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GENERATING NON-JUMPING NUMBERS OF HYPERGRAPHS

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Abstract. The concept of jump concerns the distribution of Turán densities. A number $\alpha \in [0, 1)$ is a *jump* for r if there exists a constant $c > 0$ such that if the Turán density of a family $\mathscr{F}$ of r-uniform graphs is greater than α, then the Turán density of $\mathscr{F}$ is at least $\alpha + c$. To determine whether a number is a jump or non-jump has been a challenging problem in extremal hypergraph theory. In this paper, we give a way to generate non-jumps for hypergraphs. We show that if α, β are non-jumps for $r_1, r_2 \geq 2$ respectively, then $\frac{\alpha\beta(r_1+r_2)!r_1^{r_1}r_2^{r_2}}{r_1!r_2!(r_1+r_2)^{r_1+r_2}}$ is a non-jump for $r_1 + r_2$. We also apply the Lagrangian method to determine the Turán density of the extension of the $(r-3)$-fold enlargement of a 3-uniform matching.

1. Introduction

For a set V and a positive integer r we denote by $\binom{V}{r}$ the family of all r-subsets of V. An *r-uniform graph* or *r-graph* G is a set $V(G)$ of vertices together with a set $E(G) \subseteq \binom{V(G)}{r}$ of edges. The *density* of G is defined to be $d(G) = |E(G)|/|\binom{V(G)}{r}|$. An r-graph H is a *subgraph* of an r-graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. H is an *induced subgraph* of G if $E(H) = E(G) \bigcap \binom{V(H)}{r}$. Given a 3-graph G and an integer $r \geq 3$, the $(r-3)$-*fold enlargement* of G is an r-graph F obtained by taking an $(r-3)$-set D that is vertex disjoint from G and letting $F = \{e \bigcup D : e \in G\}$. The *extension* H^F of an r-graph F is obtained as follows: For each pair of vertices v_i, v_j in F not contained in an edge of F, we add a set B_{ij} of $r-2$ new vertices and the edge $\{v_i, v_j\} \bigcup B_{ij}$, where the B_{ij}'s are pairwise disjoint over all such pairs $\{i, j\}$. Given positive integers $r \geq 3$ and $t \geq 2$, let M_t^r be the r-graph with t pairwise disjoint edges, called *r-uniform t-matching*.

Let $\mathscr{F}$ be a family of r-graphs. An r-graph G is called *$\mathscr{F}$-free* if G contain no member of $\mathscr{F}$ as a subgraph. The *Turán number* of $\mathscr{F}$, denoted by $ex(n, \mathscr{F})$, is the maximum number of edges that an $\mathscr{F}$-free r-graph of order n can have.

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The *Turán density* [18] of $\mathscr{F}$, denoted by $\gamma(\mathscr{F})$, is $\lim_{n\to\infty} ex(n,\mathscr{F})/\binom{n}{r}$. The existence of this limit is guaranteed by the following fact due to Katona et al. [11].

Fact 1.1 ([11])**.** Let G be an r-graph with n vertices and $m \geq r$ be an integer. Then the average density of all induced subgraphs of G with $m \leq n$ vertices is $d(G)$.

The set of all possible Turán densities for $r \geq 2$ is denoted by Γ_r, i.e., $\Gamma_r = \{\gamma(\mathscr{F}) : \mathscr{F}$ is a family of r-uniform graphs$\}$. Erdős-Stone [4], Erdős-Simonovits [3] obtained that $\Gamma_2 = \{0, \frac{1}{2}, \frac{2}{3}, \ldots, \frac{l-1}{l}, \ldots, \}$. However, for $r \geq 3$, a good characterization of Γ_r is by far unknown.

Definition 1.2. A number $\alpha \in [0,1)$ is a jump for $r \geq 2$ if and only if there exists a constant $c > 0$ such that $\Gamma_r \cap (\alpha, \alpha + c) = \emptyset$.

Erdős-Stone [4] proved that every $\alpha \in [0,1)$ is a jump for $r = 2$. Erdős [2] proved that every $\alpha \in [0, \frac{r!}{r^r})$ is a jump for $r \geq 3$. Furthermore, Erdős conjectured that every $\alpha \in [0,1)$ is a jump for every integer $r \geq 2$. In [7], Frankl-Rödl disproved Erdős conjecture by giving an infinite sequence of non-jumps for $r \geq 3$. To determine whether a number is a jump or non-jump has been a challenging problem in extremal hypergraph theory. Frankl-Peng-Rödl-Talbot [6] showed that $\frac{5r!}{2r^r}$ is a non-jump for $r \geq 3$, and this is the smallest known non-jump at this moment. Baber-Talbot [1] showed that for $r = 3$ every $\alpha \in [0.2299, 0.2316) \bigcup [0.2871, \frac{8}{27})$ is a jump. Pikhurko [16] showed that the set of non-jumps for every $r \geq 3$ has cardinality of the continuum. More results on non-jumps were obtained in [8, 13, 14] and some other papers. Jumps and non-jumps for non-uniform hypergraphs were also introduced by Johnston-Lu in [10]. In this paper, we give a way to generate non-jumps for $r_1 + r_2$ based on non-jumps for r_1 and r_2. We show that if α, β are non-jumps for $r_1, r_2 \geq 2$ respectively, then $\frac{\alpha\beta(r_1+r_2)!r_1^{r_1}r_2^{r_2}}{r_1!r_2!(r_1+r_2)^{r_1+r_2}}$ is a non-jump for $r_1 + r_2$. The details of this part will be given in Section 2.

