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STABILITY IN THE ENERGY SPACE OF THE SUM OF N PEAKONS FOR A CAMASSA-HOLM-TYPE EQUATION WITH QUARTIC NONLINEARITY

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Abstract. Considered herein is the orbital stability in the energy space $H^1(\mathbb{R})$ of a decoupled sum of N peakons for a Camassa-Holm-type equation with quartic nonlinearity, which admits single peakon and multi-peakons. Based on our obtained result of the stability of a single peakon, then combining modulation argument with monotonicity of local energy H^1-norm, we get the stability of the sum of N peakons.

1. Introduction

In the past two decades, the Camassa-Holm (CH) equation

$$y_t + uy_x + 2u_x y = 0, \quad y = u - u_{xx},$$

attracted a great deal of attention among the nonlinear integrable equations and the communities of the PDEs. In 1993, Camassa and Holm [3] obtained the CH equation by approximating directly in the Hamiltonian for Euler's equations in the shallow water regime. It can model the unidirectional propagation of shallow water waves over a flat bottom [3, 13, 24], with $u(t,x)$ standing for the fluid velocity at time $t \geq 0$ in the spatial $x \in \mathbb{R}$ direction. Actually, the CH equation was initially introduced in 1981 by Fuchssteiner and Fokas [18] as a bi-Hamiltonian generalization of the KdV equation. The CH equation shares with the classical KdV equation the properties that it has bi-Hamiltonian structure [18] and is completely integrable [1, 7]. However, while all smooth solutions of the KdV equation are global, the CH equation admits global strong solutions [6, 9, 10] as well as *breaking waves* [6, 9–11], i.e., the wave profile remains bounded, but its slope becomes unbounded in finite time [37].

Another remarkable property of the CH equation is the presence of peaked solitary wave solutions [4], called *peakons*. They are given by $u(t,x) = \varphi_c(x - ct) = ce^{-|x-ct|}$, $c \in \mathbb{R}$, which are solitons, retaining their shape and speed

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after interacting with other peakons [2]. It is worth pointing out that the feature of peakons that their profile is smooth except for a peak at its crest, is analogous to that of the waves of greatest height, i.e., traveling waves of largest possible amplitude which are solutions to be governing equations for water waves [8,12,36]. In 2000, Constantin and Strauss [14] applied the conservation laws to give an impressive proof of stability by a direct approach. In 2009, Dika and Molinet [15] derived the stability of multi-peakons by combining the proof of stability of single peakon with a property of almost monotonicity of the local energy norm. Then they [16] also considered the stability of ordered trains of anti-peakons and peakons.

Recently, the great interest in the CH equation has inspired the search for various CH-type equations with cubic or higher-order nonlinearity. Indeed, two integrable CH-type equations with cubic nonlinearity which admit peakons have been extensively studied recently. One is the following modified CH equation:

$$y_t + \big((u^2 - u_x^2)y\big)_x = 0, \quad y = u - u_{xx},$$

and another one is the Novikov equation:

$$y_t + u^2 y_x + 3uu_x y = 0, \quad y = u - u_{xx}.$$

The modified CH equation was derived independently in [17,20,31,32]. It has a bi-Hamiltonian structure [33], and is completely integrable [31]. Fu et al. [19] studied the Cauchy problem of the modified CH equation in Besov spaces and the blow-up scenario. Gui et al. [21] considered the formulation of singularities of solutions and showed that some solutions with certain initial data would blow up in finite time. Then the blow-up phenomena were systematically investigated in [5,28]. The modified CH equation admits peakons of the form $u(t,x) = \varphi_c(t,x) = \sqrt{\frac{3c}{2}}e^{-|x-ct|}$, $c > 0$. The single peakon and train of peakons for the modified CH equation are orbitally stable [34] and [26], respectively. The Novikov equation was proposed in [30] and its integrability, well-posedness, blow-up phenomena, global weak solutions, peakons and their stability were extensively investigated in [22,23,25,30,38].

