{L^\infty(\mathbb{R})} t \int_0^t \|\partial_x u^{(n-1)}(s)\|_{L^\infty(\mathbb{R})} ds,
\end{aligned} \tag{12}$$

where $C > 0$ is a constant depending on T, $\|u_0\|_{L^\infty(\mathbb{R})}$, $\|u_0'\|_{L^\infty(\mathbb{R})}$, $\|u_1\|_{L^\infty(\mathbb{R})}$, $\|f_0\|_{L^1(\mathbb{R}^2)}$ and $\|h_0\|_{L^1(\mathbb{R})}$. By Gronwall's lemma, it follows

$$\|\partial_x u^{(n)}(t)\|_{L^\infty(\mathbb{R})} \le C,$$

where $C > 0$ is a constant depending on T, $\|u_0\|_{L^\infty(\mathbb{R})}$, $\|u_0'\|_{L^\infty(\mathbb{R})}$, $\|u_1\|_{L^\infty(\mathbb{R})}$, $\|f_0\|_{L^1(\mathbb{R}^2)}$, $\|h_0\|_{L^1(\mathbb{R})}$ and $\|h_0\|_{L^\infty(\mathbb{R})}$. Similarly, in view of (10), we deduce for any $t \in [0, T]$

$$
\begin{aligned}
(13)\qquad & \|\partial_t u^{(n)}(t)\|_{L^\infty(\mathbb{R})} \\
&\le (ct + \frac{t}{2} + ct^3)\|u_0\|_{L^\infty(\mathbb{R})} + \|u_0'\|_{L^\infty(\mathbb{R})} \\
&\quad + (1 + ct^2)\|u_1\|_{L^\infty(\mathbb{R})} + \frac{ct^2}{2}\|f_0\|_{L^1(\mathbb{R})} \\
&\quad + \int_0^t \Big(\|h_0\|_{L^1(\mathbb{R})} + 2\|h_0\|_{L^\infty(\mathbb{R})} \int_0^s \|\partial_x u^{(n-1)}(\tau)\|_{L^\infty(\mathbb{R})} d\tau\Big) ds \\
&\le C + 2\|h_0\|_{L^\infty(\mathbb{R})} t \int_0^t \|\partial_x u^{(n-1)}(s)\|_{L^\infty(\mathbb{R})} ds \le C,
\end{aligned}
$$

where $C > 0$ is a constant depending on T, $\|u_0\|_{L^\infty(\mathbb{R})}$, $\|u_0'\|_{L^\infty(\mathbb{R})}$, $\|u_1\|_{L^\infty(\mathbb{R})}$, $\|f_0\|_{L^1(\mathbb{R}^2)}$, $\|h_0\|_{L^1(\mathbb{R})}$ and $\|h_0\|_{L^\infty(\mathbb{R})}$. Hence, we obtain $\|\rho^{(n)}(t)\|_{L^\infty(\mathbb{R})} \le C$.

Next, for $\partial_{xx} u^{(n)}$, we have the following expression

$$
\begin{aligned}
(14)\qquad & \partial_{xx} u^{(n)}(t, x) \\
&= \frac{t^2}{16}(u_0(x+t) + u_0(x-t)) - \frac{t}{4}(u_0'(x+t) - u_0'(x-t)) \\
&\quad + \frac{1}{2}(u_0''(x+t) + u_0''(x-t)) \\
&\quad + \frac{1}{2}(u_1'(x+t) - u_1'(x-t)) + \frac{t}{4}(u_1(x+t) + u_1(x-t)) \\
&\quad - \frac{t}{2}\int_{x-t}^{x+t} \frac{J_2(\sqrt{t^2 - |x-y|^2})}{t^2 - |x-y|^2} u_0(y) dy \\
&\quad - \frac{t}{2}\int_{x-t}^{x+t} \frac{J_3(\sqrt{t^2 - |x-y|^2})}{(\sqrt{t^2 - |x-y|^2})^{\frac{3}{2}}} (x-y)^2 u_0(y) dy \\
&\quad + \frac{1}{2}\int_{x-t}^{x+t} \frac{J_2(\sqrt{t^2 - |x-y|^2})}{t^2 - |x-y|^2} (x-y)^2 u_1(y) dy \\
&\quad + \frac{1}{2}\int_{x-t}^{x+t} \frac{J_1(\sqrt{t^2 - |x-y|^2})}{\sqrt{t^2 - |x-y|^2}} u_1(y) dy \\
&\quad - \frac{1}{2}\partial_x\Big(\int_0^t [\rho^{(n)}(s, x + (t-s)) - \rho^{(n)}(s, x - (t-s))] ds\Big) \\
&\quad - \frac{1}{4}\int_0^t \Big[\rho^{(n)}(s, x + (t-s)) - \rho^{(n)}(s, x - (t-s))\Big](t-s) ds
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{2}\int_0^t\int_{x-(t-s)}^{x+t-s}\rho^{(n)}(s,y)\frac{J_1(\sqrt{(t-s)^2-|x-y|^2})}{\sqrt{(t-s)^2-|x-y|^2}}dyds\\
&-\frac{1}{2}\int_0^t\int_{x-(t-s)}^{x+t-s}\rho^{(n)}(s,y)\frac{J_2(\sqrt{(t-s)^2-|x-y|^2})}{(t-s)^2-|x-y|^2}(x-y)^2dyds.
\end{aligned}
$$

