Bull. Korean Math. Soc. **0** (0), No. 0, pp. 1–0
https://doi.org/10.4134/BKMS.b180561
pISSN: 1015-8634 / eISSN: 2234-3016

EXTENSION OF PHASE-ISOMETRIES BETWEEN THE UNIT SPHERES OF ATOMIC L_p-SPACES FOR $p > 0$

Xujian Huang and Xihong Jin

Abstract. In this paper, we prove that for every surjective phase-isometry between the unit spheres of real atomic L_p-spaces for $p > 0$, its positive homogeneous extension is a phase-isometry which is phase equivalent to a linear isometry.

1. Introduction

Let X and Y be real normed spaces. A mapping $f : X \to Y$ is called a *phase-isometry* if f satisfies the functional equation

$$\{\|f(x) + f(y)\|, \|f(x) - f(y)\|\} = \{\|x + y\|, \|x - y\|\} \quad (x, y \in X).$$

Let us say that a mapping $f : X \to Y$ is *phase equivalent to a linear isometry* if there exists a phase function $\varepsilon : X \to \{-1, 1\}$ such that εf is a linear isometry. The notation of phase-isometry is linked to the famous Wigner's theorem, which plays a fundamental role in quantum mechanics and in representation theory in physics. There are several equivalent formulations of Wigner's theorem, see [1,4,5,8,10,12] to list just some of them. The real version of Wigner's theorem [10] says that a mapping $f : H \to K$ satisfies the functional equation

$$|\langle f(x), f(y)\rangle| = |\langle x, y\rangle| \quad (x, y \in H)$$

is phase equivalent to a linear isometry provided that H and K are real inner product spaces. This is equivalent to saying that every phase-isometry from the real inner product space H into K is phase equivalent to a linear isometry. Recently, Huang and Tan [6] showed that every surjective phase-isometry between real atomic L_p-spaces for $p > 0$ is phase equivalent to a linear isometry, which generalizes Wigner's theorem to real atomic L_p-spaces for $p > 0$.

In 1987, D. Tingley [11] proposed the following question: Let f be a surjective isometry between the unit spheres S_X and S_Y of real normed spaces X and Y, respectively. Is it true that $f : S_X \to S_Y$ extends to a linear isometry

Received June 9, 2018; Revised March 6, 2019; Accepted April 1, 2019.
2010 *Mathematics Subject Classification.* 46B04, 46B20.
Key words and phrases. extension of phase-isometries, unit sphere, atomic L_p-spaces.
The authors are supported by the Natural Science Foundation of China, Grant No. 11371201, 11201337, 11201338.

$F : X \to Y$ of the corresponding spaces? This problem is known as the Tingly's problem or isometric extension problem. We refer the reader to the introduction of [9] for more information and recent development on this problem. The survey of Ding [3] is one of the good references for understanding the history of the problem. Let us consider the natural positive homogeneous extension F of f, where F is given by

$$F(x) = \begin{cases} \|x\| f(\frac{x}{\|x\|}), & \text{if} \quad x \neq 0, \\ 0, & \text{if} \quad x = 0. \end{cases} \tag{1}$$

Then Tingley's problem can be solved in positive for pairs (X, Y) if and only if the natural positive homogeneous extension F is a (linear) isometry. Inspired by Tingly's problem, it is natural to ask the following question:

Problem 1.1. Let f be a surjective phase-isometry between the unit spheres S_X and S_Y of real normed spaces X and Y, respectively. Is it true that the natural positive homogeneous extension F is a phase-isometry?

In this paper, we answer Problem 1.1 in positive for real atomic L_p-spaces for $p > 0$. That is for every phase-isometry from the unit sphere $S_{l_p(\Gamma)}$ onto $S_{l_p(\Delta)}$ of real atomic L^p-spaces for $p > 0$, the natural positive homogeneous extension is phase equivalent to a linear isometry, and therefore actually a phase-isometry. We also show that Problem 1.1 is solved in positive for real inner product spaces.

2. Results

We first discuss the phase-isometric extension problem on real inner product spaces and show that Problem 1.1 is solved in positive for such spaces.