Since a small perturbation of a peakon yields another one traveling with a different speed and phase shift, the appropriate notion of stability here is that of *orbital stability*, i.e., a wave starting close to a peakon remains close to some translate of it for all later times. It is shown above that one of the main remarkable features of the CH equation (with quadratic nonlinearity), the modified CH and Novikov equations (with cubic nonlinearity) is the existence of orbitally stable peakons. Hence, a natural idea is to extend such study to other CH-type equations with higher-order nonlinearity. By generalizing one of the Hamiltonian structures of the CH equation, Recio and Anco [35] derived the following generalized CH equation:

$$y_t + u_x(u^2 - u_x^2)^{n-1}y + \big(u(u^2 - u_x^2)^{n-1}y\big)_x = 0, \quad y = u - u_{xx}. \tag{1.1}$$

Obviously, Eq. (1.1) can be reduced to the classical CH equation as $n = 1$. It possesses weak solutions given by multi-peakons, which are a linear superposition of peakons with time-dependent amplitudes and positions. Very recently, we have considered the stability of a single peakon for the following CH-type equation with quartic nonlinearity [27]:

$$y_t + u_x(u^2 - u_x^2)y + \big(u(u^2 - u_x^2)y\big)_x = 0, \quad y = u - u_{xx}, \tag{1.2}$$

which is the case $n = 2$ of the generalized CH equation (1.1). In this manuscript, we continue to study the orbital stability of the sum of N sufficiently decoupled peakons for Eq. (1.2). Using our obtained result of the stability of a single peakon [27], and the general strategy introduced by Martel, Merle and Tsai in [29] for the generalized KdV equation, we get the stability of the sum of N peakons in the present paper, which is stated as follows:

Theorem 1.1. *Let be given N velocities $c_1, \ldots, c_N$ such that $0 < c_1 < \cdots < c_N$. There exist $A > 0$, $L_0 > 0$ and $\varepsilon_0 > 0$ only depending on the speeds $(c_i)_{i=1}^N$, such that for any $u(0,x) := u_0(x) \in H^s(\mathbb{R})$, $s > \frac{5}{2}$, if*

$$0 \not\equiv y_0(x) = (1 - \partial_x^2)u_0(x) \geq 0, \tag{1.3}$$

and

$$\|u_0 - \sum_{i=1}^N \varphi_{c_i}(\cdot - z_i^0)\|_{H^1(\mathbb{R})} \leq \varepsilon^2 \quad \text{with} \quad 0 < \varepsilon < \varepsilon_0, \tag{1.4}$$

for some $(z_i^0)_{i=1}^N$, satisfying

$$z_1^0 < \cdots < z_N^0 \text{ and } z_i^0 - z_{i-1}^0 > L \quad \text{with} \quad L > L_0 > 0, \ i = 2, \ldots, N, \tag{1.5}$$

then for the corresponding solution

$$u(t,x) \in C([0,T); H^s(\mathbb{R})) \cap C^1([0,T); H^{s-1}(\mathbb{R}))$$

associated to the Cauchy problem of Eq. (1.2) with u_0 and the maximal existence time $T > 0$, there exist $x_1(t), \ldots, x_N(t)$ such that

$$\sup_{t\in[0,T)} \|u(t,\cdot) - \sum_{i=1}^N \varphi_{c_i}(\cdot - x_i(t))\|_{H^1(\mathbb{R})} \leq A(\sqrt{\varepsilon} + L^{-\frac{1}{8}}), \quad \forall t \in [0,T), \tag{1.6}$$

and

$$x_i(t) - x_{i-1}(t) > \frac{L}{2}, \quad \forall t \in [0,T), \ i = 2, \ldots, N. \tag{1.7}$$

As commented by Dika and Molinet in [15, 16], the general method in [29] developed for the generalized KdV equation indicates that there are two crucial ingredients to prove the stability of the sum of N peakons. One is a dynamical proof of the stability of a single peakon, and the other is a property of almost monotonicity, which says for a solution close to φ_c, the part of the energy traveling at the right of $\varphi_c(\cdot - ct)$ is almost decreasing with respect to time. Our approach to prove Theorem 1.1 is try to follow this method. Since the conservation law $E(u)$ of Eq. (1.2) is the same as the CH equation, we also expect orbital stability of peakons in the sense of the energy space $H^1(\mathbb{R})$. While the other conservation law $F(u)$ of Eq. (1.2) is much more complicated than the cases of the CH, modified CH and Novikov equations due to its quartic nonlinearity. There are mainly two difficulties encountered by $F(u)$. First, following [27], by introducing a polynomial of degree 3 as the functional h (see Lemma 3.3), we thus derive a localized version of an estimate, which establishes the connection between the localized version of the conservation laws E_i and F_i by a polynomial inequality. The second difficulty, involving the proof of the almost monotonicity result on the part of energy $E(\cdot)$ at the right of each peakon, is to estimate the term u_x^4, which is quite different from the cases of the CH and modified CH equations. Of course this new difficulty is caused by the complicated nonlinear structure and higher-order conservation laws. To overcome it, by exploiting the characteristic ODE related to Eq. (1.2) to get the positivity of the solution u under the assumption on the initial data $y_0 = (1-\partial_x^2)u_0 \geq 0$, and the inequality $|u_x| \leq u$, we thus establish the crucial monotonicity result (see Lemma 3.2).