Since $\frac{J_3(\cdot)}{(\cdot)^{\frac{3}{2}}}$ is uniformly bounded, we get

$$
\begin{aligned}
(15)\quad &\|\partial_{xx}u^{(n)}(t)\|_{L^\infty(\mathbb{R})}\\
&\le (ct^2+ct^4)\|u_0\|_{L^\infty(\mathbb{R})}+\frac{t}{2}\|u_0'\|_{L^\infty(\mathbb{R})}+\|u_0''\|_{L^\infty(\mathbb{R})}+(ct+ct^3)\|u_1\|_{L^\infty(\mathbb{R})}\\
&\quad+(\frac{ct}{2}+\frac{ct^3}{6})\|f_0\|_{L^1(\mathbb{R})}-\frac{1}{2}\partial_x\Big(\int_0^t\rho^{(n)}(s,x+(t-s))ds\Big)\\
&\quad+\frac{1}{2}\partial_x\Big(\int_0^t\rho^{(n)}(s,x-(t-s))ds\Big)+\frac{t}{2}\int_0^t\|\rho^{(n)}(s)\|_{L^\infty(\mathbb{R})}ds\\
&\le C+\frac{1}{2}\Big|\partial_x\Big(\int_0^t\rho^{(n)}(s,x+(t-s))ds\Big)\Big|\\
&\quad+\frac{1}{2}\Big|\partial_x\Big(\int_0^t\rho^{(n)}(s,x-(t-s))ds\Big)\Big|,
\end{aligned}
$$

where $C>0$ is a constant depending on T, $\|u_0\|_{L^\infty(\mathbb{R})}$, $\|u_0'\|_{L^\infty(\mathbb{R})}$, $\|u_1\|_{L^\infty(\mathbb{R})}$, $\|f_0\|_{L^1(\mathbb{R}^2)}$, $\|h_0\|_{L^1(\mathbb{R})}$ and $\|h_0\|_{L^\infty(\mathbb{R})}$. Taking $D_n^{\pm}(t,x)=\int_0^t\rho^{(n)}(s,x\pm(t-s))ds$, it remains to estimate the L^∞ norm of $\partial_x D_n^{\pm}$. For any test function $\eta(x)\in C_c^1(\mathbb{R})$, we have

$$
\begin{aligned}
&\int_{\mathbb{R}}D_n^{\pm}(t,x)\eta'(x)dx\\
&=\int_0^t\int_{\mathbb{R}^2}f^{(n)}(s,x\pm(t-s),v)\eta'(x)dxdvds\\
&=\int_0^t\int_{\mathbb{R}^2}f^{(n)}(s,x,v)\eta'(x\mp(t-s))dxdvds\\
&=\int_{\mathbb{R}^2}f_0(x,v)\int_0^t\eta'(X^{(n)}(s,0,x,v)\mp(t-s))dsdxdv.
\end{aligned}
$$

For simplicity, we denote $X^{(n)}(s)=X^{(n)}(s,0,x,v), V^{(n)}(s)=V^{(n)}(s,0,x,v)$. For any $t\in[0,T]$ and $\eta\in C_c^1(\mathbb{R})$, we have

$$
\begin{aligned}
&\int_0^t\eta'(X^{(n)}(s,0,x,v)\mp(t-s))ds\\
&=\int_0^t\Big[\pm(1+|V^{(n)}(s)|^2)-V^{(n)}(s)\sqrt{1+|V^{(n)}(s)|^2}\Big]\\
&\quad\cdot\frac{d}{ds}\Big\{\eta(X^{(n)}(s)\mp(t-s))\Big\}ds
\end{aligned}
$$

$$
\begin{aligned}
&= \Big[\pm (1+|V^{(n)}(t)|^2) - V^{(n)}(t)\sqrt{1+|V^{(n)}(t)|^2}\Big]\eta(X^{(n)}(t))\\
&\quad - \Big[\pm (1+|v|^2) - v\sqrt{1+|v|^2}\Big]\eta(x\mp t)\\
&\quad - \int_0^t \frac{d}{ds}\Big[\pm (1+|V^{(n)}(s)|^2) - V^{(n)}(s)\sqrt{1+|V^{(n)}(s)|^2}\Big]\\
&\quad \cdot \Big\{\eta(X^{(n)}(s)\mp(t-s))\Big\}ds\\
&\le 2(1+|V^{(n)}(t)|^2)\eta(X^{(n)}(t)) + 2(1+|v|^2)\eta(x\mp t)\\
&\quad + 4\int_0^t (1+|V^{(n)}(s)|)\|\partial_x u^{(n-1)}(s)\|_{L^\infty(\mathbb{R})}|\eta(X^{(n)}(s)\mp(t-s))|ds.
\end{aligned}
$$

Since $\|\partial_x u^{(n-1)}(s)\|_{L^\infty(\mathbb{R})}$ is bounded, it comes

$$
\begin{aligned}
(16)\qquad &\Big|\int_{\mathbb{R}} D_n^{\pm}(t,x)\eta'(x)dx\Big|\\
&\le 2\int_{\mathbb{R}^2} f_0(x,v)(1+|V^{(n)}(t)|^2)|\eta(X^{(n)}(t))|dxdv\\
&\quad + 2\int_{\mathbb{R}^2} f_0(x,v)(1+|v|^2)|\eta(x\mp t)|dxdv\\
&\quad + C\int_0^t\int_{\mathbb{R}^2} f_0(x,v)(1+|V^{(n)}(s)|)|\eta(X^{(n)}(s)\mp(t-s))|dxdvds,
\end{aligned}
$$

where C is a positive constant depending on T. To proceed further, we need the following lemma:

Lemma 3.1. *Assume that $\int_{\mathbb{R}} |v|^\gamma h_0(v)dv < +\infty$. Then for any $\gamma \in \mathbb{N}$, we have $\| \int_{\mathbb{R}} |v|^\gamma\ f^{(n)}(\cdot,\cdot,v)dv\|_{L^\infty((0,T)\times\mathbb{R})} \le C$, where $f^{(n)}$ is the characteristic solution of system* (6).

