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GRADIENT ESTIMATES OF A NONLINEAR ELLIPTIC EQUATION FOR THE V-LAPLACIAN

Fanqi Zeng

Abstract. In this paper, we consider gradient estimates for positive solutions to the following nonlinear elliptic equation on a complete Riemannian manifold:

$$\Delta_V u + cu^{\alpha} = 0,$$

where c, α are two real constants and $c \neq 0$. By applying Bochner formula and the maximum principle, we obtain local gradient estimates for positive solutions of the above equation on complete Riemannian manifolds with Bakry-Émery Ricci curvature bounded from below, which generalize some results of [8].

1. Introduction

Let (M^n, g) be an n-dimensional complete Riemannian manifold. The V-Laplacian is defined by

$$\Delta_V \cdot = \Delta + \langle V, \nabla \cdot \rangle,$$

where V is a smooth vector field on M. Here ∇ and Δ are the Levi-Civita connection and Laplacian with respect to metric g, respectively. The V-Laplacian is an important generalization of the Laplacian, as well as V-harmonic maps introduced in [2]. We define the ∞-Bakry-Émery curvature and N-Bakry-Émery curvature as follows: [2, 6]

$$\mathrm{Ric_V} = \mathrm{Ric} - \frac{1}{2}\mathcal{L}_V g, \tag{1.1}$$

$$\mathrm{Ric_V^N} = \mathrm{Ric_V} - \frac{1}{N} V \otimes V, \tag{1.2}$$

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where $N > 0$ is a natural number, Ric is the Ricci curvature of M and $\mathcal{L}_V$ denotes the Lie derivative along the direction V. In particular, we use the convention that $N = 0$ if and only if $V \equiv 0$.

In this paper, we want to study positive solutions of the nonlinear elliptic equation with the V-Laplacian

$$\Delta_V u + cu^\alpha = 0 \tag{1.3}$$

on an n-dimensional complete Riemannian manifold (M^n, g), where c, α are two real constants and $c \neq 0$. When $V = 0$, the above equation (1.3) reduces to

$$\Delta u + cu^\alpha = 0. \tag{1.4}$$

For c a function, the equation (1.4) is studied by Gidas and Spruck in [3] with $1 \le \alpha \le \frac{n+2}{n-2}$ when $n > 2$ and lather it is studied by Li in [5] to achieve gradient estimates and Liouville type results with $1 \le \alpha \le \frac{n}{n-2}$ when $n > 2$. If $c < 0$ and $\alpha < 0$, the equation (1.4) on a bounded smooth domain in $\mathbb{R}^n$ is known as the thin film equation, which describes a steady state of the thin film (see [4]). More progress of this and related equations can be found in [7, 9, 10, 12] and the references therein.

Recently, inspired by the methods used by Yau in [11] and Brighton in [1], Ma, Huang and Luo [8] derived local gradient estimates for positive solutions of equations (1.4). We want to generalize their results to equation (1.3) and we obtain the following results.

Theorem 1.1. *Let (M^n, g) be an n-dimensional Riemannian manifold with $\mathrm{Ric}_V^N(B_p(2R)) \ge -K$, where K is a non-negative constant. Suppose that u is a positive solution to the equation (1.3) on $B_p(2R)$. Then on $B_p(R)$, we have the following inequalities.*

(1) *If $c < 0$ and $\alpha > 0$, then we have*

$$\begin{aligned} &|\nabla u|(x) \\ \le &\frac{M}{\epsilon\sqrt{C_1}}\sqrt{2K + \frac{1}{R^2}\left[\left(R\sqrt{(n-1)K} + n - 1\right)c_1 + c_2 + \left(2 + \frac{C_2^2}{C_1}\right)c_1^2\right]}, \end{aligned} \tag{1.5}$$

where $M = \sup_{x \in B_p(2R)} u(x)$, the c_1 and c_2 are positive constants, and the positive constants C_1 and C_2 are given by

$$C_1 = \frac{(\epsilon - 1)^2}{(n+N)\epsilon^2} - \frac{\epsilon - 1}{\epsilon},\ C_2 = \frac{1-\epsilon}{\epsilon},$$

respectively. Here $\epsilon \in (0, 1)$ is close enough to 1.

(2) *If $c > 0$ and $\frac{n+N+2}{2(n+N-1)} < \alpha < \frac{2(n+N)^2 + 9(n+N) + 6}{2(n+N)(n+N+2)}$ with $n \ge 3$, then we have*

(1.6) $|\nabla u|(x)$

$$\leq \frac{M}{\tilde{\epsilon}\sqrt{C_3}}\sqrt{2K + \frac{1}{R^2}\left[\left(R\sqrt{(n-1)K} + n - 1\right)c_1 + c_2 + \left(2 + \frac{C_4^2}{C_3}\right)c_1^2\right]},$$

where M, c_1 and c_2 are the same as (1.5), and the positive constants C_3 and C_4 are given by

$$C_3 = \frac{1}{2}\left[\left(\frac{(\tilde{\epsilon}-1)^2}{(n+N)\tilde{\epsilon}^2} - \frac{\tilde{\epsilon}-1}{\tilde{\epsilon}}\right) - \frac{n+N}{\tilde{\epsilon}^2}\left(\frac{(n+N)+2}{n+N}(\tilde{\epsilon}-1) + \alpha\right)^2\right],$$

$$C_4 = \frac{4(\alpha-1)(n+N)(n+N+2) + (n+N)[2(n+N)+5]}{[5(n+N)+6] - 4(\alpha-1)(n+N)(n+N+2)},$$

respectively. Here $\tilde{\epsilon} = \frac{[5(n+N)+6] - 2(\alpha-1)[(n+N)^2 + 2(n+N)]}{2[(n+N)^2 + 5(n+N) + 3]}$.