Very few exact results are known for hypergraph Turán densities. Lagrangian has been an important tool in hypergraph extremal problems. In [12], Motzkin-Straus determined the Lagrangian of any graph and gave a new proof of Turán's theorem on the Turán's density of a complete graph. In 1980's, Sidorenko [17] and Frankl-Füredi [5] further developed the Lagrangian method in hypergraph Turán densities. The well-known Erdős-sós conjecture says that if T is a k-vertex tree or forest then $ex(n,T) \leq \frac{n(k-2)}{2}$. In [17], Sidorenko obtained the Turán's density of the extension of the $(r-2)$-fold enlargement of T which is a graph satisfying the Erdős-sós conjecture. Recently, the connection between the Lagrangian density of a hypergraph and the Turán number of its extension has been studied actively in a number of papers. In this paper, we will apply the Lagrangian method to determine the Turán density of the extension of the $(r-3)$-fold enlargement of a 3-uniform matching. The details of this part will be given in Section 3.

2. Generating non-jump numbers

We first introduce some definitions, results and facts.

Definition 2.1 (Equivalent Definition of Jump [7]). A real number $\alpha \in [0,1)$ is a jump for an integer $r \geq 2$ if there exists a constant $c > 0$ such that for any $\epsilon > 0$ and any integer m, $m \geq r$, there exists an integer n_0 such that any r-graph with $n \geq n_0$ vertices and density $\geq \alpha + \epsilon$ contains a subgraph with m vertices and density $\geq \alpha + c$.

In addition to the known jumps or non-jumps mentioned in the previous results, the following results give more non-jumps.

Theorem 2.2 ([14]). *Let $p \geq r \geq 3$ be positive integers. If $c \cdot \frac{r!}{r^r}$ is a non-jump for r, then $c \cdot \frac{p!}{p^p}$ is a non-jump for p.*

Peng-Zhao [15] generalised the concept of jump to strong jump and obtained some non-strong-jump numbers.

Definition 2.3 ([15]). A real number $\alpha \in [0,1)$ is a strong-jump for an integer $r \geq 2$ if there exists a constant $c > 0$ such that for any integer $m \geq r$, there exists an integer n_0 such that any r-graph with $n \geq n_0$ vertices and density $> \alpha$ contains a subgraph with m vertices and density $\geq \alpha + c$.

Theorem 2.4 ([15]). *Let $p \geq r \geq 2$ be positive integers. If $c \cdot \frac{r!}{r^r}$ is a non-strong-jump for r, then $c \cdot \frac{p!}{p^p}$ is a non-strong-jump for p.*

Theorem 2.5 ([15]). *$\frac{r!}{r^r}$ is a non-strong-jump for $r \geq 3$.*

Fact 2.6 ([15]). Let l be a positive integer. Then every number in $(1 - \frac{1}{l}, 1 - \frac{1}{l+1})$ is a strong jump and $1 - \frac{1}{l}$ is a non-strong-jump for $r = 2$.

We obtain the following results in this section.

Lemma 2.7. *Let $r_1, r_2 \geq 2$ be positive integers. If α, β are non-jumps for $r_1, r_2 \geq 2$, respectively, then $\frac{\alpha\beta(r_1+r_2)! r_1^{r_1} r_2^{r_2}}{r_1! r_2! (r_1+r_2)^{r_1+r_2}}$ is a non-jump for $r_1 + r_2$.*

Lemma 2.8. *Let $r_1, r_2 \geq 2$ be positive integers. If α, β are non-strong-jumps for $r_1, r_2 \geq 2$, respectively, then $\frac{\alpha\beta(r_1+r_2)! r_1^{r_1} r_2^{r_2}}{r_1! r_2! (r_1+r_2)^{r_1+r_2}}$ is a non-strong-jump for $r_1 + r_2$.*

Theorem 2.9. *Let $r_1, r_2 \geq 2$ be positive integers. If α is a non-strong-jump for $r_1 \geq 2$ and β is a non-jump for $r_2 \geq 2$, then $\frac{\alpha\beta(r_1+r_2)! r_1^{r_1} r_2^{r_2}}{r_1! r_2! (r_1+r_2)^{r_1+r_2}}$ is a non-jump for $r_1 + r_2$.*

Remark 2.10. Let $r_1 = p - r$, $r_2 = r$, $\alpha = \frac{(p-r)!}{(p-r)^{(p-r)}}$ and $\beta = c\frac{r!}{r^r}$. Theorem 2.9 and Lemma 2.8 imply Theorem 2.2 and Theorem 2.4 respectively.

Since the proofs of Lemmas 2.7 and 2.8 are similar to that of Theorem 2.9, we omit their proofs. Let us turn to the proof of Theorem 2.9.