Proof. Applying the similar argument in (11), since $\partial_x u^{(n-1)}$ is uniformly bounded we get

$$
\begin{aligned}
&\int_{\mathbb{R}} |v|^\gamma f^{(n)}(t,x,v)dv\\
&= \int_{\mathbb{R}} |v|^\gamma f_0(X^{(n)}(0,t,x,v),V^{(n)}(0,t,x,v))dv\\
&\le \int_{\mathbb{R}} |v|^\gamma h_0(V^{(n)}(0,t,x,v))dv\\
&\le \int_{-\infty}^{-R(t)} |v|^\gamma h_0(v+R(t))dv + \int_{R(t)}^{+\infty} |v|^\gamma h_0(v-R(t))dv + \int_{-R(t)}^{R(t)} |v|^\gamma h_0 dv\\
&\le C\int_{-\infty}^{-R(t)} \Big(|v+R(t)|^\gamma + R(t)^\gamma\Big)h_0(v+R(t))dv
\end{aligned}
$$

$$
\begin{aligned}
&+C\int_{R(t)}^{+\infty}\Big(|v-R(t)|^{\gamma}+R(t)^{\gamma}\Big)h_0(v-R(t))dv+\int_{-R(t)}^{R(t)}|v|^{\gamma}h_0dv\\
\le{}& C\int_{\mathbb{R}}|v|^{\gamma}h_0(v)dv+C\|h_0\|_{L^1(\mathbb{R})}+\|h_0\|_{L^\infty(\mathbb{R})}\int_{-R(t)}^{R(t)}|v|^{\gamma}dv\\
\le{}& C\int_{\mathbb{R}}|v|^{\gamma}h_0(v)dv+C\Big(\|h_0\|_{L^1(\mathbb{R})}+\|h_0\|_{L^\infty(\mathbb{R})}\Big),
\end{aligned}
$$

where C only depends on T, $\|h_0\|_{L^1(\mathbb{R})}$, $\|h_0\|_{L^\infty(\mathbb{R})}$ and $\||v|^{\gamma}h_0\|_{L^1(\mathbb{R})}$. □

In view of Lemma 3.1, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^2}f_0(x,v)(1+|V^{(n)}(t)|^2)|\eta(X^{(n)}(t))|dxdv\\
={}&\int_{\mathbb{R}}|\eta(X^{(n)}(t))|\int_{\mathbb{R}}f^{(n)}(t,X^{(n)}(t),V^{(n)}(t))\\
&\cdot(1+|V^{(n)}(t)|^2)dV^{(n)}(t)dX^{(n)}(t)\\
\le{}& C\|\eta\|_{L^1},
\end{aligned}
\tag{17}
$$

where C is a positive constant depending on T and $\|h_0(1+|v|^2)\|_{L^1(\mathbb{R})}$. The last term in (16) is analogous. In addition, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^2}f_0(x,v)(1+|v|^2)|\eta(x\mp t)|dxdv\\
\le{}&\int_{\mathbb{R}^2}h_0(v)(1+|v|^2)dv|\eta(x\mp t)|dx\\
\le{}&\|\eta\|_{L^1(\mathbb{R})}\int_{\mathbb{R}}(1+|v|^2)h_0(v)dv.
\end{aligned}
\tag{18}
$$

Taken together, for any $\eta\in C_c^1(\mathbb{R})$ we get

$$\Big|\int_{\mathbb{R}}D_n^{\pm}(t,x)\eta'(x)dx\Big|\le C\|\eta\|_{L^1},$$

which implies $\|\partial_x D_n^{\pm}\|_{L^\infty(\mathbb{R})}\le C$ where C is a positive constant depending on T and $\|h_0(1+|v|^2)\|_{L^1(\mathbb{R})}$. Therefore, by (15) we obtain

$$\|\partial_{xx}u^{(n)}(t)\|_{L^\infty(\mathbb{R})}\le C,$$

where $C>0$ is a constant depending on T, $\|u_0\|_{L^\infty(\mathbb{R})}$, $\|u_0'\|_{L^\infty(\mathbb{R})}$, $\|u_0''\|_{L^\infty(\mathbb{R})}$, $\|u_1\|_{L^\infty(\mathbb{R})}$, $\|f_0\|_{L^1(\mathbb{R}^2)}$, $\|h_0\|_{L^\infty(\mathbb{R})}$ and $\|h_0(1+|v|^2)\|_{L^1(\mathbb{R})}$. Similarly, we have $\|\partial_{tx}u^{(n)}(t)\|_{L^\infty(\mathbb{R})}\le C$. By (7), it comes

$$\|\partial_t^2u^{(n)}(t)\|_{L^\infty(\mathbb{R})}\le\|\partial_{xx}u^{(n)}(t)\|_{L^\infty(\mathbb{R})}+\|u^{(n)}(t)\|_{L^\infty(\mathbb{R})}+\|\rho^{(n)}(t)\|_{L^\infty(\mathbb{R})}\le C,$$

where $C>0$ is a constant depending on T, $\|u_0\|_{L^\infty(\mathbb{R})}$, $\|u_0'\|_{L^\infty(\mathbb{R})}$, $\|u_0''\|_{L^\infty(\mathbb{R})}$, $\|u_1\|_{L^\infty(\mathbb{R})}$, $\|f_0\|_{L^1(\mathbb{R}^2)}$, $\|h_0\|_{L^\infty(\mathbb{R})}$ and $\|h_0(1+|v|^2)\|_{L^1(\mathbb{R})}$. Hence, we get

$$\|\partial_t^2u^{(n)}(t)\|_{L^\infty(\mathbb{R})}+\|\partial_{tx}u^{(n)}(t)\|_{L^\infty(\mathbb{R})}+\|\partial_{xx}u^{(n)}(t)\|_{L^\infty(\mathbb{R})}$$

$$+\Big\|\int_{\mathbb{R}} f^{(n)}(\cdot,\cdot,v)|v|^2dv\Big\|_{L^\infty((0,T)\times\mathbb{R})}<\infty.$$

Step 2. Convergence of the iterations. In this step, we devote to proving that the sequence $\{u^{(n)}\}$ is a Cauchy sequence in $C^1([0,T]\times\mathbb{R})$ and converges to some u. We fix two indices m and n. $f^{(m)}$ and $f^{(n)}$ are denoted to be the characteristic solution of equation (6) corresponding to $u^{(m)}$ and $u^{(n)}$. In consideration of (8), for any $\zeta\in L^1(\mathbb{R})$ and $t\in[0,T]$, we have