Letting $R \to \infty$ in (1.5) and (1.6), we obtain the following gradient estimates on complete noncompact Riemannian manifolds:

Corollary 1.2. *Let (M^n, g) be an n-dimensional complete noncompact Riemannian manifold with $\mathrm{Ric}_{\mathrm{V}}^{\mathrm{N}} \geq -K$, where K is a non-negative constant. Let u be a positive solution to the equation (1.3). Then, we have the following inequalities.*

(1) *If $c < 0$ and $\alpha > 0$, then we have*

$$|\nabla u|(x) \leq \frac{M}{\epsilon\sqrt{C_1}}\sqrt{2K}; \tag{1.7}$$

(2) *If $c > 0$ and $\frac{n+N+2}{2(n+N-1)} < \alpha < \frac{2(n+N)^2 + 9(n+N) + 6}{2(n+N)(n+N+2)}$ with $n \geq 3$, then we have*

$$|\nabla u|(x) \leq \frac{M}{\tilde{\epsilon}\sqrt{C_3}}\sqrt{2K}, \tag{1.8}$$

where $M = \sup_{x \in M} u(x)$.

We can also obtain similar results under the assumption that $\mathrm{Ric}_{\mathrm{V}}$ is bounded by below.

Theorem 1.3. *Let (M^n, g) be an n-dimensional Riemannian manifold with $\mathrm{Ric}_{\mathrm{V}}(B_p(2R)) \geq -\widetilde{K}$, and $|V| \leq L$, where $\widetilde{K}$ and L are non-negative constants. Suppose that u is a positive solution to the equation (1.3) on $B_p(2R)$. Then on $B_p(R)$, we have the following inequalities.*

(1) *If $c < 0$ and $\alpha > 0$, then we have*

$$|\nabla u|(x) \leq \frac{M}{\epsilon\sqrt{\widetilde{C}_1}}\sqrt{2\widetilde{K} + \frac{1}{R^2}\left[\left(R\sqrt{(n-1)\widetilde{K}} + RL + n - 1\right)c_1 + c_2 + \left(2 + \frac{\widetilde{C}_2^2}{\widetilde{C}_1}\right)c_1^2\right]}, \tag{1.9}$$

where $M = \sup_{x \in B_p(2R)} u(x)$*, the* c_1 *and* c_2 *are positive constants, and the positive constants* $\widetilde{C}_1$ *and* $\widetilde{C}_2$ *are given by*

$$\widetilde{C}_1 = \frac{(\epsilon-1)^2}{n\epsilon^2} - \frac{\epsilon-1}{\epsilon},\ \widetilde{C}_2 = \frac{1-\epsilon}{\epsilon},$$

respectively. Here $\epsilon \in (0,1)$ *is close enough to* 1*.*

(2) *If* $c > 0$ *and* $\frac{n+2}{2(n-1)} < \alpha < \frac{2n^2+9n+6}{2n(n+2)}$ *with* $n \geq 3$*, then we have*

$$|\nabla u|(x) \leq \frac{M}{\tilde{\epsilon}\sqrt{\widetilde{C}_3}} \sqrt{2\widetilde{K} + \frac{1}{R^2}\left[\left(R\sqrt{(n-1)\widetilde{K}} + RL + n - 1\right)c_1 + c_2 + \left(2 + \frac{\widetilde{C}_4^2}{\widetilde{C}_3}\right)c_1^2\right]}, \tag{1.10}$$

where M*,* c_1 *and* c_2 *are the same as* (1.9)*, and the positive constants* $\widetilde{C}_3$ *and* $\widetilde{C}_4$ *are given by*

$$\widetilde{C}_3 = \frac{1}{2}\left[\left(\frac{(\tilde{\epsilon}-1)^2}{n\tilde{\epsilon}^2} - \frac{\tilde{\epsilon}-1}{\tilde{\epsilon}}\right) - \frac{n}{\tilde{\epsilon}^2}\left(\frac{n+2}{n}(\tilde{\epsilon}-1) + \alpha\right)^2\right],$$

$$\widetilde{C}_4 = \frac{4(\alpha-1)n(n+2) + n(2n+5)}{(5n+6) - 4(\alpha-1)n(n+2)},$$

respectively. Here $\tilde{\epsilon} = \frac{(5n+6)-2(\alpha-1)(n^2+2n)}{2(n^2+5n+3)}$*.*