Proof of Theorem 2.9. If α is a non-strong-jump for r_1, then for any $c > 0$, there exists an integer m_1 such that for any integer n_0, there exists $n_1 \geq \max\{n_0, n_{01}\}$ (where n_{01} is a sufficiently large number satisfying inequality (2.11)) and an r_1-graph $G_1^{(r_1)}$ on n_1 vertices such that

$$d(G_1^{(r_1)}) > \alpha, \tag{2.1}$$

and any subgraph $H_1^{(r_1)}$ of $G_1^{(r_1)}$ with m_1 vertices has density $d(H_1^{(r_1)}) < \alpha + \frac{c}{4}$. By Fact 1.1, this implies that any subgraph $H_1^{(r_1)}$ of $G_1^{(r_1)}$ with at least m_1 vertices has density

$$d(H_1^{(r_1)}) < \alpha + \frac{c}{4}. \tag{2.2}$$

Similarly, if β is a non-jump for r_2, then for any $c > 0$, there exists an $\epsilon_1 > 0$ and an integer m_2 such that for any integer n_0, there exists $n_2 \geq \max\{n_0, n_{02}\}$ (where $n_{02} = \frac{r_2}{r_1} n_{01}$) and an r_2-graph $G_2^{(r_2)}$ on n_2 vertices such that

$$d(G_2^{(r_2)}) \geq \beta + \epsilon_1, \tag{2.3}$$

and any subgraph $H_2^{(r_2)}$ of $G_2^{(r_2)}$ with m_2 vertices has density $d(H_2^{(r_2)}) < \beta + \frac{c}{4}$. By Fact 1.1, this implies that any subgraph $H_2^{(r_2)}$ of $G_2^{(r_2)}$ with at least m_2 vertices has density

$$d(H_2^{(r_2)}) < \beta + \frac{c}{4}. \tag{2.4}$$

We are going to show that for any $c > 0$ there exists an $\epsilon > 0$ and an integer M such that for any integer n_0, there exists an integer $N \geq n_0$ and an $(r_1 + r_2)$ -graph $G^{(r_1+r_2)}$ on N vertices such that $d(G^{(r_1+r_2)}) > \frac{\alpha\beta(r_1+r_2)! r_1^{r_1} r_2^{r_2}}{r_1! r_2! (r_1+r_2)^{r_1+r_2}} + \epsilon$ and any subgraph $H^{(r_1+r_2)}$ of $G^{(r_1+r_2)}$ with M vertices has density $d(H^{(r_1+r_2)}) < \frac{\alpha\beta(r_1+r_2)! r_1^{r_1} r_2^{r_2}}{r_1! r_2! (r_1+r_2)^{r_1+r_2}} + c$.

Take a sufficiently large integer $M (\geq m_1 + m_2)$ satisfying

$$m_1 \leq \frac{\alpha^{\frac{1}{r_1}} r_1 r_2^{\frac{r_2}{r_1}}}{(r_1 + r_2)^{1+\frac{r_2}{r_1}}} M, \tag{2.5}$$

$$m_2 \leq \frac{\beta^{\frac{1}{r_2}} r_1^{\frac{r_1}{r_2}} r_2}{(r_1 + r_2)^{1+\frac{r_1}{r_2}}} M, \tag{2.6}$$

and

$$\frac{M^{r_1+r_2}}{M(M-1)\cdots(M - r_1 - r_2 + 1)} < 1 + \frac{c}{4}. \tag{2.7}$$

This is possible since the left hand side of (2.7) approaches 1 as $M \to \infty$.

Take $\epsilon = \frac{\alpha\epsilon_1 (r_1+r_2)! r_1^{r_1} r_2^{r_2}}{r_1! r_2! (r_1+r_2)^{r_1+r_2}}$. Based on the r_1-graph $G_1^{(r_1)}$ on n_1 vertices and the r_2-graph $G_2^{(r_2)}$ on n_2 vertices, we are going to construct an $(r_1 + r_2)$-graph

$G^{(r_1+r_2)}$ on $N = n_1 + n_2$ vertices such that $d(G^{(r_1+r_2)}) > \frac{\alpha\beta(r_1+r_2)!r_1^{r_1}r_2^{r_2}}{r_1!r_2!(r_1+r_2)^{r_1+r_2}} + \epsilon$ and any subgraph $H^{(r_1+r_2)}$ of $G^{(r_1+r_2)}$ with M vertices has density $d(H^{(r_1+r_2)}) < \frac{\alpha\beta(r_1+r_2)!r_1^{r_1}r_2^{r_2}}{r_1!r_2!(r_1+r_2)^{r_1+r_2}} + c$. If this can be done, then $\frac{\alpha\beta(r_1+r_2)!r_1^{r_1}r_2^{r_2}}{r_1!r_2!(r_1+r_2)^{r_1+r_2}}$ is a non-jump for $r_1 + r_2$.