(19)

$$\begin{aligned}
&\int_{\mathbb{R}}(u^{(m)}-u^{(n)})\zeta(y)dy\\
=&-\frac{1}{2}\int_{\mathbb{R}}\int_0^t\int_{\mathbb{R}^2}(f^{(m)}-f^{(n)})(s,x,v)\zeta(y)\\
&\cdot\mathbf{1}_{\{|x-y|<t-s\}}J_0(\sqrt{(t-s)^2-|x-y|^2})dxdvdsdy\\
=&-\frac{1}{2}\int_0^t\int_{\mathbb{R}^2}(f^{(m)}-f^{(n)})(s,x,v)\int_{x-(t-s)}^{x+(t-s)}\zeta(y)\\
&\cdot J_0(\sqrt{(t-s)^2-|x-y|^2})dydxdvds\\
=&-\frac{1}{2}\int_0^t\int_{\mathbb{R}^2}f_0(x,v)\Big[\int_{X^{(m)}(s)-(t-s)}^{X^{(m)}(s)+(t-s)}\zeta(y)J_0(\sqrt{(t-s)^2-|X^{(m)}(s)-y|^2})\\
&\cdot dy-\int_{X^{(n)}(s)-(t-s)}^{X^{(n)}(s)+(t-s)}\zeta(y)J_0(\sqrt{(t-s)^2-|X^{(n)}(s)-y|^2})dy\Big]dxdvds.
\end{aligned}$$

Noting the uniform boundedness of $J_0(\cdot)$, we deduce

(20)
$$\begin{aligned}
&\Big|\int_{X^{(m)}(s)-(t-s)}^{X^{(m)}(s)+(t-s)}\zeta(y)J_0(\sqrt{(t-s)^2-|X^{(m)}(s)-y|^2})dy\\
&-\int_{X^{(n)}(s)-(t-s)}^{X^{(n)}(s)+(t-s)}\zeta(y)J_0(\sqrt{(t-s)^2-|X^{(n)}(s)-y|^2})dy\Big|\\
\le&\Big|\int_{X^{(m)}(s)-(t-s)}^{X^{(m)}(s)+(t-s)}\zeta(y)\Big[J_0(\sqrt{(t-s)^2-|X^{(m)}(s)-y|^2})\\
&-J_0(\sqrt{(t-s)^2-|X^{(n)}(s)-y|^2})\Big]dy\Big|\\
&+\Big|\int_{X^{(n)}(s)-(t-s)}^{X^{(m)}(s)-(t-s)}\zeta(y)J_0(\sqrt{(t-s)^2-|X^{(n)}(s)-y|^2})dy\Big|\\
&+\Big|\int_{X^{(n)}(s)+(t-s)}^{X^{(m)}(s)+(t-s)}\zeta(y)J_0(\sqrt{(t-s)^2-|X^{(n)}(s)-y|^2})dy\Big|\\
\le&\Big|\int_{X^{(m)}(s)-(t-s)}^{X^{(m)}(s)+(t-s)}\zeta(y)\Big[J_0(\sqrt{(t-s)^2-|X^{(m)}(s)-y|^2})
\end{aligned}$$

$$- J_0(\sqrt{(t-s)^2 - |X^{(n)}(s) - y|^2})\Big] dy\Big|$$
$$+ c\Big|\int_{X^{(n)}(s)-(t-s)}^{X^{(m)}(s)-(t-s)} \zeta(y)dy\Big| + c\Big|\int_{X^{(n)}(s)+(t-s)}^{X^{(m)}(s)+(t-s)} \zeta(y)dy\Big|.$$

Let $g(x) = \sqrt{(t-s)^2 - |x-y|^2}$, then we have

$$\begin{aligned}
&|J_0(g(X^{(m)}(s))) - J_0(g(X^{(n)}(s)))| \\
\le\ & c|X^{(m)}(s) - X^{(n)}(s)| \\
\le\ & e^{Qt} \int_0^t \|\partial_x u^{(m-1)}(\tau) - \partial_x u^{(n-1)}(\tau)\|_{L^\infty(\mathbb{R})}.
\end{aligned} \tag{21}$$

Combining (21), (20) and (19), we have

(22)

$$\begin{aligned}
&\int_{\mathbb{R}} (u^{(m)} - u^{(n)})\zeta(y)dy \\
\le\ & c\|\zeta\|_{L^1(\mathbb{R})} e^{Qt} \int_0^t \|\partial_x u^{(m-1)}(\tau) - \partial_x u^{(n-1)}(\tau)\|_{L^\infty(\mathbb{R})} d\tau \\
&+ c\int_0^t \int_{\mathbb{R}^2} f_0(x,v)\Big|\int_{X^{(n)}(s)-(t-s)}^{X^{(m)}(s)-(t-s)} \zeta(y)dy\Big| dxdvds \\
&+ c\int_0^t \int_{\mathbb{R}^2} f_0(x,v)\Big|\int_{X^{(n)}(s)+(t-s)}^{X^{(m)}(s)+(t-s)} \zeta(y)dy\Big| dxdvds \\
\le\ & c\|\zeta\|_{L^1(\mathbb{R})} e^{Qt} \int_0^t \|\partial_x u^{(m-1)}(\tau) - \partial_x u^{(n-1)}(\tau)\|_{L^\infty(\mathbb{R})} d\tau \\
&+ c\int_0^t \int_{\mathbb{R}} |\zeta(y)| \int_{\mathbb{R}^2} f_0(x,v)\mathbf{1}_{\{|y - X^{(n)}(s)+(t-s)| < |X^{(m)}(s) - X^{(n)}(s)|\}} dxdvdyds \\
&+ c\int_0^t \int_{\mathbb{R}} |\zeta(y)| \int_{\mathbb{R}^2} f_0(x,v)\mathbf{1}_{\{|y - X^{(n)}(s)-(t-s)| < |X^{(m)}(s) - X^{(n)}(s)|\}} dxdvdyds.
\end{aligned}$$