The remainder of this paper is organized as follows. In Section 2, we briefly recall the well-posedness, two conservation laws, and the existence of peakons for Eq. (1.2). In Section 3, we complete the proof of Theorem 1.1, which is divided into four subsections for convenience. In Section 4, we end our paper with an appendix devoted to the proofs of existence of C^1-functions $(\tilde{x}_i(t))_{i=1}^N$ in Lemma 3.1 and identity (3.16) in Lemma 3.2.

2. Preliminaries

In this section, we firstly recall the local well-posedness result regarding the Cauchy problem associated to Eq. (1.2), some properties of the strong solutions and two useful conservation laws, which will be frequently used in the rest of the paper.

Lemma 2.1 ([27])**.** *Let* $u_0(x) \in H^s(\mathbb{R})$ *with* $s > \frac{5}{2}$*. Then there exists* $T > 0$ *such that the Cauchy problem* (1.2) *has a unique strong solution* $u(t,x) \in C([0,T); H^s(\mathbb{R})) \cap C^1([0,T); H^{s-1}(\mathbb{R}))$ *and the map* $u_0 \mapsto u$ *is continuous from a neighborhood of* u_0 *in* $H^s(\mathbb{R})$ *into* $C([0,T); H^s(\mathbb{R})) \cap C^1([0,T); H^{s-1}(\mathbb{R}))$*.*

Lemma 2.2 ([27])**.** *If the initial data* $u_0 \in H^s(\mathbb{R})$ *with* $s > \frac{5}{2}$*, then the following two functionals*

$$E(u) = \int_{\mathbb{R}} \big(u^2 + u_x^2\big)dx \quad \text{and} \quad F(u) = \int_{\mathbb{R}} \big(u^5 + 2u^3u_x^2 - \frac{1}{3}uu_x^4\big)dx \tag{2.1}$$

are invariants for Eq. (1.2)*. Moreover, if* $y_0(x) = (1-\partial_x^2)u_0(x)$ *does not change sign, then* $y(t,x)$ *will not change sign for all* $t \in [0,T)$*. It follows that if* $y_0 \geq 0$*, then the solution* u *of Eq.* (1.2) *is positive for* $(t,x) \in [0,T) \times \mathbb{R}$*, and satisfies*

$$|u_x(t,x)| \leq u(t,x) \quad \text{for all} \quad (t,x) \in [0,T) \times \mathbb{R}. \tag{2.2}$$

In order to understand the meaning of a peakon solution to Eq. (1.2), applying the operator $(1-\partial_x^2)^{-1}$ to the both sides of Eq. (1.2), we deduce

$$u_t + (u^3 - \frac{1}{3}uu_x^2)u_x + (1-\partial_x^2)^{-1}\partial_x\big(u^4 + \frac{3}{2}u^2u_x^2 - \frac{1}{12}u_x^4\big) + (1-\partial_x^2)^{-1}(\frac{1}{3}uu_x^3) = 0. \tag{2.3}$$

Recall that if $p(x) := \frac{1}{2}e^{-|x|}$, $x \in \mathbb{R}$, then $(1-\partial_x^2)^{-1}f = p * f$ for all $f \in L^2$.

We thus have the following notion of weak solutions.