Corollary 1.4. *Let* (M^n, g) *be an* n*-dimensional complete noncompact Riemannian manifold with* $\mathrm{Ric_V} \geq -\widetilde{K}$*, and* $|V| \leq L$*, where* $\widetilde{K}$ *and* L *are nonnegative constants. Let* u *be a positive solution to the equation* (1.3)*. Then, we have the following inequalities.*

(1) *If* $c < 0$ *and* $\alpha > 0$*, then we have*

$$|\nabla u|(x) \leq \frac{M}{\epsilon\sqrt{\widetilde{C}_1}}\sqrt{2\widetilde{K}}; \tag{1.11}$$

(2) *If* $c > 0$ *and* $\frac{n+2}{2(n-1)} < \alpha < \frac{2n^2+9n+6}{2n(n+2)}$ *with* $n \geq 3$*, then we have*

$$|\nabla u|(x) \leq \frac{M}{\tilde{\epsilon}\sqrt{\widetilde{C}_3}}\sqrt{2\widetilde{K}}, \tag{1.12}$$

where $M = \sup_{x \in M} u(x)$*.*

Remark 1.1. Clearly, our results generalize some results of [8] with respect to the nonlinear elliptic equation (1.3) with $V = 0$.

2. The proof of theorems

We firstly give the following lemma.

Lemma 2.1. *Let (M^n, g) be an n-dimensional complete Riemannian manifold with $\mathrm{Ric}_V^N(B_p(2R)) \geq -K$, where K is a nonnegative constant. Assuming that u is a positive solution to nonlinear elliptic equation (1.3) on $B_p(2R)$. Denote $h = u^\epsilon$ with $\epsilon \neq 0$. Then on $B_p(R)$, the following inequalities hold.*

(a) *If $c < 0$ and $\alpha > 0$, then there exists $\epsilon \in (0, 1)$ such that*

$$\frac{1}{2}\Delta_V |\nabla h|^2 \geq \left(\frac{(\epsilon-1)^2}{(n+N)\epsilon^2} - \frac{\epsilon-1}{\epsilon} \right) \frac{|\nabla h|^4}{h^2} + \frac{\epsilon-1}{\epsilon}\frac{\nabla h}{h}\nabla(|\nabla h|^2) - K|\nabla h|^2. \tag{2.1}$$

(b) *If $c > 0$ and for a fixed α, there exist two positive constants ϵ, δ such that*

$$c\left[\frac{n+N+2}{n+N}(\epsilon-1)+\alpha\right] > 0 \tag{2.2}$$

and

$$\frac{c^2\epsilon^2}{n+N} - \frac{c}{\delta}\left(\frac{n+N+2}{n+N}(\epsilon-1)+\alpha\right) > 0, \tag{2.3}$$

then we have

$$\frac{1}{2}\Delta_V |\nabla h|^2 \geq \left[\frac{(\epsilon-1)^2}{(n+N)\epsilon^2} - \frac{\epsilon-1}{\epsilon} - c\delta\left(\frac{n+N+2}{n+N}(\epsilon-1)+\alpha\right)\right] \frac{|\nabla h|^4}{h^2} + \frac{\epsilon-1}{\epsilon}\frac{\nabla h}{h}\nabla(|\nabla h|^2) - K|\nabla h|^2. \tag{2.4}$$

Proof. Let $h = u^\epsilon$, where $\epsilon \neq 0$ is a constant to be determined. Then we have

$$\log h = \log u^\epsilon = \epsilon \log u.$$

A simple calculation implies

$$\begin{aligned} \Delta_V h &= \Delta(u^\epsilon) + \langle V, \nabla(u^\epsilon)\rangle \\ &= \epsilon(\epsilon-1)u^{\epsilon-2}|\nabla u|^2 + \epsilon u^{\epsilon-1}\Delta_V u \\ &= \epsilon(\epsilon-1)u^{\epsilon-2}|\nabla u|^2 - c\epsilon u^{\alpha+\epsilon-1} \\ &= \frac{\epsilon-1}{\epsilon}\frac{|\nabla h|^2}{h} - c\epsilon h^{\frac{\alpha+\epsilon-1}{\epsilon}}. \end{aligned} \tag{2.5}$$

Therefore we get

$$\begin{aligned} &\nabla h \nabla \Delta_V h \\ &= \nabla h \nabla \left(\frac{\epsilon-1}{\epsilon}\frac{|\nabla h|^2}{h} - c\epsilon h^{\frac{\alpha+\epsilon-1}{\epsilon}} \right) \\ &= \frac{\epsilon-1}{\epsilon}\nabla h \nabla \frac{|\nabla h|^2}{h} - c(\alpha+\epsilon-1)h^{\frac{\alpha+\epsilon-1}{\epsilon}}\frac{|\nabla h|^2}{h} \end{aligned} \tag{2.6}$$

$$= \frac{\epsilon-1}{\epsilon h}\nabla h\nabla(|\nabla h|^2) - \frac{\epsilon-1}{\epsilon}\frac{|\nabla h|^4}{h^2} - c(\alpha+\epsilon-1)h^{\frac{\alpha+\epsilon-1}{\epsilon}}\frac{|\nabla h|^2}{h}.$$