We may take large enough $n_1, n_2 \geq n_0$ such that $\frac{n_2}{n_1} = \frac{r_2}{r_1}$ and $N = n_1 + n_2$ large enough so that (2.11) is satisfied. This is possible since we may take n_1, n_2 as large as we want to satisfy (2.1) and (2.2) or (2.3) and (2.4), respectively. If $n_2 > \frac{r_2}{r_1}n_1$, then by Fact 1.1, there exists an induced subgraph with $\frac{r_2}{r_1}n_1$ vertices satisfying (2.3) and (2.4). Clearly, $n_1 = \frac{r_1 N}{r_1+r_2}, n_2 = \frac{r_2 N}{r_1+r_2}$. Let $V(G^{(r_1+r_2)}) = V(G_1^{(r_1)}) \cup V(G_2^{(r_2)})$. Note that $|V(G^{(r_1+r_2)})| = n_1 + n_2 = N$. An (r_1+r_2)-subset of $V(G^{(r_1+r_2)})$ is an edge of $G^{(r_1+r_2)}$ if and only if it consists of r_1 vertices in $E(G_1^{(r_1)})$ and r_2 vertices in $E(G_2^{(r_2)})$. In other words, $E(G^{(r_1+r_2)}) = \{\{e_1, e_2\} \,|\, e_1 \in E(G_1^{(r_1)}), e_2 \in E(G_2^{(r_2)})\}$. Then

$$|E(G^{(r_1+r_2)})| = |E(G_1^{(r_1)})||E(G_2^{(r_2)})|. \tag{2.8}$$

The assumption $d(G_1^{(r_1)}) > \alpha$, $d(G_2^{(r_2)}) \geq \beta + \epsilon_1$ implies that

$$|E(G_1^{(r_1)})| > \alpha \binom{n_1}{r_1}, \tag{2.9}$$

$$|E(G_2^{(r_2)})| \geq (\beta + \epsilon_1) \binom{n_2}{r_2}. \tag{2.10}$$

Combining (2.8), (2.9) and (2.10), we have

$$|E(G^{(r_1+r_2)})| > \alpha(\beta+\epsilon_1) \binom{n_1}{r_1} \binom{n_2}{r_2}.$$

Therefore,

$$\begin{aligned}
d(G^{(r_1+r_2)}) &> \frac{\alpha(\beta+\epsilon_1)\binom{n_1}{r_1}\binom{n_2}{r_2}}{\binom{N}{r_1+r_2}} \\
&= \frac{n_1(n_1-1)\cdots(n_1-r_1+1)n_2(n_2-1)\cdots(n_2-r_2+1)}{N(N-1)\cdots(N-r_1-r_2+1)} \\
&\quad \cdot \frac{\alpha(\beta+\epsilon_1)(r_1+r_2)!}{r_1!r_2!} \\
&= \frac{(1-\frac{r_1+r_2}{r_1}\frac{1}{N})\cdots(1-\frac{r_1+r_2}{r_1}\frac{r_1-1}{N})(1-\frac{r_1+r_2}{r_2}\frac{1}{N})\cdots(1-\frac{r_1+r_2}{r_2}\frac{r_2-1}{N})}{(1-\frac{1}{N})\cdots(1-\frac{r_1+r_2-1}{N})} \\
&\quad \cdot \frac{\alpha(\beta+\epsilon_1)(r_1+r_2)!(\frac{r_1}{r_1+r_2})^{r_1}(\frac{r_2}{r_1+r_2})^{r_2}}{r_1!r_2!} \\
&= \frac{(1-\frac{(r_1+r_2)(r_1-1)}{2N}+o(\frac{1}{N}))(1-\frac{(r_1+r_2)(r_2-1)}{2N}+o(\frac{1}{N}))}{1-\frac{(r_1+r_2)(r_1+r_2-1)}{2N}+o(\frac{1}{N})} \\
&\quad \cdot \frac{\alpha(\beta+\epsilon_1)(r_1+r_2)!r_1^{r_1}r_2^{r_2}}{r_1!r_2!(r_1+r_2)^{r_1+r_2}} \\
&= \frac{\alpha(\beta+\epsilon_1)(r_1+r_2)!r_1^{r_1}r_2^{r_2}(1-\frac{(r_1+r_2)(r_1+r_2-2)}{2N}+o(\frac{1}{N}))}{r_1!r_2!(r_1+r_2)^{r_1+r_2}(1-\frac{(r_1+r_2)(r_1+r_2-1)}{2N}+o(\frac{1}{N}))}
\end{aligned}$$

$$> \tfrac{\alpha\beta(r_1+r_2)!r_1^{r_1}r_2^{r_2}}{r_1!r_2!(r_1+r_2)^{r_1+r_2}} + \epsilon \tag{2.11}$$

for N large enough.

The proof will be completed by showing the following claim.