Proposition 2.1. *Let H and K be inner product spaces, and let $f : S_H \to S_K$ be a phase-isometry. Then the positive homogeneous extension F of f is a phase-isometry.*

Proof. Since H and K are inner product spaces, by the polarization identity, we have

$$\langle x, y\rangle = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2),$$
$$\langle f(x), f(y)\rangle = \frac{1}{4}(\|f(x)+f(y)\|^2 - \|f(x)-f(y)\|^2)$$

for all $x, y \in S_H$. By the assumption of f, we have $|\langle f(x), f(y)\rangle| = |\langle x, y\rangle|$ for all $x, y \in S_H$. Hence,

$$\begin{aligned} |\langle F(x), F(y)\rangle| &= |\langle \|x\| f(\frac{x}{\|x\|}), \|y\| f(\frac{y}{\|y\|})\rangle| \\ &= \|x\|\|y\| |\langle f(\frac{x}{\|x\|}), f(\frac{y}{\|y\|})\rangle| = |\langle x, y\rangle| \end{aligned}$$

for all $x, y \in H$ with $x, y \neq 0$. It follows from Wigner's Theorem that F is phase equivalent to a linear isometry, and this completes the proof. $\square$

Recall that every real atomic L_p-space for $p > 0$ is linearly isometric to $l_p(\Gamma)$ for some nonempty index set Γ, that is,

$$l_p(\Gamma) = \{x = \sum_{\gamma\in\Gamma} \xi_\gamma e_\gamma : \|x\| = (\sum_{\gamma\in\Gamma} |\xi_\gamma|^p)^{\frac{1}{p}} < \infty,\ \xi_\gamma \in \mathbb{R}\}.$$

The unit sphere of $l_p(\Gamma)$ is $\{x \in l_p(\Gamma) : \|x\| = 1\}$ and is denoted by $S_{l_p(\Gamma)}$. For every $x = \sum_{\gamma\in\Gamma} \xi_\gamma e_\gamma \in l_p(\Gamma)$, we denote the support of x by Γ_x, i.e.,

$$\Gamma_x = \{\gamma \in \Gamma : \xi_\gamma \neq 0\}.$$

Then x can be rewritten in the form $x = \sum_{\gamma\in\Gamma_x} \xi_\gamma e_\gamma \in l_p(\Gamma)$. For $x, y \in l_p(\Gamma)$, we use the symbol $xy = 0$ to represent $\Gamma_x \cap \Gamma_y = \emptyset$. It is well-known that $xy = 0$ if and only if $\|x + y\| = \|x - y\|$ for all $x, y \in l_2(\Gamma)$. We also need the following well-known result which can be found from [7, Corollary 2.1] (noting that Banach used it in his book [2] already). The statement is that $xy = 0$ if and only if $\|x+y\|^p + \|x-y\|^p = 2(\|x\|^p + \|y\|^p)$ for all $x, y \in l_p(\Gamma)$ with $p > 0$, $p \neq 2$. By this one can conclude the following result.

Lemma 2.2. *Let $X = l_p(\Gamma)$ and $Y = l_p(\Delta)$ for $p > 0$. Suppose that $f : S_X \to S_Y$ is a phase-isometry. Then $xy = 0$ if and only if $f(x)f(y) = 0$ for all $x, y \in S_X$.*

Our next lemma will show that every surjective phase-isometry between the unit spheres of real atomic L_p-space for $p > 0$ necessarily maps antipodal points to antipodal points. So the positive homogeneous extension is homogeneous for the negative scalars as well.

Lemma 2.3. *Let $X = l_p(\Gamma)$ and $Y = l_p(\Delta)$ for $p > 0$. Suppose that $f : S_X \to S_Y$ is a surjective phase-isometry. Then f is injective and $f(-x) = -f(x)$ for every $x \in S_X$. Moreover, for every $\gamma \in \Gamma$, there exists $\delta \in \Delta$ such that $f(e_\gamma) = \pm e_\delta$.*

Proof. Let us take $x \in S_X$. Since f is surjective, we can pick $y \in S_X$ such that $f(y) = -f(x)$. Notice that f is a phase-isometry, we have

$$\{\|x + y\|, \|x - y\|\} = \{\|f(x) + f(y)\|, \|f(x) - f(y)\|\} = \{0, 2\}$$

which implies that $y = \pm x$. If $y = x$, then $f(x) = f(y) = -f(x)$, which is impossible. Hence we get $y = -x$ and so $f(-x) = -f(x)$. On the other hand, suppose that $f(z) = f(x)$ for some $z \in S_X$. By the assumption of f, we have

$$\{\|x + z\|, \|x - z\|\} = \{\|f(x) + f(z)\|, \|f(x) - f(z)\|\} = \{2, 0\}.$$

This means that $z = x$ and f is injective.