Definition 2.1. Let $u_0 \in W^{1,4}(\mathbb{R})$ be given. If $u(t,x) \in L^\infty_{loc}([0,T); W^{1,4}_{loc}(\mathbb{R}))$ and satisfies

$$\int_0^T \int_{\mathbb{R}} \Big(u\phi_t + \frac{1}{4}u^4\phi_x + \frac{1}{3}uu_x^3\phi + p * \big(u^4 + \frac{3}{2}u^2u_x^2 - \frac{1}{12}u_x^4\big)\phi_x - p * (\frac{1}{3}uu_x^3)\phi\Big)dxdt + \int_{\mathbb{R}} u_0(x)\phi(0,x)dx = 0$$

for all functions $\phi \in C_c^\infty([0,T) \times \mathbb{R})$, then $u(t,x)$ is called a weak solution to Eq. (1.2). If u is a weak solution on $[0,T)$ for every $T > 0$, then it is called a global weak solution.

Based on the above definition of weak solution, we have proved the following existence result of single peakon of Eq. (1.2).

Lemma 2.3 ([27])**.** *For any $a > 0$, the peaked functions of the form*

$$\varphi_c(t,x) = ae^{-|x-ct|}, \quad \textit{where} \quad c = \frac{2}{3}a^3, \tag{2.4}$$

is a global weak solution to Eq. (1.2).

3. Proof of Theorem 1.1

In this section, we will break the proof of Theorem 1.1 into four subsections for convenience. For $\alpha > 0$, and $L > 0$, we define the following neighborhood of size α of the superpositon of N peakons of speed $c_1, \ldots, c_N$, with spatial shifts z_i that satisfied $z_i - z_{i-1} \geq L$,

$$U(\alpha, L) = \left\{ u \in H^1(\mathbb{R});\ \inf_{z_i - z_{i-1} > L} \|u - \sum_{i=1}^N \varphi_{c_i}(\cdot - z_i)\|_{H^1(\mathbb{R})} < \alpha \right\}.$$

By a standard continuity argument, owing to the continuity of $u(t,x)$ in $H^s(\mathbb{R}) \hookrightarrow H^1(\mathbb{R})$, $s > \frac{5}{2}$, to prove Theorem 1.1, it is sufficient to show that there exist $A > 0$, $L_0 > 0$ and $\varepsilon_0 > 0$ such that for all $L > L_0$ and $0 < \varepsilon < \varepsilon_0$, if u_0 satisfies (1.3)-(1.5) and if for some $0 < t_0 < T$,

$$u(t) \in U\big(A(\sqrt{\varepsilon} + L^{-\frac{1}{8}}), \frac{L}{2}\big), \qquad \forall t \in [0, t_0], \tag{3.1}$$

then

$$u(t_0) \in U\big(\frac{A}{2}(\sqrt{\varepsilon} + L^{-\frac{1}{8}}), \frac{2L}{3}\big). \tag{3.2}$$

Therefore, to complete the proof of Theorem 1.1, we only need to verify (3.2) under the assumption (3.1) for some $L > L_0$ and $0 < \varepsilon < \varepsilon_0$, with A, L_0, and ε_0 to be specified later.

3.1. Modulation

In this subsection, we prove that if u stays in some neighbourhood $U(\alpha, \frac{L}{2})$ of the sum of N peakons, then we can decompose the solution u as the sum of N modulated peakons plus a function $v(t)$ which remains small in $H^1(\mathbb{R})$, in the following way: $u(t) = \sum_{i=1}^{N} \varphi_{c_i}(\cdot - \tilde{x}_i(t)) + v(t)$. Moreover, we show that the different bumps of u that are individually close to a peakon get away from each others as time is increasing.

Lemma 3.1. *Let* u_0 *satisfy the conditions* (1.3)-(1.5). *There exist* $0 < \alpha_0 \ll 1$ *and* $L_0 \gg 1$ *depending only on* $(c_i)_{i=1}^{N}$ *such that for all* $0 < \alpha < \alpha_0$ *and* $L_0 < L$, *if the corresponding solution* $u(t) \in U(\alpha, \frac{L}{2})$ *on* $[0, t_0]$ *for some* $0 < t_0 < T$, *then there exist unique* C^1*-functions* $\tilde{x}_i : [0, t_0] \mapsto \mathbb{R}$, $i = 1, \ldots, N$, *such that*

$$\big\|u(t) - \sum_{i=1}^{N} \varphi_{c_i}(\cdot - \tilde{x}_i(t))\big\|_{H^1(\mathbb{R})} \leq O(\sqrt{\alpha}), \tag{3.3}$$

$$\dot{\tilde{x}}(t) := \frac{d}{dt}\tilde{x}_i = c_i + O(\sqrt{\alpha}) + O(L^{-1}), \quad i = 1, \ldots, N, \tag{3.4}$$

and

$$\tilde{x}_i(t) - \tilde{x}_{i-1}(t) \geq \frac{3}{4}L + \frac{c_i - c_{i-1}}{2}t, \quad i = 2, \ldots, N. \tag{3.5}$$