Claim 2.11. Let $H^{(r_1+r_2)}$ be a subgraph of $G^{(r_1+r_2)}$ with $M \geq m_1 + m_2$ vertices. Then

$$d(H^{(r_1+r_2)}) < \frac{\alpha\beta(r_1+r_2)!r_1^{r_1}r_2^{r_2}}{r_1!r_2!(r_1+r_2)^{r_1+r_2}} + c.$$

Proof of Claim 2.11. Let $U_1 = V(H^{(r_1+r_2)}) \cap V(G_1^{(r_1)})$, $U_2 = V(H^{(r_1+r_2)}) \cap V(G_2^{(r_2)})$. Let $|U_1| = t_1$ and $|U_2| = t_2$. Note that $t_1 + t_2 = M$ and

$$|E(H^{(r_1+r_2)})| = |E(G_1^{(r_1)}[U_1])||E(G_2^{(r_2)}[U_2])|. \tag{2.12}$$

Recall that $M \geq m_1 + m_2$. So there are three possible cases as discussed below.

Case 1. If $t_1 \geq m_1$, $t_2 \geq m_2$, then by (2.2) and (2.4), we have

$$|E(G_1^{(r_1)}[U_1])| < \left(\alpha + \frac{c}{4}\right)\binom{t_1}{r_1}, \tag{2.13}$$

$$|E(G_2^{(r_2)}[U_2])| < \left(\beta + \frac{c}{4}\right)\binom{t_2}{r_2}. \tag{2.14}$$

Combining (2.12), (2.13) and (2.14), we have

$$\begin{aligned} |E(H^{(r_1+r_2)})| &< \left(\alpha + \frac{c}{4}\right)\left(\beta + \frac{c}{4}\right)\binom{t_1}{r_1}\binom{t_2}{r_2} \\ &\leq \left(\alpha + \frac{c}{4}\right)\left(\beta + \frac{c}{4}\right)\frac{t_1^{r_1}}{r_1!}\frac{t_2^{r_2}}{r_2!} \\ &= \frac{\left(\alpha + \frac{c}{4}\right)\left(\beta + \frac{c}{4}\right) r_1^{r_1} r_2^{r_2} (\frac{t_1}{r_1})^{r_1} (\frac{t_2}{r_2})^{r_2}}{r_1!r_2!}. \end{aligned}$$

Since geometric mean is no more than arithmetic mean, then

$$\begin{aligned} |E(H^{(r_1+r_2)})| &< \frac{(\alpha + \frac{c}{4})(\beta + \frac{c}{4})r_1^{r_1}r_2^{r_2}(\frac{\frac{t_1}{r_1}r_1 + \frac{t_2}{r_2}r_2}{r_1+r_2})^{r_1+r_2}}{r_1!r_2!} \\ &= \frac{(\alpha + \frac{c}{4})(\beta + \frac{c}{4})r_1^{r_1}r_2^{r_2}M^{r_1+r_2}}{r_1!r_2!(r_1+r_2)^{r_1+r_2}}. \end{aligned}$$

Therefore,

$$d(H^{(r_1+r_2)}) < \frac{\frac{(\alpha + \frac{c}{4})(\beta + \frac{c}{4})r_1^{r_1}r_2^{r_2}M^{r_1+r_2}}{r_1!r_2!(r_1+r_2)^{r_1+r_2}}}{\binom{M}{r_1+r_2}}.$$

Applying (2.7), we have

$$d(H^{(r_1+r_2)}) < \frac{(\alpha + \frac{c}{4})(\beta + \frac{c}{4})(r_1+r_2)!r_1^{r_1}r_2^{r_2}}{r_1!r_2!(r_1+r_2)^{r_1+r_2}}\left(1 + \frac{c}{4}\right)$$

$$\leq \frac{\alpha\beta(r_1+r_2)!r_1^{r_1}r_2^{r_2}}{r_1!r_2!(r_1+r_2)^{r_1+r_2}} + c.$$

Case 2. If $t_1 < m_1$, $t_2 \geq m_2$, then by (2.4), we have

$$|E(G_2^{(r_2)}[U_2])| < \left(\beta + \frac{c}{4}\right)\binom{t_2}{r_2}.$$

Combining (2.12) and (2.5), we have

$$\begin{aligned} |E(H^{(r_1+r_2)})| &< \left(\beta + \frac{c}{4}\right)\binom{t_1}{r_1}\binom{t_2}{r_2} \\ &\leq \left(\beta + \frac{c}{4}\right)\frac{t_1^{r_1}}{r_1!}\frac{t_2^{r_2}}{r_2!} \\ &< \left(\beta + \frac{c}{4}\right)\frac{m_1^{r_1}}{r_1!}\frac{M^{r_2}}{r_2!} \\ &\leq \frac{(\beta+\frac{c}{4})\left[\frac{\alpha^{\frac{1}{r_1}}r_1r_2^{\frac{r_2}{r_1}}}{(r_1+r_2)^{1+\frac{r_2}{r_1}}}M\right]^{r_1}M^{r_2}}{r_1!r_2!} \\ &= \frac{\alpha(\beta+\frac{c}{4})r_1^{r_1}r_2^{r_2}M^{r_1+r_2}}{r_1!r_2!(r_1+r_2)^{r_1+r_2}}. \end{aligned}$$

Therefore,

$$d(H^{(r_1+r_2)}) < \frac{\frac{\alpha(\beta+\frac{c}{4})r_1^{r_1}r_2^{r_2}M^{r_1+r_2}}{r_1!r_2!(r_1+r_2)^{r_1+r_2}}}{\binom{M}{r_1+r_2}}.$$