Applying Lemma 2.1 and Lemma 3.1, it comes for any $s \in [0,t]$

$$\begin{aligned}
&\int_{\mathbb{R}^2} f_0(x,v)\mathbf{1}_{\{|y - X^{(n)}(s)-(t-s)| < |X^{(m)}(s) - X^{(n)}(s)|\}} dxdv \\
=\ & \int_{\mathbb{R}} \mathbf{1}_{\{|y - X^{(n)}(s)-(t-s)| < e^{Qs}\int_0^t \|\partial_x u^{(m-1)}(\tau) - \partial_x u^{(n-1)}(\tau)\|_{L^\infty(\mathbb{R})} d\tau\}} \\
&\cdot \int_{\mathbb{R}} f^{(n)}(s, X^{(n)}(s), V^{(n)}(s)) dV^{(n)}(s) dX^{(n)}(s) \\
\le\ & \|\int_{\mathbb{R}} f^{(n)}(\cdot,\cdot,v)dv\|_{L^\infty((0,T)\times\mathbb{R})} \\
&\cdot e^{Qs}\int_0^t \|\partial_x u^{(m-1)}(\tau) - \partial_x u^{(n-1)}(\tau)\|_{L^\infty(\mathbb{R})} d\tau
\end{aligned} \tag{23}$$

$$\leq C\int_0^t \|\partial_x u^{(m-1)}(\tau)-\partial_x u^{(n-1)}(\tau)\|_{L^\infty(\mathbb{R})}d\tau,$$

where $C>0$ is a constant depending on T, $\|u_0\|_{L^\infty(\mathbb{R})}$, $\|u_0'\|_{L^\infty(\mathbb{R})}$, $\|u_0''\|_{L^\infty(\mathbb{R})}$, $\|u_1\|_{L^\infty(\mathbb{R})}$, $\|f_0\|_{L^1(\mathbb{R}^2)}$, $\|h_0\|_{L^\infty(\mathbb{R})}$ and $\|h_0(1+|v|^2)\|_{L^1(\mathbb{R})}$. Substituting (23) into (22), for any $t\in[0,T]$ and $\zeta\in L^1(\mathbb{R})$, we obtain

$$\int_{\mathbb{R}}(u^{(m)}-u^{(n)})\zeta(y)dy\leq c\|\zeta\|_{L^1(\mathbb{R})}\int_0^t \|\partial_x u^{(m-1)}(\tau)-\partial_x u^{(n-1)}(\tau)\|_{L^\infty(\mathbb{R})}d\tau.$$

Thus, we have for any $t\in[0,T]$

$$\|u^{(m)}(t)-u^{(n)}(t)\|_{L^\infty(\mathbb{R})}\leq C\int_0^t \|\partial_x u^{(m-1)}(\tau)-\partial_x u^{(n-1)}(\tau)\|_{L^\infty(\mathbb{R})}d\tau, \tag{24}$$

where $C>0$ is a constant depending on T, $\|u_0\|_{L^\infty(\mathbb{R})}$, $\|u_0'\|_{L^\infty(\mathbb{R})}$, $\|u_0''\|_{L^\infty(\mathbb{R})}$, $\|u_1\|_{L^\infty(\mathbb{R})}$, $\|f_0\|_{L^1(\mathbb{R}^2)}$, $\|h_0\|_{L^\infty(\mathbb{R})}$ and $\|h_0(1+|v|^2)\|_{L^1(\mathbb{R})}$.

As for the spatial derivative in term of $u^{(m)}-u^{(n)}$, for any $\zeta\in L^1(\mathbb{R})$ and $t\in[0,T]$, using (9) we have

$$\begin{aligned}
&\int_{\mathbb{R}}(\partial_x u^{(m)}-\partial_x u^{(n)})\zeta(x)dx \\
&=\frac{1}{2}\int_0^t\int_{\mathbb{R}^2}(f^{(m)}-f^{(n)})(s,x-(t-s),v)\zeta(x)dxdvds \\
&\quad-\frac{1}{2}\int_0^t\int_{\mathbb{R}^2}(f^{(m)}-f^{(n)})(s,x+(t-s),v)\zeta(x)dxdvds \\
&\quad-\frac{1}{2}\int_0^t\int_{\mathbb{R}}\int_{x-(t-s)}^{x+t-s}\int_{\mathbb{R}}(f^{(m)}-f^{(n)})(s,y,v)dv \\
&\quad\cdot\frac{J_1(\sqrt{(t-s)^2-|x-y|^2})}{\sqrt{(t-s)^2-|x-y|^2}}(x-y)dy\zeta(x)dxds \\
&=\frac{1}{2}(I_1+I_2+I_3).
\end{aligned}\tag{25}$$

Next, we estimate I_1, I_2 and I_3 separately as follows. For I_1, we have

$$\begin{aligned}
I_1 &= \int_{\mathbb{R}^2}f_0(x,v)\int_0^t\Big[\zeta\big(X^{(m)}(s)+t-s\big)-\zeta\big(X^{(n)}(s)+t-s\big)\Big]dsdxdv \\
&= -\int_{\mathbb{R}^2}f_0\int_0^t\Big[1+|V^{(m)}(s)|^2+V^{(m)}(s)\sqrt{1+|V^{(m)}(s)|^2}\Big] \\
&\quad\cdot\frac{d}{ds}\int_{X^{(m)}(t)}^{X^{(m)}(s)+(t-s)}\zeta(u)dudsdxdv \\
&\quad+\int_{\mathbb{R}^2}f_0\int_0^t\Big[1+|V^{(n)}(s)|^2+V^{(n)}(s)\sqrt{1+|V^{(n)}(s)|^2}\Big]
\end{aligned}$$