We will prove the "moreover" part. Let δ be in the support of $f(e_\gamma)$ and pick $x \in S_X$ such that $f(x) = e_\delta$. Applying Lemma 2.2 we have

$$e_\gamma e_{\gamma'} = 0 \Rightarrow f(e_\gamma)f(e_{\gamma'}) = 0 \Rightarrow f(x)f(e_{\gamma'}) = 0 \Rightarrow xe_{\gamma'} = 0$$

for all $\gamma' \in \Gamma$ with $\gamma' \neq \gamma$. It follows that $x = \pm e_\gamma$, and so $f(e_\gamma) = \pm e_\delta$. □

Now we derive the representation theorem of surjective phase-isometries between the unit spheres of real atomic L_p-spaces for $p > 0$, $p \neq 2$.

Theorem 2.4. *Let $X = l_p(\Gamma)$ and $Y = l_p(\Delta)$ for $p > 0$, $p \neq 2$. Suppose that $f : S_X \to S_Y$ is a surjective phase-isometry. Then for every $x = \sum_{\gamma\in\Gamma} \xi_\gamma e_\gamma \in S_X$, we have $f(x) = \sum_{\gamma\in\Gamma} \eta_\gamma f(e_\gamma)$, where $|\xi_\gamma| = |\eta_\gamma|$ for all $\gamma \in \Gamma$.*

Proof. Let x be in S_X and write $x = \sum_{\gamma\in\Gamma_x} \xi_\gamma e_\gamma$, where $\sum_{\gamma\in\Gamma_x} |\xi_\gamma|^p = 1$ and $\xi_\gamma \neq 0$ for all $\gamma \in \Gamma_x$. According to Lemma 2.3, we can set

$$M := \{\delta \in \Delta : f(e_\gamma) = \pm e_\delta,\ \forall \gamma \in \Gamma_x\}.$$

Choose $y \in S_X$ such that $f(y) = e_\delta$ for some $\delta \in \Delta \setminus M$. Applying Lemma 2.2, we have

$$f(e_\gamma)f(y) = 0 \Rightarrow e_\gamma y = 0 \Rightarrow xy = 0 \Rightarrow f(x)f(y) = f(x)e_\delta = 0$$

for all $\gamma \in \Gamma_x$. Thus we can write $f(x) = \sum_{\gamma\in\Gamma_x} \eta_\gamma f(e_\gamma)$, where $\sum_{\gamma\in\Gamma_x} |\eta_\gamma|^p = 1$. By the assumption of f,

$$\begin{aligned}
&\|f(x) + f(e_\gamma)\|^p + \|f(x) - f(e_\gamma)\|^p \\
&= \|x + e_\gamma\|^p + \|x - e_\gamma\|^p \\
&= 1 - |\xi_\gamma|^p + |\xi_\gamma + 1|^p + 1 - |\xi_\gamma|^p + |\xi_\gamma - 1|^p \\
&= |1 + \xi_\gamma|^p + |1 - \xi_\gamma|^p - 2|\xi_\gamma|^p + 2.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\|f(x) + f(e_\gamma)\|^p + \|f(x) - f(e_\gamma)\|^p \\
&= 1 - |\eta_\gamma|^p + |\eta_\gamma + 1|^p + 1 - |\eta_\gamma|^p + |\eta_\gamma - 1|^p \\
&= |1 + \eta_\gamma|^p + |1 - \eta_\gamma|^p - 2|\eta_\gamma|^p + 2.
\end{aligned}$$

It follows that

$$|1 + \xi_\gamma|^p + |1 - \xi_\gamma|^p - 2|\xi_\gamma|^p = |1 + \eta_\gamma|^p + |1 - \eta_\gamma|^p - 2|\eta_\gamma|^p.$$

Notice that the function $\varphi(t) = (1 + t)^p + (1 - t)^p - 2t^p$ is strictly decreasing (increasing) on $[0, 1]$ for $0 < p < 2$ $(p > 2)$ (Here, we need the fact that $(s + r)^p < s^p + r^p$ for $0 < p < 1$ and $(s + r)^p > s^p + r^p$ for $p > 1$ whenever $s, r > 0$). Consequently, we obtain $|\xi_\gamma| = |\eta_\gamma|$ for all $\gamma \in \Gamma_x$. □

Our next results are devoted to determining the behaviour of surjective phase-isometries between the unit spheres of real atomic L_p-spaces for $p > 0$, $p \neq 2$ on vectors which are linear combinations of two zero-product norm-one vectors.