Applying (2.5) and (2.6) into the famous Bochner formula to h, we have

$$\begin{aligned}
(2.7)\qquad & \frac{1}{2}\Delta_V|\nabla h|^2 \\
&= |\nabla^2 h|^2 + \nabla h\nabla\Delta_V h + \mathrm{Ric_V}(\nabla h,\nabla h) \\
&\geq \frac{1}{n+N}(\Delta_V h)^2 + \nabla h\nabla\Delta_V h + \mathrm{Ric_V^N}(\nabla h,\nabla h) \\
&\geq \frac{1}{n+N}\left(\frac{\epsilon-1}{\epsilon}\frac{|\nabla h|^2}{h} - c\epsilon h^{\frac{\alpha+\epsilon-1}{\epsilon}}\right)^2 + \nabla h\nabla\Delta_V h - K|\nabla h|^2 \\
&= \left(\frac{(\epsilon-1)^2}{(n+N)\epsilon^2} - \frac{\epsilon-1}{\epsilon}\right)\frac{|\nabla h|^4}{h^2} \\
&\quad - c\left(\frac{n+N+2}{n+N}(\epsilon-1)+\alpha\right)h^{\frac{\alpha+\epsilon-1}{\epsilon}}\frac{|\nabla h|^2}{h} \\
&\quad + \frac{c^2\epsilon^2}{n+N}h^{\frac{2(\alpha+\epsilon-1)}{\epsilon}} + \frac{\epsilon-1}{\epsilon}\frac{\nabla h}{h}\nabla(|\nabla h|^2) - K|\nabla h|^2.
\end{aligned}$$

First, we prove (a).

In (2.7), if $c<0$ and $\alpha>0$, we can choose $\epsilon\in(0,1)$ close enough to 1 such that

$$-c\left(\frac{n+N+2}{n+N}(\epsilon-1)+\alpha\right)\geq 0,$$

and then (2.1) follows directly.

Next, we prove (b).

For a fixed point p, if there exists a positive constant δ such that $h^{\frac{\alpha+\epsilon-1}{\epsilon}} \leq \delta\frac{|\nabla h|^2}{h}$, according to (2.2), then (2.7) becomes

$$\begin{aligned}
(2.8)\qquad & \frac{1}{2}\Delta_V|\nabla h|^2 \\
&\geq \left[\frac{(\epsilon-1)^2}{(n+N)\epsilon^2} - \frac{\epsilon-1}{\epsilon} - c\delta\left(\frac{n+N+2}{n+N}(\epsilon-1)+\alpha\right)\right]\frac{|\nabla h|^4}{h^2} \\
&\quad + \frac{c^2\epsilon^2}{n+N}h^{\frac{2(\alpha+\epsilon-1)}{\epsilon}} + \frac{\epsilon-1}{\epsilon}\frac{\nabla h}{h}\nabla(|\nabla h|^2) - K|\nabla h|^2 \\
&\geq \left[\frac{(\epsilon-1)^2}{(n+N)\epsilon^2} - \frac{\epsilon-1}{\epsilon} - c\delta\left(\frac{n+N+2}{n+N}(\epsilon-1)+\alpha\right)\right]\frac{|\nabla h|^4}{h^2} \\
&\quad + \frac{\epsilon-1}{\epsilon}\frac{\nabla h}{h}\nabla(|\nabla h|^2) - K|\nabla h|^2.
\end{aligned}$$

On the contrary, at the point p, if $h^{\frac{\alpha+\epsilon-1}{\epsilon}} \geq \delta\frac{|\nabla h|^2}{h}$, then (2.7) becomes

$$(2.9)\qquad \frac{1}{2}\Delta_V|\nabla h|^2$$

$$
\begin{aligned}
&\geq \left(\frac{(\epsilon-1)^2}{(n+N)\epsilon^2}-\frac{\epsilon-1}{\epsilon}\right)\frac{|\nabla h|^4}{h^2}\\
&\quad+\left[\frac{c^2\epsilon^2}{n+N}-\frac{c}{\delta}\left(\frac{n+N+2}{n+N}(\epsilon-1)+\alpha\right)\right]h^{\frac{2(\alpha+\epsilon-1)}{\epsilon}}\\
&\quad+\frac{\epsilon-1}{\epsilon}\frac{\nabla h}{h}\nabla(|\nabla h|^2)-K|\nabla h|^2\\
&\geq \left\{\left(\frac{(\epsilon-1)^2}{(n+N)\epsilon^2}-\frac{\epsilon-1}{\epsilon}\right)\right.\\
&\quad\left.+\delta^2\left[\frac{c^2\epsilon^2}{n+N}-\frac{c}{\delta}\left(\frac{n+N+2}{n+N}(\epsilon-1)+\alpha\right)\right]\right\}\frac{|\nabla h|^4}{h^2}\\
&\quad+\frac{\epsilon-1}{\epsilon}\frac{\nabla h}{h}\nabla(|\nabla h|^2)-K|\nabla h|^2\\
&\geq \left[\frac{(\epsilon-1)^2}{(n+N)\epsilon^2}-\frac{\epsilon-1}{\epsilon}-c\delta\left(\frac{n+N+2}{n+N}(\epsilon-1)+\alpha\right)\right]\frac{|\nabla h|^4}{h^2}\\
&\quad+\frac{\epsilon-1}{\epsilon}\frac{\nabla h}{h}\nabla(|\nabla h|^2)-K|\nabla h|^2
\end{aligned}
$$