$$
\begin{aligned}
&\cdot \frac{d}{ds}\int_{X^{(n)}(t)}^{X^{(n)}(s)+(t-s)} \zeta(u)dudsdxdv \\
=&\int_{\mathbb{R}^2} f_0(x,v)\Big[\int_{X^{(m)}(t)}^{x+t}\zeta(u)du\Big(1+|v|^2+v\sqrt{1+|v|^2}\Big) \\
&+\int_0^t \frac{d}{ds}\Big(V^{(m)}(s)\sqrt{1+|V^{(m)}(s)|^2}+1+|V^{(m)}(s)|^2\Big)\int_{X^{(m)}(t)}^{X^{(m)}(s)+(t-s)} \\
&\cdot\zeta(u)duds-\int_{X^{(n)}(t)}^{x+t}\zeta(u)du\Big(1+|v|^2+v\sqrt{1+|v|^2}\Big) \\
&-\int_0^t \frac{d}{ds}\Big(V^{(n)}(s)\sqrt{1+|V^{(n)}(s)|^2}+1+|V^{(n)}(s)|^2\Big) \\
&\cdot\int_{X^{(n)}(t)}^{X^{(n)}(s)+(t-s)}\zeta(u)duds\Big]dxdv \\
=&\int_{\mathbb{R}^2} f_0(x,v)\int_{X^{(m)}(t)}^{X^{(n)}(t)}\zeta(u)du\Big[1+|v|^2+v\sqrt{1+|v|^2}\Big]dxdv \\
&-\int_{\mathbb{R}^2} f_0(x,v)\int_0^t\Big[\Big(2V^{(m)}(s)+\frac{1+2|V^{(m)}(s)|^2}{\sqrt{1+|V^{(m)}(s)|^2}}\Big) \\
&\cdot\partial_x u^{(m-1)}(s,X^{(m)}(s))\int_{X^{(m)}(t)}^{X^{(m)}(s)+(t-s)}\zeta(u)du \\
&-\Big(2V^{(n)}(s)+\frac{1+2|V^{(n)}(s)|^2}{\sqrt{1+|V^{(n)}(s)|^2}}\Big) \\
&\cdot\partial_x u^{(n-1)}(s,X^{(n)}(s))\int_{X^{(n)}(t)}^{X^{(n)}(s)+(t-s)}\zeta(u)du\Big]dsdxdv.
\end{aligned}
$$

Note that

$$
\begin{aligned}
&\Big(2V^{(m)}(s)+\frac{1+2|V^{(m)}(s)|^2}{\sqrt{1+|V^{(m)}(s)|^2}}\Big)\partial_x u^{(m-1)}(s,X^{(m)}(s)) \\
&\cdot\int_{X^{(m)}(t)}^{X^{(m)}(s)+(t-s)}\zeta(u)du-\Big(2V^{(n)}(s)+\frac{1+2|V^{(n)}(s)|^2}{\sqrt{1+|V^{(n)}(s)|^2}}\Big) \\
&\cdot\partial_x u^{(n-1)}(s,X^{(n)}(s))\int_{X^{(n)}(t)}^{X^{(n)}(s)+(t-s)}\zeta(u)du \\
\le\ & C(1+|V^{(m)}(s)|)\Big|\int_{X^{(n)}(t)}^{X^{(m)}(t)}\zeta(u)du\Big| \\
&+C(1+|V^{(m)}(s)|)\Big|\int_{X^{(m)}(s)+(t-s)}^{X^{(n)}(s)+(t-s)}\zeta(u)du\Big|
\end{aligned}
$$

$$
\begin{aligned}
&+\Big|\int_{X^{(n)}(t)}^{X^{(n)}(s)+(t-s)}\zeta(u)du\Big| \\
&\cdot\Big|\Big(2V^{(m)}(s)+\frac{1+2|V^{(m)}(s)|^2}{\sqrt{1+|V^{(m)}(s)|^2}}\Big)\partial_x u^{(m-1)}(s,X^{(m)}(s)) \\
&-\partial_x u^{(n-1)}(s,X^{(n)}(s))\Big(2V^{(n)}(s)+\frac{1+2|V^{(n)}(s)|^2}{\sqrt{1+|V^{(n)}(s)|^2}}\Big)\Big|
\end{aligned}
$$

and

$$
\begin{aligned}
&\Big(2V^{(m)}(s)+\frac{1+2|V^{(m)}(s)|^2}{\sqrt{1+|V^{(m)}(s)|^2}}\Big)\partial_x u^{(m-1)}(s,X^{(m)}(s)) \\
&-\Big(2V^{(n)}(s)+\frac{1+2|V^{(n)}(s)|^2}{\sqrt{1+|V^{(n)}(s)|^2}}\Big)\partial_x u^{(n-1)}(s,X^{(n)}(s)) \\
\le\ & 4\Big(1+|V^{(m)}(s)|\Big)\Big|\partial_x u^{(m-1)}(s,X^{(m)}(s))-\partial_x u^{(n-1)}(s,X^{(n)}(s))\Big| \\
&+c|\partial_x u^{(n-1)}(s,X^{(n)}(s))||V^{(m)}(s)-V^{(n)}(s)| \\
\le\ & C\Big(1+|V^{(m)}(s)|\Big)\Big(\|\partial_x u^{(m-1)}(s)-\partial_x u^{(n-1)}(s)\|_{L^\infty(\mathbb{R})} \\
&+|X^{(m)}(s)-X^{(n)}(s)|+|V^{(m)}(s)-V^{(n)}(s)|\Big),
\end{aligned}
$$

where $C>0$ is a constant depending on T, $\|u_0\|_{L^\infty(\mathbb{R})}$, $\|u_0'\|_{L^\infty(\mathbb{R})}$, $\|u_0''\|_{L^\infty(\mathbb{R})}$, $\|u_1\|_{L^\infty(\mathbb{R})}$, $\|f_0\|_{L^1(\mathbb{R}^2)}$, $\|h_0\|_{L^\infty(\mathbb{R})}$ and $\|h_0(1+|v|^2)\|_{L^1(\mathbb{R})}$. Then we get