Moreover, setting $\mathcal{J}_i := [y_i(t), y_{i+1}(t)]$, $i = 1, \ldots, N$, *with*

$$\begin{cases} y_1 = -\infty, \quad y_{N+1} = +\infty, \\ y_i(t) = \frac{\tilde{x}_i(t) + \tilde{x}_{i-1}(t)}{2}, \quad i = 2, \ldots, N, \end{cases} \tag{3.6}$$

it holds

$$|x_i(t) - \tilde{x}_i(t)| \leq \frac{L}{12}, \quad i = 1, \ldots, N, \tag{3.7}$$

where $x_1(t), \ldots, x_N(t)$ *are any points such that*

$$u(t, x_i(t)) = \max_{x \in \mathcal{J}_i(t)} u(t, x), \quad i = 1, \ldots, N. \tag{3.8}$$

Proof. Following the approach in [15, 16] developed for the CH equation, we can similarly apply the implicit function theorem and modulation argument to construct N C^1-functions $\tilde{x}_1(t), \ldots, \tilde{x}_N(t)$ on $[0, t_0]$ satisfying a suitable orthogonality condition (4.3). The detail of its proof is given in Appendix A.1

in Section 4. Therefore, to complete the proof of this lemma, it remains for us to prove (3.3)-(3.8).

For $0 < \alpha < \alpha_0$ with $\alpha_0 \ll 1$, using that $u(t) \in U(\alpha, \frac{L}{2})$ and (4.1), we infer that

$$\begin{aligned}
&\big\|u(t) - \sum_{i=1}^{N} \varphi_{c_i}(\cdot - \tilde{x}_i(t))\big\|_{H^1} \\
&\leq \big\|u(t) - \sum_{i=1}^{N} \varphi_{c_i}(\cdot - z_i)\big\|_{H^1} + \sum_{i=1}^{N} \big\|\varphi_{c_i}(\cdot - z_i) - \varphi_{c_i}(\cdot - z_i - y_i(u(t)))\big\|_{H^1} \\
&\leq \alpha + \sqrt{2}\sum_{i=1}^{N} \big(E(\varphi_{c_i}) - \int_{\mathbb{R}} \varphi_{c_i}(x - z_i)\varphi_{c_i}(x - z_i - y_i(u(t)))dx \\
&\quad - \int_{\mathbb{R}} \partial_x\varphi_{c_i}(x - z_i)\partial_x\varphi_{c_i}(x - z_i - y_i(u(t)))dx\big)^{\frac{1}{2}} \\
&\leq \alpha + 2\sum_{i=1}^{N} a_i|y_i(u(t))|^{\frac{1}{2}} \leq O(\sqrt{\alpha}),
\end{aligned}$$

which proves (3.3).

Next let us show that the speed of $\tilde{x}_i$ stays close to c_i. We set

$$R_j(t) := \varphi_{c_j}(\cdot - \tilde{x}_j(t)) \quad \text{and} \quad v(t) := u(t) - \sum_{j=1}^{N} R_j(t).$$

Noticing that

$$\partial_x^2 R_i(t) = -2a_i\delta(\tilde{x}_i(t)) + R_i(t) \ \text{ with } a_i = \sqrt[3]{\frac{3c_i}{2}}. \tag{3.9}$$

Differentiating the orthogonality condition (4.3) with respect to time t, it follows from (3.9) that

$$\begin{aligned}
\int_{\mathbb{R}} v_t(t)\partial_x R_i(t)dx &= \dot{\tilde{x}}_i(t)\langle \partial_x^2 R_i(t), v(t)\rangle_{H^{-1},H^1} \\
&= \dot{\tilde{x}}_i(t)\big(\int_{\mathbb{R}} R_i(t)v(t)dx - 2a_i v(t, \tilde{x}_i(t))\big),
\end{aligned}$$

and thus

$$\big|\int_{\mathbb{R}} v_t(t)\partial_x R_i(t)dx\big| \leq |\dot{\tilde{x}}_i|O(\|v\|_{H^1}) \leq |\dot{\tilde{x}}_i - c_i|O(\|v\|_{H^1}) + O(\|v\|_{H^1}). \tag{3.10}$$