Applying (2.7), we have

$$\begin{aligned} d(H^{(r_1+r_2)}) &< \frac{\alpha(\beta+\frac{c}{4})(r_1+r_2)!r_1^{r_1}r_2^{r_2}}{r_1!r_2!(r_1+r_2)^{r_1+r_2}}\left(1+\frac{c}{4}\right) \\ &\leq \frac{\alpha\beta(r_1+r_2)!r_1^{r_1}r_2^{r_2}}{r_1!r_2!(r_1+r_2)^{r_1+r_2}} + c. \end{aligned}$$

Case 3. If $t_1 \geq m_1$, $t_2 < m_2$, then, similar to Case 2, we have

$$d(H^{(r_1+r_2)}) < \frac{\alpha\beta(r_1+r_2)!r_1^{r_1}r_2^{r_2}}{r_1!r_2!(r_1+r_2)^{r_1+r_2}} + c.$$

This completes the proof of Claim 2.11. □

Consequently, the proof of Theorem 2.9 is completed. □

3. The Turán density of extension of the $(r-3)$-fold enlargement of M_t^3

The well-known Erdős-sós conjecture says that if T is a k-vertex tree or forest then $ex(n,T) \leq \frac{n(k-2)}{2}$. In [17], Sidorenko obtained the Turán' density of the extension of the $(r-2)$-fold enlargement of T which is a graph satisfying the Erdős-sós conjecture.

Define the following function

$$f_r(x) = \frac{\prod_{i=1}^{r-1}(x+i-2)}{(x+r-3)^r}.$$

Note that $f_r(x) > 0$ on $[2,\infty)$ and $\lim_{x\to\infty} f_r(x) = 0$. Let A_r denote the last maximal of the function f_r on the interval $[2,\infty)$, so $f_r(x)$ is strictly decreasing on $[A_r,\infty)$. As pointed out in [17], A_r is non-decreasing in r.

In this section, we mainly prove the following result.

Theorem 3.1. *Let $r \geq 4$, $t \geq \max\{\frac{A_r}{3}, 3\}$ and $T = M_t^3$. Then $\gamma(\widetilde{T}) = \frac{(3t+1)^3(3t-2)(3t-3)}{(3t)^4} f_r(3t)$, where $\widetilde{T}$ is the extension of the $(r-3)$-fold enlargement of T.*

We will apply the Lagrangian method to prove Theorem 3.1.

Definition 3.2. For an r-graph G with the vertex set $[n]$, edge set $E(G)$ and a weighting $\vec{x} = (x_1,\dots,x_n) \in \mathbb{R}^n$, define

$$\lambda(G,\vec{x}) = \sum_{e\in E(G)} \prod_{i\in e} x_i.$$

The *Lagrangian* of G, denoted by $\lambda(G)$, is defined as

$$\lambda(G) = \max\{\lambda(G,\vec{x}) : \vec{x} \in \Delta\},$$

where

$$\Delta = \{\vec{x} = (x_1,x_2,\dots,x_n) \in \mathbb{R}^n : \sum_{i=1}^n x_i = 1, x_i \geq 0 \text{ for every } i \in [n]\}.$$

The value x_i is called the *weight* of the vertex i and any weighting $\vec{x} \in \Delta$ is called a *legal weighting*. An r-graph G is *dense* if and only if every proper subgraph G' of G satisfies $\lambda(G') < \lambda(G)$.

Consider now an r-graph G and one of its vertices v. Delete all edges not containing v from G and delete v from all edges containing it. The $(r-1)$-graph obtained this way is the *link* of G at vertex v. Given $0 < b < 1$ and an r-graph G on $[n]$, a *local b-bound weight assignment* of G is a weighting vector $\vec{x} = (x_1,x_2,\dots,x_n)$ of G such that $\vec{x} \in \Delta$ and $\max\{x_i : i \in [n]\} = b$. Let $\lambda_b(G) = \max\{\lambda(G,\vec{x}) : \vec{x}$ is a local b-bound weight assignment of $G\}$. Given positive integers $r \geq 3$ and $t \geq 2$, let L_t^r be the r-graph with t edges intersecting at a fixed vertex, called *r-uniform t-linear star*.

We will apply the following results given by Sidorenko.

Lemma 3.3 ([17])**.** *For every r-graph $G(r \geq 3)$ there exists a link G' of G with*

$$\lambda_b(G) \leq \frac{1}{r}(1-b)^{r-1} \max_{\delta \leq b/(1-b)} \lambda_\delta(G').$$

Lemma 3.4 ([17])**.** *If the r-graph $\widetilde{H}$ is the extension of the r-graph H, then $\gamma(\widetilde{H}) = r!\sup\lambda(G)$, where the supremum is taken over all dense H-free r-graphs G.*

The following results given in [9] will be also applied.

Lemma 3.5 ([9])**.** *Let $t \geq 2$ be a positive integer. Let G be an L_t^4-free 4-graph. Then*

$$\lambda(G) \leq \lambda(K_{3t}^4) = \frac{(3t-1)(3t-2)(3t-3)}{24 \cdot (3t)^3}.$$

Furthermore, the equality holds if and only if $K_{3t}^4 \subseteq G$.