Lemma 2.5. *Let $X = l_p(\Gamma)$ and $Y = l_p(\Delta)$ for $p > 0$, $p \neq 2$. Suppose that $f : S_X \to S_Y$ is a surjective phase-isometry. Let $x, y \in S_X$ with $xy = 0$ and $\lambda \in \mathbb{R}$. Then there exist two real numbers α, β with $|\alpha| = |\beta| = 1$ such that*

$$\|x + \lambda y\| f\left(\frac{x + \lambda y}{\|x + \lambda y\|}\right) = \alpha f(x) + \beta \lambda f(y).$$

Proof. Suppose that $x = \sum_{\gamma \in \Gamma_x} \xi_\gamma e_\gamma$ and $y = \sum_{\gamma \in \Gamma_y} \eta_\gamma e_\gamma$, and that $0 \neq \lambda \in \mathbb{R}$. By Theorem 2.4 we can write

$$f(x) = \sum_{\gamma \in \Gamma_x} {\xi'}_\gamma f(e_\gamma), \quad f(y) = \sum_{\gamma \in \Gamma_y} {\eta'}_\gamma f(e_\gamma),$$
$$\|x + \lambda y\| f\left(\frac{x + \lambda y}{\|x + \lambda y\|}\right) = \sum_{\gamma \in \Gamma_x} {\xi''}_\gamma f(e_\gamma) + \lambda \sum_{\gamma \in \Gamma_y} {\eta''}_\gamma f(e_\gamma),$$

where $|{\xi'}_\gamma| = |{\xi''}_\gamma| = |\xi_\gamma|$ and $|{\eta'}_\gamma| = |{\eta''}_\gamma| = |\eta_\gamma|$ for all $\gamma \in \Gamma_x \cup \Gamma_y$. To simplify the writing, we take $A = \frac{1}{\|x+\lambda y\|} = \frac{1}{(1+|\lambda|^p)^{\frac{1}{p}}}$. Since f is a phase-isometry,

$$\begin{aligned}
&\{(A+1)^p + (A|\lambda|)^p, (1-A)^p + (A|\lambda|)^p\} \\
= &\left\{\left\|\frac{x + \lambda y}{\|x + \lambda y\|} + x\right\|^p, \left\|\frac{x + \lambda y}{\|x + \lambda y\|} - x\right\|^p\right\} \\
= &\left\{\left\|f\left(\frac{x + \lambda y}{\|x + \lambda y\|}\right) + f(x)\right\|^p, \left\|f\left(\frac{x + \lambda y}{\|x + \lambda y\|}\right) - f(x)\right\|^p\right\} \\
= &\left\{\sum_{\gamma \in \Gamma_x} |A{\xi''}_\gamma + {\xi'}_\gamma|^p + (A|\lambda|)^p, \sum_{\gamma \in \Gamma_x} |A{\xi''}_\gamma - {\xi'}_\gamma|^p + (A|\lambda|)^p\right\}.
\end{aligned}$$

This shows that

$$(A+1)^p \in \left\{\sum_{\gamma \in \Gamma_x} |A{\xi''}_\gamma + {\xi'}_\gamma|^p, \sum_{\gamma \in \Gamma_x} |A{\xi''}_\gamma - {\xi'}_\gamma|^p\right\}.$$

Notice that

$$\sum_{\gamma \in \Gamma_x} |A{\xi''}_\gamma \pm {\xi'}_\gamma|^p \leq \sum_{\gamma \in \Gamma_x} (|A{\xi''}_\gamma| + |{\xi'}_\gamma|)^p = (A+1)^p.$$

Then we obtain ${\xi''}_\gamma = {\xi'}_\gamma$ for all $\gamma \in \Gamma_x$, or ${\xi''}_\gamma = -{\xi'}_\gamma$ for all $\gamma \in \Gamma_x$. It follows that $\sum_{\gamma \in \Gamma_x} {\xi''}_\gamma e_\gamma = \pm f(x)$. Similar argument yields $\sum_{\gamma \in \Gamma_y} {\eta''}_\gamma e_\gamma = \pm f(y)$. The proof is complete. □

In [13] Wang proved that for every surjective isometry between unit spheres of real atomic L_p-spaces for $p > 0$, $p \neq 2$, its natural positive homogeneous extension is a linear isometry on the whole space. By this result, we are now ready to present main result of this paper.