On the other hand, substituting u by $v(t) + \sum_{j=1}^{N} R_j(t)$ into (2.3) and using the following equation of $R_j(t)$:

$$\begin{aligned}
&\partial_t R_j + (\dot{\tilde{x}}_j - c_j)\partial_x R_j + \frac{1}{4}\partial_x(R_j^4) - \frac{1}{3}R_j(\partial_x R_j)^3 \\
&+ (1-\partial_x^2)^{-1}\partial_x(R_j^4 + \frac{3}{2}R_j^2(\partial_x R_j)^2 - \frac{1}{12}(\partial_x R_j)^4) + \frac{1}{3}(1-\partial_x^2)^{-1}R_j(\partial_x R_j)^3 = 0.
\end{aligned}$$

Then we deduce that v satisfies on $[0,t_0]$:

$$\begin{aligned}
&v_t-\sum_{j=1}^{N}(\dot{\tilde{x}}_j-c_j)\partial_x R_j\\
=&\ -\frac{1}{4}\partial_x\big((v+\sum_{j=1}^{N}R_j)^4-\sum_{j=1}^{N}R_j^4\big)\\
&+\frac{1}{3}\big((v+\sum_{j=1}^{N}R_j)(v_x+\sum_{j=1}^{N}\partial_x R_j)^3-\sum_{j=1}^{N}R_j(\partial_x R_j)^3\big)\\
&-(1-\partial_x^2)^{-1}\partial_x\Big(\big((v+\sum_{j=1}^{N}R_j)^4-\sum_{j=1}^{N}R_j^4\big)+\frac{3}{2}\big((v+\sum_{j=1}^{N}R_j)^2(v_x+\sum_{j=1}^{N}\partial_x R_j)^2\\
&-\sum_{j=1}^{N}R_j^2(\partial_x R_j)^2\big)-\frac{1}{12}\big((v_x+\sum_{j=1}^{N}\partial_x R_j)^4-\sum_{j=1}^{N}(\partial_x R_j)^4\big)\Big)\\
&-\frac{1}{3}(1-\partial_x^2)^{-1}\big((v+\sum_{j=1}^{N}R_j)(v_x+\sum_{j=1}^{N}\partial_x R_j)^3-\sum_{j=1}^{N}R_j(\partial_x R_j)^3\big).
\end{aligned}$$

Taking the L^2-scalar product with $\partial_x R_i$, and integrating by parts, we obtain for $t\in[0,t_0]$

(3.11)

$$\begin{aligned}
&-(\dot{\tilde{x}}_i-c_i)\int_{\mathbb{R}}(\partial_x R_i)^2dx\\
=&\ -\int_{\mathbb{R}}v_t\partial_x R_i dx+\sum_{1\le j\le N,j\neq i}(\dot{\tilde{x}}_j-c_j)\int_{\mathbb{R}}(\partial_x R_j)(\partial_x R_i)dx\\
&+\frac{1}{4}\int_{\mathbb{R}}\big((v+\sum_{j=1}^{N}R_j)^4-\sum_{j=1}^{N}R_j^4\big)\partial_x^2 R_i dx\\
&+\frac{1}{3}\int_{\mathbb{R}}\big((v+\sum_{j=1}^{N}R_j)(v_x+\sum_{j=1}^{N}\partial_x R_j)^3-\sum_{j=1}^{N}R_j(\partial_x R_j)^3\big)\partial_x R_i dx\\
&+\int_{\mathbb{R}}(1-\partial_x^2)^{-1}\Big(\big((v+\sum_{j=1}^{N}R_j)^4-\sum_{j=1}^{N}R_j^4\big)+\frac{3}{2}\big((v+\sum_{j=1}^{N}R_j)^2(v_x+\sum_{j=1}^{N}\partial_x R_j)^2\\
&-\sum_{j=1}^{N}R_j^2(\partial_x R_j)^2\big)-\frac{1}{12}\big((v_x+\sum_{j=1}^{N}\partial_x R_j)^4-\sum_{j=1}^{N}(\partial_x R_j)^4\big)\Big)\partial_x^2 R_i dx\\
&-\frac{1}{3}\int_{\mathbb{R}}(1-\partial_x^2)^{-1}\big((v+\sum_{j=1}^{N}R_j)(v_x+\sum_{j=1}^{N}\partial_x R_j)^3-\sum_{j=1}^{N}R_j(\partial_x R_j)^3\big)\partial_x R_i dx
\end{aligned}$$