Lemma 3.6 ([9])**.** *Let $t \geq 3$ be an integer and b a real with $0 < b < \frac{1}{3t-1}$. Let G be an M_t^3-free 3-graph with $n \geq 3t$ vertices. Then*

$$\lambda_b(G) \leq \frac{t-1}{2} b(1 - 3b + 7b^2).$$

Lemma 3.7. *Let $t \geq 3$ and $0 < b < \frac{1}{3t}$. Let $h(b) = \frac{3b(t-1)(11b^2-5b+1)}{f_4(\max\{\frac{1}{b}-1, 3t\})}$. Then $h(b) < \frac{(3t+1)^3(3t-2)(3t-3)}{(3t)^4}$.*

Proof. Recall that $f_r(x) = (x+r-3)^{-r}\prod_{i=1}^{r-1}(x+i-2)$. Clearly, $f_4(x) = \frac{x(x-1)}{(x+1)^3}$. If $b < \frac{1}{3t+1}$, then $f_4(\max\{\frac{1}{b}-1, 3t\}) = f_4(\frac{1}{b}-1)$. If $\frac{1}{3t+1} \leq b < \frac{1}{3t}$, then $f_4(\max\{\frac{1}{b}-1, 3t\}) = f_4(3t)$. Hence

$$h(b) = \begin{cases} \frac{3b(t-1)(11b^2-5b+1)}{f_4(\frac{1}{b}-1)} & \text{if } 0 < b < \frac{1}{3t+1}; \\ \frac{3b(t-1)(11b^2-5b+1)}{f_4(3t)} & \text{if } \frac{1}{3t+1} \leq b < \frac{1}{3t}. \end{cases}$$

Note that $h(b)$ is a continuous function on $(0, \frac{1}{3t})$.

Case 1. $0 < b < \frac{1}{3t+1}$.

Since $f_4(\frac{1}{b}-1) = b(1-b)(1-2b)$, then $h(b) = \frac{3(t-1)(11b^2-5b+1)}{(1-b)(1-2b)}$.

Let $g(b) = \frac{11b^2-5b+1}{(1-b)(1-2b)}$. Then $h(b) = 3(t-1)g(b)$. We consider the derivative of $g(b)$ and get

$$g'(b) = \frac{-23b^2 + 18b - 2}{(1 + 2b^2 - 3b)^2}.$$

Note that if $t \geq 3$ and $0 < b < \frac{1}{3t+1}$, then $-23b^2 + 18b - 2 < 0$ and $g'(b) < 0$. Hence, $h(b)$ is decreasing on $(0, \frac{1}{3t+1})$ and thus

$$h(b) \leq \lim_{b \to 0} h(b) = 3(t-1) < \frac{(3t+1)^3(3t-2)(3t-3)}{(3t)^4}. \tag{3.15}$$

Case 2. $\frac{1}{3t+1} \leq b < \frac{1}{3t}$.

Note that $f_4(3t) = \frac{3t(3t-1)}{(3t+1)^3}$. Then

$$h(b) = \frac{(t-1)(3t+1)^3 b(11b^2 - 5b + 1)}{t(3t-1)}.$$

It is easy to verify that $h'(b) > 0$ and $h(b)$ is increasing on $[\frac{1}{3t+1}, \frac{1}{3t}]$. Hence

$$h(b) \le h(\frac{1}{3t}) = \frac{3(t-1)(3t+1)^3(9t^2-15t+11)}{(3t)^4(3t-1)}$$
$$< \frac{(3t+1)^3(3t-2)(3t-3)}{(3t)^4}, \tag{3.16}$$

where the last inequality holds since $t \ge 3$.

Inequalities (3.15) and (3.16) implies that $h(b) < \frac{(3t+1)^3(3t-2)(3t-3)}{(3t)^4}$. This completes the proof. □

Lemma 3.8. *Let $r \ge 4$, $t \ge \max\{\frac{A_{r-1}}{3}, 3\}$ and $T = M_t^3$. If the r-graph G contains no $(r-3)$-fold enlargement of T as a subgraph, then $\lambda_b(G) \le \frac{(3t+1)^3(3t-2)(3t-3)}{r!(3t)^4} f_r(x)$, where $x = \max\{\frac{1}{b} - r + 3, 3t\}$.*

Proof. We use induction on r. For the basis step, let $r = 4$. If $b \ge \frac{1}{3t}$, then $f_4(\max\{\frac{1}{b} - 1, 3t\}) = f_4(3t) = \frac{3t(3t-1)}{(3t+1)^3}$, $\frac{(3t+1)^3(3t-2)(3t-3)}{4!(3t)^4} f_4(\max\{\frac{1}{b} - 1, 3t\}) = \frac{(3t-1)(3t-2)(3t-3)}{24\cdot(3t)^3}$. By Lemma 3.5, the lemma holds. Therefore, we assume that $0 < b < \frac{1}{3t}$. By Lemma 3.3, we have

$$\lambda_b(G) \le \frac{1}{4}(1-b)^3 \max_{\delta \le \frac{b}{1-b}} \lambda_\delta(G'),$$