Theorem 2.6. *Let $X = l_p(\Gamma)$ and $Y = l_p(\Delta)$ for $p > 0$. Suppose that $f : S_X \to S_Y$ is a surjective phase-isometry. Then the positive extension F of f is phase equivalent to a linear isometry.*

Proof. Proposition 2.1 proves the case $p = 2$. We need only consider the case $p > 0, p \neq 2$. Set $Z := \{x \in X : xe_{\gamma_0} = 0\}$ and $W := \{w \in Y : wf(e_{\gamma_0}) = 0\}$ for some $\gamma_0 \in \Gamma$. It is not hard to check that $S_X = \{\frac{z+\lambda e_{\gamma_0}}{\|z+\lambda e_{\gamma_0}\|} : z \in S_Z, \lambda \in \mathbb{R}\} \cup \{\pm e_{\gamma_0}\}$. By Lemma 2.5 we can write

$$\begin{aligned}&\|z + \lambda e_{\gamma_0}\| f\left(\frac{z + \lambda e_{\gamma_0}}{\|z + \lambda e_{\gamma_0}\|}\right) = \alpha(z,\lambda) f(z) + \beta(z,\lambda)\lambda f(e_{\gamma_0}),\\ &|\alpha(z,\lambda)| = |\beta(z,\lambda)| = 1\end{aligned}$$

for all $z \in S_Z$ and $\lambda \in \mathbb{R}$. Define a mapping $g : S_X \to S_Y$ as follows:

$$\begin{aligned}&g(e_{\gamma_0}) = f(e_{\gamma_0}), \quad g(-e_{\gamma_0}) = -f(e_{\gamma_0}), \quad g(z) = \alpha(z,1)\beta(z,1)f(z),\\ &\|z + \lambda e_{\gamma_0}\| g\left(\frac{z + \lambda e_{\gamma_0}}{\|z + \lambda e_{\gamma_0}\|}\right) = \alpha(z,\lambda)\beta(z,\lambda) f(z) + \lambda f(e_{\gamma_0})\end{aligned}$$

for all $z \in S_Z$ and $0 \neq \lambda \in \mathbb{R}$. Then g is a phase-isometry, which is phase equivalent to f. Since $f(S_Z) = S_W$ by Theorem 2.4, we deduce that $g(S_Z) \subset S_W$.

Next, we will show that $g|S_Z : S_Z \to S_W$ is a surjective isometry. Let us take $z \in S_Z$ and $0 \neq \lambda \in \mathbb{R}$. Set $A := \frac{1}{\|z+e_{\gamma_0}\|}$ and $B := \frac{1}{\|z+\lambda e_{\gamma_0}\|}$. Since g is a phase-isometry,

$$\begin{aligned}&\{|A+B|^p + |A+B\lambda|^p, |A-B|^p + |A-B\lambda|^p\}\\ =&\left\{\left\|\frac{z+e_{\gamma_0}}{\|z+e_{\gamma_0}\|} + \frac{z+\lambda e_{\gamma_0}}{\|z+\lambda e_{\gamma_0}\|}\right\|^p, \left\|\frac{z+e_{\gamma_0}}{\|z+e_{\gamma_0}\|} - \frac{z+|\lambda e_{\gamma_0}}{\|z+\lambda e_{\gamma_0}\|}\right\|^p\right\}\\ =&\left\{\left\|g\left(\frac{z+e_{\gamma_0}}{\|z+e_{\gamma_0}\|}\right) + g\left(\frac{z+\lambda e_{\gamma_0}}{\|z+\lambda e_{\gamma_0}\|}\right)\right\|^p, \left\|g\left(\frac{z+e_{\gamma_0}}{\|z+e_{\gamma_0}\|}\right) - g\left(\frac{z+\lambda e_{\gamma_0}}{\|z+\lambda e_{\gamma_0}\|}\right)\right\|^p\right\}\\ =& \{|A\alpha(z,1)\beta(z,1) + B\alpha(z,\lambda)\beta(z,\lambda)|^p + |A+B\lambda|^p,\\ &|A\alpha(z,1)\beta(z,1) - B\alpha(z,\lambda)\beta(z,\lambda)|^p + |A-B\lambda|^p\}.\end{aligned}$$