as long as

$$
\frac{c^2\epsilon^2}{n+N}-\frac{c}{\delta}\left(\frac{n+N+2}{n+N}(\epsilon-1)+\alpha\right)>0. \tag{2.10}
$$

In both cases, (2.4) holds always. We complete the proof of Lemma 2.1. □

In order to obtain the upper bound of $|\nabla h|$ by using the maximum principle, it is sufficient to choose the coefficient of $\frac{|\nabla h|^4}{h^2}$ in (2.1) and (2.4) such that it is positive. In (2.4) of Lemma 2.1, we need to choose appropriate ϵ, δ such that

$$
\frac{(\epsilon-1)^2}{(n+N)\epsilon^2}-\frac{\epsilon-1}{\epsilon}-\delta c\left(\frac{n+N+2}{n+N}(\epsilon-1)+\alpha\right)>0. \tag{2.11}
$$

Under the assumption of (2.2), the inequality (2.3) becomes

$$
\delta>\frac{(n+N)c}{c^2\epsilon^2}\left(\frac{n+N+2}{n+N}(\epsilon-1)+\alpha\right) \tag{2.12}
$$

and (2.11) becomes

$$
\delta<\frac{\frac{(\epsilon-1)^2}{(n+N)\epsilon^2}-\frac{\epsilon-1}{\epsilon}}{c\left(\frac{n+N+2}{n+N}(\epsilon-1)+\alpha\right)}. \tag{2.13}
$$

In order to ensure we can choose a positive δ, from (2.12) and (2.13), we need choose an ϵ satisfying

$$
\frac{(n+N)c}{c^2\epsilon^2}\left(\frac{n+N+2}{n+N}(\epsilon-1)+\alpha\right)<\frac{\frac{(\epsilon-1)^2}{(n+N)\epsilon^2}-\frac{\epsilon-1}{\epsilon}}{c\left(\frac{n+N+2}{n+N}(\epsilon-1)+\alpha\right)}, \tag{2.14}
$$

which is equivalent to

$$\begin{aligned}(2.15)\quad & [(n+N)^2+5(n+N)+3]\epsilon^2+\{2(\alpha-1)[(n+N)^2+2(n+N)] \\ & -[5(n+N)+6]\}\epsilon+(\alpha-1)^2(n+N)^2-4(\alpha-1)(n+N)+3<0.\end{aligned}$$

By a direct calculation, under the condition

$$\begin{aligned}(2.16)\quad & \frac{-(n+N-4)-\sqrt{(n+N)^2+5(n+N)+3}}{2(n+N-1)} \\ & <\alpha-1 \\ & <\frac{-(n+N-4)+\sqrt{(n+N)^2+5(n+N)+3}}{2(n+N-1)},\end{aligned}$$

we have

$$\begin{aligned}(2.17)\quad & \{2(\alpha-1)[(n+N)^2+2(n+N)]-[5(n+N)+6]\}^2 \\ & -4[(n+N)^2+5(n+N)+3] \\ & \times[(\alpha-1)^2(n+N)^2-4(\alpha-1)(n+N)+3] \\ = & (n+N)^2\{-4(n+N-1)(\alpha-1)^2-4(n+N-4)(\alpha-1)+13\}>0,\end{aligned}$$

which shows the quadratic inequality (2.15) with respect to ϵ has two real roots.

Now we are ready to prove the following proposition which plays a key role in the proof of main results.

Proposition 2.2. *Let (M^n, g) be an n-dimensional complete Riemannian manifold with $\mathrm{Ric}_V^N(B_p(2R)) \geq -K$, where K is a nonnegative constant. Assuming that u is a positive solution to nonlinear elliptic equation (1.3) on $B_p(2R)$. Denote $h = u^\epsilon$ with $\epsilon \neq 0$. Then on $B_p(R)$ the following inequalities hold.*

(c) *If $c < 0$ and $\alpha > 0$, then we have*

$$(2.18)\quad \frac{1}{2}\Delta_V|\nabla h|^2 \geq C_1\frac{|\nabla h|^4}{h^2} - C_2\frac{\nabla h}{h}\nabla(|\nabla h|^2) - K|\nabla h|^2,$$

where positive constants C_1 and C_2 are given by

$$C_1 = \frac{(\epsilon-1)^2}{(n+N)\epsilon^2} - \frac{\epsilon-1}{\epsilon},$$
$$C_2 = \frac{1-\epsilon}{\epsilon},$$

respectively.