$$
\begin{aligned}
I_1 \le\ & C\int_{\mathbb{R}^2} f_0(x,v)\Big[1+|v|^2+\int_0^t(1+|V^{(m)}(s)|)ds\Big]\Big|\int_{X^{(m)}(t)}^{X^{(n)}(t)}\zeta(u)du\Big|dxdv \\
&+C\int_{\mathbb{R}^2} f_0(x,v)\int_0^t\Big(|V^{(m)}(s)|+1\Big)\Big|\int_{X^{(m)}(t)+(t-s)}^{X^{(n)}(s)+(t-s)}\zeta(u)du\Big|dsdxdv \\
&+C\int_{\mathbb{R}^2} f_0(x,v)\int_0^t\Big|\int_{X^{(n)}(t)}^{X^{(n)}(s)+(t-s)}\zeta(u)du\Big|(1+|V^{(m)}(s)|)\cdot \qquad (26)\\
&\Big(\|\partial_x u^{(m-1)}(s)-\partial_x u^{(n-1)}(s)\|_{L^\infty(\mathbb{R})}+|X^{(m)}(s)-X^{(n)}(s)| \\
&+|V^{(m)}(s)-V^{(n)}(s)|\Big)dsdxdv \\
=\ & I_{11}+I_{12}+I_{13}.
\end{aligned}
$$

Noting that $\int_0^t(1+|V^{(m)}(s)|)ds\le c(1+|v|)$, $1+|v|^2\le C(1+|V^{(m)}(t)|^2)$, Lemma 2.1 and Lemma 3.1, we obtain

$$
\begin{aligned}
I_{11}\le\ & C\int_{\mathbb{R}^2} f_0(x,v)(1+|V^{(m)}(t)|^2)\int_{\mathbb{R}}\Big|\zeta(y)\Big| \\
&\cdot\mathbf{1}_{\{|y-X^{(m)}(t)|<|X^{(m)}(t)-X^{(n)}(t)|\}}dydxdv
\end{aligned}
$$

$$
\begin{aligned}
&\leq C\int_{\mathbb{R}}\Big|\zeta(y)\Big|\int_{\mathbb{R}^2} f^{(m)}(t,X^{(m)}(t),V^{(m)}(t))(1+|V^{(m)}(t)|^2)\\
&\quad\cdot \mathbf{1}_{\{|y-X^{(m)}(t)|<e^{Qt}\int_0^t \|\partial_x u^{(m-1)}(\tau)-\partial_x u^{(n-1)}(\tau)\|_{L^\infty(\mathbb{R})}d\tau\}}dydxdv\\
&\leq C\|\zeta\|_{L^1(\mathbb{R})}\Big\|\int_{\mathbb{R}}(1+|v|^2)f(\cdot,\cdot,v)dv\Big\|_{L^\infty((0,T)\times\mathbb{R})}\\
&\quad\cdot\int_0^t \|\partial_x u^{(m-1)}(\tau)-\partial_x u^{(n-1)}(\tau)\|_{L^\infty(\mathbb{R})}d\tau\\
&\leq C\|\zeta\|_{L^1(\mathbb{R})}\int_0^t \|\partial_x u^{(m-1)}(\tau)-\partial_x u^{(n-1)}(\tau)\|_{L^\infty(\mathbb{R})}d\tau
\end{aligned}
$$

and

$$
\begin{aligned}
I_{13}\leq\ & c\int_{\mathbb{R}^2} f_0(x,v)(1+|V^{(m)}(t)|)\Big|\int_{X^{(n)}(t)}^{X^{(n)}(s)+(t-s)}\zeta(u)du\Big|\\
&\cdot\int_0^t\Big[\|\partial_x u^{(m-1)}(s)-\partial_x u^{(n-1)}(s)\|_{L^\infty(\mathbb{R})}\\
&+e^{Qs}\int_0^t \|\partial_x u^{(m-1)}(\tau)-\partial_x u^{(n-1)}(\tau)\|_{L^\infty(\mathbb{R})}d\tau\Big]dsdxdv\\
\leq\ & C\|\zeta\|_{L^1(\mathbb{R})}\int_0^t \|\partial_x u^{(m-1)}(\tau)-\partial_x u^{(n-1)}(\tau)\|_{L^\infty(\mathbb{R})}d\tau,
\end{aligned}
$$

where $C>0$ is a constant depending on T, $\|u_0\|_{L^\infty(\mathbb{R})}$, $\|u_0'\|_{L^\infty(\mathbb{R})}$, $\|u_0''\|_{L^\infty(\mathbb{R})}$, $\|u_1\|_{L^\infty(\mathbb{R})}$, $\|f_0\|_{L^1(\mathbb{R}^2)}$, $\|h_0\|_{L^\infty(\mathbb{R})}$ and $\|h_0(1+|v|^2)\|_{L^1(\mathbb{R})}$. Since I_{12} is similar to I_{11}, it is not difficult to find that

$$
I_{12}\leq C\|\zeta\|_{L^1(\mathbb{R})}\int_0^t \|\partial_x u^{(m-1)}(\tau)-\partial_x u^{(n-1)}(\tau)\|_{L^\infty(\mathbb{R})}d\tau.
$$