If $\alpha(z,1)\beta(z,1) = -\alpha(z,\lambda)\beta(z,\lambda)$, then

$$\begin{aligned}&\{|A-B|^p + |A+B\lambda|^p, |A+B|^p + |A-B\lambda|^p\}\\ =& \{|A+B|^p + |A+B\lambda|^p, |A-B|^p + |A-B\lambda|^p\}.\end{aligned}$$

This leads to a contradiction for $\lambda \neq 0$. It follows that $\alpha(z,1)\beta(z,1) = \alpha(z,\lambda)\beta(z,\lambda)$, and hence

$$\|z + \lambda e_{\gamma_0}\| g\left(\frac{z + \lambda e_{\gamma_0}}{\|z + \lambda e_{\gamma_0}\|}\right) = g(z) + \lambda g(e_{\gamma_0})$$

for all $z \in S_Z$ and $\lambda \in \mathbb{R}$. Let z_1, z_2 be in S_Z and $\lambda > \|z_1 - z_2\|/2$. Clearly,

$$\frac{1}{1+\lambda^p}\{\|g(z_1) + g(z_2)\|^p + (2\lambda)^p, \|g(z_1) - g(z_2)\|^p\}$$

$$= \left\{\left\|g\left(\frac{z_1+\lambda e_{\gamma_0}}{\|z_1+\lambda e_{\gamma_0}\|}\right)+g\left(\frac{z_2+\lambda e_{\gamma_0}}{\|z_2+\lambda e_{\gamma_0}\|}\right)\right\|^p, \left\|g\left(\frac{z_1+\lambda e_{\gamma_0}}{\|z_1+\lambda e_{\gamma_0}\|}\right)-g\left(\frac{z_2+\lambda e_{\gamma_0}}{\|z_2+\lambda e_{\gamma_0}\|}\right)\right\|^p\right\}$$

$$= \left\{\left\|\frac{z_1+\lambda e_{\gamma_0}}{\|z_1+\lambda e_{\gamma_0}\|}+\frac{z_2+\lambda e_{\gamma_0}}{\|z_2+\lambda e_{\gamma_0}\|}\right\|^p, \left\|\frac{z_1+\lambda e_{\gamma_0}}{\|z_1+\lambda e_{\gamma_0}\|}-\frac{z_2+\lambda e_{\gamma_0}}{\|z_2+\lambda e_{\gamma_0}\|}\right\|^p\right\}$$

$$= \frac{1}{1+\lambda^p}\{\|z_1+z_2\|^p+(2\lambda)^p, \|z_1-z_2\|^p\}.$$

This implies that $\|g(z_1)-g(z_2)\| = \|z_1-z_2\|$ for all $z_1, z_2 \in S_Z$. On the other hand,

$$\frac{1}{2}\{\|g(z)+g(-z)\|^p, \|g(z)-g(-z)\|^p+2^p\}$$

$$=\left\{\left\|g\left(\frac{z+e_{\gamma_0}}{\|z+e_{\gamma_0}\|}\right)+g\left(\frac{-z-e_{\gamma_0}}{\|z+e_{\gamma_0}\|}\right)\right\|^p, \left\|g\left(\frac{z+e_{\gamma_0}}{\|z+e_{\gamma_0}\|}\right)-g\left(\frac{-z-e_{\gamma_0}}{\|z+e_{\gamma_0}\|}\right)\right\|^p\right\}$$

$$=\left\{\left\|\frac{z+e_{\gamma_0}}{\|z+e_{\gamma_0}\|}+\frac{-z-e_{\gamma_0}}{\|z+e_{\gamma_0}\|}\right\|^p, \left\|\frac{z+e_{\gamma_0}}{\|z+e_{\gamma_0}\|}-\frac{-z-e_{\gamma_0}}{\|z+e_{\gamma_0}\|}\right\|^p\right\}$$

$$= \frac{1}{2}\{0, 2^p\}$$

for all $z \in S_Z$. This shows that $g(-z) = -g(z)$ for all $z \in S_Z$. Since g is phase equivalent to f, we see that $g|S_Z : S_Z \to S_W$ is a surjective isometry.