where G' is a link of G. If we apply Lemma 3.6 to the link G', then

$$\lambda_b(G) \le \frac{3}{4!}(t-1)(1-b)^3\delta(1 - 3\delta + 7\delta^2).$$

Let $f(\delta) = \delta(1 - 3\delta + 7\delta^2)$. It is easy to verify that $f(\delta)$ is increasing in $(0, \frac{b}{1-b}) \subseteq (0, \frac{1}{3t-1})$. So

$$\lambda_b(G) \le \frac{3(t-1)(1-b)^3 \frac{b}{1-b}(1 - 3\frac{b}{1-b} + 7(\frac{b}{1-b})^2)}{24 f_4(\max\{\frac{1}{b} - 1, 3t\})} f_4(x)$$
$$= \frac{3b(t-1)(11b^2 - 5b + 1)}{24 f_4(\max\{\frac{1}{b} - 1, 3t\})} f_4(x)$$

where $x = \max\{\frac{1}{b} - r + 3, 3t\} = \max\{\frac{1}{b} - 1, 3t\}$ when $r = 4$. Applying Lemma 3.7, we have

$$\lambda_b(G) < \frac{(3t+1)^3(3t-2)(3t-3)}{24(3t)^4} f_4(x).$$

Now we prove the induction step from $r-1$ to r. Since G contains no $(r-3)$-fold enlargement of M_t^3 as a subgraph and A_r is non-decreasing as r increases, then link G' contains no $(r-4)$-fold enlargement of M_t^3 as a subgraph and $t \ge \max\{\frac{A_{r-2}}{3}, 3\}$. So G' satisfies the induction hypothesis for $r-1$, and $\lambda_\delta(G') \le \frac{(3t+1)^3(3t-2)(3t-3)}{(r-1)!(3t)^4} f_{r-1}(y)$, where $y = \max\{\frac{1}{\delta} - r + 4, 3t\}$. If $\delta \le \frac{b}{1-b}$, then $\frac{1}{\delta} \geq \frac{1}{b} - 1$ and $y \geq x = \max\{\frac{1}{b} - r + 3, 3t\}$. Recall that $f_{r-1}(z)$ decreases when $z \geq A_{r-1}$, so $f_{r-1}(y) \leq f_{r-1}(x)$. By Lemma 3.3, we get

$$\begin{aligned}
\lambda_b(G) &\leq \frac{1}{r}(1-b)^{r-1} \max_{\delta \leq \frac{b}{1-b}} \lambda_\delta(G') \\
&\leq \frac{1}{r}(1 - \frac{1}{x+r-3})^{r-1} \max_{\delta \leq \frac{b}{1-b}} \frac{(3t+1)^3(3t-2)(3t-3)}{(r-1)!(3t)^4} f_{r-1}(y) \\
&\leq \frac{(3t+1)^3(3t-2)(3t-3)}{r!(3t)^4}(1 - \frac{1}{x+r-3})^{r-1} f_{r-1}(x) \\
&\leq \frac{(3t+1)^3(3t-2)(3t-3)}{r!(3t)^4} f_r(x).
\end{aligned}$$

□

Now we are going to prove Theorem 3.1.

Proof of Theorem 3.1. Let $r \geq 4$, $t \geq \max\{\frac{A_r}{3}, 3\}$ and $T = M_t^3$. Clearly, $3t \geq A_r$. Let G be an r-graph which contains no $(r-3)$-fold enlargement of T as a subgraph. By lemma 3.8, if $\frac{1}{b} - r + 3 \leq 3t$, then

$$\lambda_b(G) \leq \frac{(3t+1)^3(3t-2)(3t-3)}{r!(3t)^4} f_r(3t).$$

If $3t < \frac{1}{b} - r + 3$, since $f_r(x)$ is decreasing in $[A_r, \infty)$, then

$$\begin{aligned}
\lambda_b(G) &\leq \frac{(3t+1)^3(3t-2)(3t-3)}{r!(3t)^4} f_r\left(\frac{1}{b} - r + 3\right) \\
&< \frac{(3t+1)^3(3t-2)(3t-3)}{r!(3t)^4} f_r(3t).
\end{aligned}$$

Therefore,

$$\lambda(G) \leq \max_{0<b<1}\{\lambda_b(G)\} = \frac{(3t+1)^3(3t-2)(3t-3)}{r!(3t)^4} f_r(3t).$$

By Lemma 3.4, if $\widetilde{T}$ is the extension of the $(r-3)$-fold enlargement of T, then we have

$$\begin{aligned}
\lambda(\widetilde{T}) = r! \sup \lambda(G) &= r! \frac{(3t+1)^3(3t-2)(3t-3)}{r!(3t)^4} f_r(3t) \\
&= \frac{(3t+1)^3(3t-2)(3t-3)}{(3t)^4} f_r(3t).
\end{aligned}$$

This completes the proof of Theorem 3.1. □

4. Remarks

It would be interesting to study whether results similar to Theorem 2.7 hold for Turán densities. In general, is there a way to generate a number in Γ_{r+s} from numbers in Γ_r and Γ_s?

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