Thus,

$$
I_1\leq C\|\zeta\|_{L^1(\mathbb{R})}\int_0^t \|\partial_x u^{(m-1)}(\tau)-\partial_x u^{(n-1)}(\tau)\|_{L^\infty(\mathbb{R})}d\tau,
$$

where $C>0$ is a constant depending on T, $\|u_0\|_{L^\infty(\mathbb{R})}$, $\|u_0'\|_{L^\infty(\mathbb{R})}$, $\|u_0''\|_{L^\infty(\mathbb{R})}$, $\|u_1\|_{L^\infty(\mathbb{R})}$, $\|f_0\|_{L^1(\mathbb{R}^2)}$, $\|h_0\|_{L^\infty(\mathbb{R})}$ and $\|h_0(1+|v|^2)\|_{L^1(\mathbb{R})}$. Analogously, we can establish similar estimates for I_2. The last term I_3,

$$
\begin{aligned}
I_3=\ &-\int_0^t\int_{\mathbb{R}}\int_{x-(t-s)}^{x+(t-s)}\int_{\mathbb{R}} f_0(z,w)\Big[\frac{J_1(\sqrt{(t-s)^2-|x-X^{(m)}(s)|^2})}{\sqrt{(t-s)^2-|x-X^{(m)}(s)|^2}}\\
&(x-X^{(m)}(s))-\frac{J_1(\sqrt{(t-s)^2-|x-X^{(n)}(s)|^2})}{\sqrt{(t-s)^2-|x-X^{(n)}(s)|^2}}(x-X^{(n)}(s))\Big]dzdw\zeta(x)dxds\\
&\leq C\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}^2} f_0(z,w)|X^{(m)}(s)-X^{(n)}(s)|dzdw|\zeta(x)|dxds\\
&\leq C\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}^2} f_0(z,w)e^{Qt}\int_0^t \|\partial_x u^{(m-1)}(\tau)-\partial_x u^{(n-1)}(\tau)\|_{L^\infty(\mathbb{R})}
\end{aligned}
$$

$$\cdot\, d\tau dz dw |\zeta(x)| dx ds$$
$$\le C\|\zeta\|_{L^1(\mathbb{R})}\int_0^t \|\partial_x u^{(m-1)}(\tau) - \partial_x u^{(n-1)}(\tau)\|_{L^\infty(\mathbb{R})} d\tau,$$

where $C > 0$ is a constant depending on T and $\|f_0\|_{L^1(\mathbb{R}^2)}$. Hence, we obtain

$$\|\partial_x u^{(m)}(t) - \partial_x u^{(n)}(t)\|_{L^\infty(\mathbb{R})} \le C\int_0^t \|\partial_x u^{(m-1)}(\tau) - \partial_x u^{(n-1)}(\tau)\|_{L^\infty(\mathbb{R})} d\tau,$$

where $C > 0$ is a constant depending on T, $\|u_0\|_{L^\infty(\mathbb{R})}$, $\|u_0'\|_{L^\infty(\mathbb{R})}$, $\|u_0''\|_{L^\infty(\mathbb{R})}$, $\|u_1\|_{L^\infty(\mathbb{R})}$, $\|f_0\|_{L^1(\mathbb{R}^2)}$, $\|h_0\|_{L^\infty(\mathbb{R})}$ and $\|h_0(1+|v|^2)\|_{L^1(\mathbb{R})}$. By induction, we have

$$\|\partial_x u^{(m)}(t) - \partial_x u^{(n)}(t)\|_{L^\infty(\mathbb{R})} \le C\frac{t^n}{n!} \to 0 \quad \text{as} \quad n \to \infty, \tag{27}$$

for any $n \in \mathbb{N}$ and $t \in [0,T]$.

For the time derivative in term of $u^{(m)} - u^{(n)}$, we have this expression

$$\begin{aligned}\partial_t u^{(m)} - \partial_t u^{(n)} = & -\frac{1}{2}\int_0^t \int_{\mathbb{R}} (f^{(m)} - f^{(n)})(s, x-(t-s), v) dv ds \\ & -\frac{1}{2}\int_0^t \int_{\mathbb{R}} (f^{(m)} - f^{(n)})(s, x+(t-s), v) dv ds \\ & +\frac{1}{2}\int_0^t \int_{x-(t-s)}^{x+t-s}\int_{\mathbb{R}} (f^{(m)} - f^{(n)})(s, y, v) dv \\ & \cdot \frac{J_1(\sqrt{(t-s)^2 - |x-y|^2})}{\sqrt{(t-s)^2-|x-y|^2}}(x-y) dy ds.\end{aligned}$$

Using a similar argument as in $\partial_x u^{(m)}(t) - \partial_x u^{(n)}(t)$, we get the estimate

$$\begin{aligned}&\|\partial_t u^{(m)}(t) - \partial_t u^{(n)}(t)\|_{L^\infty(\mathbb{R})} \\ &\le C\int_0^t \|\partial_x u^{(m-1)}(\tau) - \partial_x u^{(n-1)}(\tau)\|_{L^\infty(\mathbb{R})} d\tau.\end{aligned} \tag{28}$$

It follows from (24), (28) and (27) in which $\{u^{(n)}\}$ is a Cauchy sequence in $C^1([0,T]\times\mathbb{R})$. Without loss of generality, we may assume that $\{u^{(n)}\}$ converges to some u as $n \to \infty$. By the definition of the mild solutions to the system (1)-(3), (f,u) solves the system (1)-(3).

Step 3. Uniqueness. Let $(f^{(k)}, u^{(k)})$ with $k = \{1,2\}$ be the two mild solutions to the system (1)-(3). Similar to the proof in Step 2, we get for any $t \in [0,T]$

$$\begin{aligned}&\|u^{(1)}(t) - u^{(2)}(t)\|_{L^\infty(\mathbb{R})} + \|\partial_x u^{(1)}(t) - \partial_x u^{(2)}(t)\|_{L^\infty(\mathbb{R})} \\ &\quad + \|\partial_t u^{(1)}(t) - \partial_t u^{(2)}(t)\|_{L^\infty(\mathbb{R})} \\ &\le C\int_0^t \|\partial_x u^{(1)}(\tau) - \partial_x u^{(2)}(\tau)\|_{L^\infty(\mathbb{R})} d\tau.\end{aligned}$$

Hence, the uniqueness follows directly by Gronwall's lemma.

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