Finally, we prove that F is phase equivalent to a linear isometry. Since the natural positive homogeneous extension G of g is phase equivalent to F, it is suffices to showing that $G : X \to Y$ is a linear isometry. By Lemma 2.3, we have $f(e_{\gamma_0}) = \pm e_{\delta_0}$ for some $\delta_0 \in \Delta$. It is easily verified that Z and W are linearly isometric to $l_p(\Gamma \setminus \{\gamma_0\})$ and $l_p(\Delta \setminus \{\delta_0\})$ respectively. From Wang's result [13], the restriction of G to Z is a linear isometry. Set $x := \frac{z}{\|z\|} + \frac{\lambda e_{\gamma_0}}{\|z\|}$ for some $0 \neq z \in Z$ and $\lambda \in \mathbb{R}$. It follows that

$$G(z+\lambda e_{\gamma_0}) = \|z\|\|x\|g\left(\frac{x}{\|x\|}\right) = \|z\|\left(g\left(\frac{z}{\|z\|}\right)+\frac{\lambda g(e_{\gamma_0})}{\|z\|}\right) = G(z)+\lambda g(e_{\gamma_0}).$$

This shows that $G : X \to Y$ is a linear isometry, which completes the proof. □

Acknowledgements. The authors cordially thank Professor Guanggui Ding for his helpful suggestions. The authors also thank the referee for several helpful comments and suggestions.

References

[1] D. F. Almeida and C. S. Sharma, *The first mathematical proof of Wigner's theorem*, J. Natur. Geom. **2** (1992), no. 2, 113–123.

[2] S. Banach, *Théorie des opérations linéaires*, reprint of the 1932 original, Éditions Jacques Gabay, Sceaux, 1993.

[3] G. Ding, *On isometric extension problem between two unit spheres*, Sci. China Ser. A **52** (2009), no. 10, 2069–2083. `https://doi.org/10.1007/s11425-009-0156-x`

[4] Gy. P. Gehér, *An elementary proof for the non-bijective version of Wigner's theorem*, Phys. Lett. A **378** (2014), no. 30-31, 2054–2057. `https://doi.org/10.1016/j.physleta.2014.05.039`

[5] M. Győry, *A new proof of Wigner's theorem*, Rep. Math. Phys. **54** (2004), no. 2, 159–167. https://doi.org/10.1016/S0034-4877(04)80012-0
[6] X. Huang and D. Tan, *Wigner's theorem in atomic L_p-spaces* ($p > 0$), Publ. Math. Debrecen **92** (2018), no. 3-4, 411–418. https://doi.org/10.5486/pmd.2018.8005
[7] J. Lamperti, *On the isometries of certain function-spaces*, Pacific J. Math. **8** (1958), 459–466. http://projecteuclid.org/euclid.pjm/1103039892
[8] L. Molnár, *Orthogonality preserving transformations on indefinite inner product spaces: generalization of Uhlhorn's version of Wigner's theorem*, J. Funct. Anal. **194** (2002), no. 2, 248–262. https://doi.org/10.1006/jfan.2002.3970
[9] A. M. Peralta and M. Cueto-Avellaneda, *The Mazur-Ulam property for commutative von Neumann algebras*, Linear and Multilinear A.; arXiv:1803.00604, 2018.
[10] J. Rätz, *On Wigner's theorem: remarks, complements, comments, and corollaries*, Aequationes Math. **52** (1996), no. 1-2, 1–9. https://doi.org/10.1007/BF01818323
[11] D. Tingley, *Isometries of the unit sphere*, Geom. Dedicata **22** (1987), no. 3, 371–378. https://doi.org/10.1007/BF00147942
[12] A. Turnšek, *A variant of Wigner's functional equation*, Aequationes Math. **89** (2015), no. 4, 1–8,
[13] J. Wang, *On extension of isometries between unit spheres of AL^p-spaces* ($0 < p < \infty$), Proc. Amer. Math. Soc. **132** (2004), no. 10, 2899–2909.

Xujian Huang
Department of Mathematics
Tianjin University of Technology
300384 Tianjin, P. R. China
Email address: huangxujian86@sina.cn

Xihong Jin
Department of Mathematics
Tianjin University of Technology
300384 Tianjin, P. R. China
Email address: jinxihong05@163.com