(d) *If $c > 0$ and $\frac{n+N+2}{2(n+N-1)} < \alpha < \frac{2(n+N)^2+9(n+N)+6}{2(n+N)(n+N+2)}$ with $n \geq 3$, then we have*

$$(2.19)\quad \frac{1}{2}\Delta_V|\nabla h|^2 \geq C_3\frac{|\nabla h|^4}{h^2} - C_4\frac{\nabla h}{h}\nabla(|\nabla h|^2) - K|\nabla h|^2,$$

where positive constants C_3 *and* C_4 *are given by*

$$C_3 = \frac{1}{2}\left[\left(\frac{(\tilde{\epsilon}-1)^2}{(n+N)\tilde{\epsilon}^2} - \frac{\tilde{\epsilon}-1}{\tilde{\epsilon}}\right) - \frac{n+N}{\tilde{\epsilon}^2}\left(\frac{(n+N)+2}{n+N}(\tilde{\epsilon}-1)+\alpha\right)^2\right],$$

$$C_4 = \frac{4(\alpha-1)(n+N)(n+N+2)+(n+N)[2(n+N)+5]}{[5(n+N)+6]-4(\alpha-1)(n+N)(n+N+2)},$$

respectively. Here $\tilde{\epsilon} = \frac{[5(n+N)+6]-2(\alpha-1)[(n+N)^2+2(n+N)]}{2[(n+N)^2+5(n+N)+3]}$.

Proof. We prove this proposition case by case.

(c) The case of $c<0$ and $\alpha>0$. In the proof of Lemma 2.1 we see that by choosing an $\epsilon\in(0,1)$ such that $\frac{n+N+2}{n+N}(\epsilon-1)+\alpha\geq 0$ we get the

$$\frac{1}{2}\Delta_V|\nabla h|^2 \geq \left(\frac{(\epsilon-1)^2}{(n+N)\epsilon^2} - \frac{\epsilon-1}{\epsilon}\right)\frac{|\nabla h|^4}{h^2} + \frac{\epsilon-1}{\epsilon}\frac{\nabla h}{h}\nabla(|\nabla h|^2) - K|\nabla h|^2. \tag{2.20}$$

Then we see that $C_1 = \frac{(\epsilon-1)^2}{(n+N)\epsilon^2} - \frac{\epsilon-1}{\epsilon} > 0$ and $C_2 = \frac{1-\epsilon}{\epsilon} > 0$.

(d) The case of $c>0$ and $\frac{n+N+2}{2(n+N-1)} < \alpha < \frac{2(n+N)^2+9(n+N)+6}{2(n+N)(n+N+2)}$ when $n\geq 3$. In this case, (2.2) is equivalent to

$$\epsilon > 1 - \frac{(n+N)\alpha}{n+N+2}. \tag{2.21}$$

We can check

$$\frac{5(n+N)+6}{2[(n+N)^2+2(n+N)]} < \frac{-(n+N-4)+\sqrt{(n+N)^2+5(n+N)+3}}{2(n+N-1)}. \tag{2.22}$$

Hence, when $n\geq 3$, for any α satisfies

$$-\frac{n+N-4}{2(n+N-1)} < \alpha - 1 < \frac{5(n+N)+6}{2[(n+N)^2+2(n+N)]} \tag{2.23}$$

which is equivalent to

$$-\frac{n+N+2}{2(n+N-1)} < \alpha < \frac{2(n+N)^2+9(n+N)+6}{2(n+N)(n+N+2)}, \tag{2.24}$$

then (2.21) is satisfied by choosing

$$\epsilon := \tilde{\epsilon} = \frac{[5(n+N)+6]-2(\alpha-1)[(n+N)^2+2(n+N)]}{2[(n+N)^2+5(n+N)+3]}, \tag{2.25}$$

and it is easy to check that $\epsilon\in(0,1)$.

In particular, we let

$$\begin{aligned}\delta &:= \tilde{\delta} \\ &= \frac{1}{2}\left[\frac{(n+N)c}{c^2\tilde{\epsilon}^2}\left(\frac{n+N+2}{n+N}(\tilde{\epsilon}-1)+\alpha\right) + \frac{\frac{(\tilde{\epsilon}-1)^2}{(n+N)\tilde{\epsilon}^2} - \frac{\tilde{\epsilon}-1}{\tilde{\epsilon}}}{c\left(\frac{n+N+2}{n+N}(\tilde{\epsilon}-1)+\alpha\right)}\right],\end{aligned} \tag{2.26}$$

then (2.10) and (2.11) are satisfied and (2.4) becomes

$$\frac{1}{2}\Delta_V|\nabla h|^2 \geq C_3\frac{|\nabla h|^4}{h^2} - C_4\frac{\nabla h}{h}\nabla(|\nabla h|^2) - K|\nabla h|^2, \tag{2.27}$$

where positive constants C_3 and C_4 are given by

$$C_3 = \frac{1}{2}\left[\left(\frac{(\tilde{\epsilon}-1)^2}{(n+N)\tilde{\epsilon}^2} - \frac{\tilde{\epsilon}-1}{\tilde{\epsilon}}\right) - \frac{n+N}{\tilde{\epsilon}^2}\left(\frac{n+N+2}{n+N}(\tilde{\epsilon}-1)+\alpha\right)^2\right],$$

$$C_4 = \frac{4(\alpha-1)(n+N)(n+N+2)+(n+N)[2(n+N)+5]}{[5(n+N)+6]-4(\alpha-1)(n+N)(n+N+2)},$$

respectively. We conclude the proof of Proposition 2.2. □

Now we begin to prove Theorem 1.1.

Proof of Theorem 1.1. We first prove the case of $c < 0$ and $\alpha > 0$. Choose a smooth function $\eta(r)$ such that $0 \leq \eta(r) \leq 1$, $\eta(r) = 1$ if $r \leq 1$, $\eta(r) = 0$ if $r \geq 2$, and

$$0 \geq \eta(r)^{-\frac{1}{2}}\eta(r)' \geq -c_1,\ \eta(r)'' \geq -c_2$$

for some $c_1,\ c_2 \geq 0$. For a fixed point $p \in M$, let $\rho(x) = dist(p,x)$ and $\psi = \eta\left(\frac{\rho(x)}{R}\right)$. Therefore,

$$\frac{|\nabla\psi|^2}{\psi} = \frac{|\nabla\eta|^2}{\eta} = \frac{1}{\eta(r)}\frac{(\eta(r)')^2}{R^2}|\nabla\rho(x)|^2 \leq \frac{c_1^2}{R^2}. \tag{2.28}$$

Since $\mathrm{Ric}_V^N \geq -K$, the Laplacian comparison theorem in [6] implies that

$$\Delta_V\rho \leq \sqrt{(n-1)K}\coth\left(\sqrt{\frac{K}{n-1}}\rho\right) \leq \sqrt{(n-1)K} + \frac{n-1}{\rho}. \tag{2.29}$$

Hence,

$$\begin{aligned}\Delta_V\psi &= \frac{\eta(r)''|\nabla\rho|^2}{R^2} + \frac{\eta(r)'\Delta_V\rho}{R} \\ &\geq \frac{-c_2}{R^2} + \frac{-c_1}{R}\left(\sqrt{(n-1)K} + \frac{n-1}{\rho}\right) \\ &\geq -\frac{R\left(\sqrt{(n-1)K}+\frac{n-1}{R}\right)c_1 + c_2}{R^2} \\ &= -\frac{\left(R\sqrt{(n-1)K}+n-1\right)c_1+c_2}{R^2}.\end{aligned} \tag{2.30}$$

Denote by $B_p(R)$ the geodesic ball centered at p with radius R. Let $G = \psi|\nabla h|^2$. Assume G achieves its maximum at the point $x_0 \in B_p(2R)$ and assume $G(x_0) > 0$ (otherwise this is obvious). Then at the point x_0, it holds that

$$\Delta_V G \leq 0,\ \nabla(|\nabla h|^2) = -\frac{|\nabla h|^2}{\psi}\nabla\psi.$$

Using (2.18) in Proposition 2.2, we obtain

$$
\begin{aligned}
(2.31)\quad 0 &\geq \Delta_V G \\
&= \psi \Delta_V (|\nabla h|^2) + |\nabla h|^2 \Delta_V \psi + 2\nabla\psi\nabla|\nabla h|^2 \\
&= \psi \Delta_V (|\nabla h|^2) + \frac{\Delta_V \psi}{\psi} G - 2\frac{|\nabla\psi|^2}{\psi^2} G \\
&\geq 2\psi \left[C_1 \frac{|\nabla h|^4}{h^2} - C_2 \frac{\nabla h}{h} \nabla(|\nabla h|^2) - K|\nabla h|^2 \right] + \frac{\Delta_V \psi}{\psi} G - 2\frac{|\nabla\psi|^2}{\psi^2} G \\
&= 2C_1 \frac{G^2}{\psi h^2} + 2C_2 \frac{G}{\psi} \nabla\psi \frac{\nabla h}{h} - 2KG + \frac{\Delta_V \psi}{\psi} G - 2\frac{|\nabla\psi|^2}{\psi^2} G.
\end{aligned}
$$

Multiplying both sides of (2.31) by $\frac{\psi}{G}$ yields

$$
2C_1 \frac{G}{h^2} \leq -2C_2 \nabla\psi \frac{\nabla h}{h} + 2\psi K - \Delta_V \psi + 2\frac{|\nabla\psi|^2}{\psi}. \tag{2.32}
$$

Using the Cauchy inequality

$$
-2C_2 \nabla\psi \frac{\nabla h}{h} \leq 2C_2 |\nabla\psi| \frac{|\nabla h|}{h} \leq \frac{C_2^2}{C_1} \frac{|\nabla\psi|^2}{\psi} + C_1 \frac{G}{h^2},
$$

into (2.32) yields

$$
C_1 \frac{G}{h^2} \leq 2\psi K - \Delta_V \psi + \left(2 + \frac{C_2^2}{C_1}\right) \frac{|\nabla\psi|^2}{\psi}. \tag{2.33}
$$

Hence, for $x \in B_p(R)$, we have

$$
\begin{aligned}
(2.34)\quad & C_1 G(x) \\
&\leq C_1 G(x_0) \\
&\leq h^2(x_0) \left\{ 2K + \frac{1}{R^2} \left[\left(R\sqrt{(n-1)K} + n - 1 \right) c_1 + c_2 + \left(2 + \frac{C_2^2}{C_1} \right) c_1^2 \right] \right\}.
\end{aligned}
$$

It shows that

$$
\begin{aligned}
(2.35)\quad & |\nabla u|^2(x) \\
&\leq \frac{M^2}{\epsilon^2 C_1} \left\{ 2K + \frac{1}{R^2} \left[\left(R\sqrt{(n-1)K} + n - 1 \right) c_1 + c_2 + \left(2 + \frac{C_2^2}{C_1} \right) c_1^2 \right] \right\},
\end{aligned}
$$

and hence,

$$
\begin{aligned}
(2.36)\quad & |\nabla u|(x) \\
&\leq \frac{M}{\epsilon\sqrt{C_1}} \sqrt{ \left\{ 2K + \frac{1}{R^2} \left[\left(R\sqrt{(n-1)K} + n - 1 \right) c_1 + c_2 + \left(2 + \frac{C_2^2}{C_1} \right) c_1^2 \right] \right\} }.
\end{aligned}
$$

It yields the desired inequality (1.5) of Theorem 1.1.

Next, we prove the case $c>0$ and $\frac{n+N+2}{2(n+N-1)}<\alpha<\frac{2(n+N)^2+9(n+N)+6}{2(n+N)(n+N+2)}$ with $n\geq 3$. In a similar way as the case $c<0$ and $\alpha>0$, on $B_p(R)$, we have

$$
\begin{aligned}
(2.37)\qquad & |\nabla u|(x)\\
& \leq \frac{M}{\epsilon\sqrt{C_3}}\sqrt{2K+\frac{1}{R^2}\left[\left(R\sqrt{(n-1)K}+n-1\right)c_1+c_2+\left(2+\frac{C_4^2}{C_3}\right)c_1^2\right]}.
\end{aligned}
$$

This concludes the proof of inequality (1.6) of Theorem 1.1. We complete the proof of Theorem 1.1. □

Now we are in the position to give a brief proof of Theorem 1.3.

Skept of the proof of Theorem 1.3. Noticing that we have the following Bochner formula to h with $\mathrm{Ric_V}$,

$$\frac{1}{2}\Delta_V|\nabla h|^2=|\nabla^2 h|^2+\nabla h\nabla\Delta_V h+\mathrm{Ric_V}(\nabla h,\nabla h),$$

then (2.7) becomes

$$
\begin{aligned}
\frac{1}{2}\Delta_V|\nabla h|^2 &= |\nabla^2 h|^2+\nabla h\nabla\Delta_V h+\mathrm{Ric_V}(\nabla h,\nabla h)\\
&\geq \frac{1}{n}\left(\frac{\epsilon-1}{\epsilon}\frac{|\nabla h|^2}{h}-c\epsilon h^{\frac{\alpha+\epsilon-1}{\epsilon}}\right)^2+\nabla h\nabla\Delta_V h-\widetilde{K}|\nabla h|^2\\
&= \left(\frac{(\epsilon-1)^2}{n\epsilon^2}-\frac{\epsilon-1}{\epsilon}\right)\frac{|\nabla h|^4}{h^2}-c\left(\frac{n+2}{n}(\epsilon-1)+\alpha\right)h^{\frac{\alpha+\epsilon-1}{\epsilon}}\frac{|\nabla h|^2}{h}\\
&\quad+\frac{c^2\epsilon^2}{n}h^{\frac{2(\alpha+\epsilon-1)}{\epsilon}}+\frac{\epsilon-1}{\epsilon}\frac{\nabla h}{h}\nabla(|\nabla h|^2)-\widetilde{K}|\nabla h|^2.
\end{aligned}
$$

Moreover, the Laplacian comparison theorem in [2] implies: if $\mathrm{Ric_V}\geq-\widetilde{K}$ and $|V|\leq L$, we have

$$\Delta_V\rho\leq\sqrt{(n-1)\widetilde{K}}+\frac{n-1}{\rho}+L.$$

So (2.28) and (2.30) also hold true in almost the same forms

$$\frac{|\nabla\psi|^2}{\psi}\leq\frac{c_1^2}{R^2}$$

and

$$\Delta_V\psi\geq-\frac{\left(R\sqrt{(n-1)\widetilde{K}}+RL+n-1\right)c_1+c_2}{R^2}.$$

Noticing the above facts, the proof of Theorem 1.3 is the same to that of Theorem 1.1, so we omit it here. □

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Fanqi Zeng
School of Mathematics and Statistics
Xinyang Normal University
Xinyang, 464000, P. R. China
Email address: fanzeng10@126.com