$$
\begin{aligned}
&= -\int_{\mathbb{R}} v_t \partial_x R_i dx + \sum_{1\le j\le N, j\ne i} (\dot{\tilde{x}}_j - c_j) \int_{\mathbb{R}} (\partial_x R_j)(\partial_x R_i) dx \\
&\quad + I_1 + I_2 + I_3 + I_4.
\end{aligned}
$$

To estimate I_1, for simplicity, we denote

$$
\begin{aligned}
\hat{I}_1(t,x) &:= (v + \sum_{j=1}^N R_j)^4 - \sum_{j=1}^N R_j^4 \\
&= v^4 + 4v^3(\sum_{j=1}^N R_j) + 6v^2(\sum_{j=1}^N R_j)^2 + 4v(\sum_{j=1}^N R_j)^3 + (\sum_{j=1}^N R_j)^4 - \sum_{j=1}^N R_j^4,
\end{aligned}
$$

which together with $\|v\|_{L^\infty(\mathbb{R})} \le \frac{\|v\|_{H^1(\mathbb{R})}}{\sqrt{2}} \le O(\sqrt{\alpha})$, (4.2) and the exponential decay of R_j gives

$$
|\hat{I}_1(t,x)| \le O(\sqrt{\alpha})(O(\sqrt{\alpha}) + O(1)) + O(e^{-\frac{L}{8}}).
$$

Thus, by (3.9), we infer that

$$
I_1 = \frac{1}{4}\Big(-2a_i \hat{I}_1(t, \tilde{x}_i(t)) + \int_{\mathbb{R}} \hat{I}_1 R_i dx \Big) \le O(\sqrt{\alpha}) + O(e^{-\frac{L}{8}}).
$$

To estimate I_2, we calculate

$$
\begin{aligned}
I_2 = \frac{1}{3}\int_{\mathbb{R}} \Big(& v v_x^3 + 3 v v_x^2 \sum_{j=1}^N \partial_x R_j + 3 v v_x \big(\sum_{j=1}^N \partial_x R_j \big)^2 + v \big(\sum_{j=1}^N \partial_x R_j \big)^3 \\
&+ v_x^3 (\sum_{j=1}^N R_j) + 3 v_x^2 (\sum_{j=1}^N R_j \cdot \sum_{j=1}^N \partial_x R_j) + 3 v_x \big(\sum_{j=1}^N R_j \cdot (\sum_{j=1}^N \partial_x R_j)^2 \big) \Big) \partial_x R_i dx \\
&+ \frac{1}{3}\int_{\mathbb{R}} \Big(\sum_{j=1}^N R_j \cdot (\sum_{j=1}^N \partial_x R_j)^3 - \sum_{j=1}^N R_j (\partial_x R_j)^3 \Big) \partial_x R_i dx.
\end{aligned}
$$

Using (2.2) and (3.3), we have

$$
\begin{aligned}
\|v_x\|_{L^\infty(\mathbb{R})} &\le \|u_x\|_{L^\infty(\mathbb{R})} + \|\sum_{j=1}^N \partial_x R_j\|_{L^\infty(\mathbb{R})} \le \|u\|_{L^\infty(\mathbb{R})} + \sum_{j=1}^N \|\partial_x R_j\|_{L^\infty(\mathbb{R})} \\
&\le \frac{1}{\sqrt{2}} \|v + \sum_{j=1}^N R_j\|_{H^1(\mathbb{R})} + \sum_{j=1}^N a_j \le O(\sqrt{\alpha}) + O(1). \tag{3.12}
\end{aligned}
$$

In view of (3.12) and (4.2), using the exponential decay of R_j and Hölder's inequality, we have the following estimate

$$
\begin{aligned}
I_2 \le C\Big(& \|v\|_{L^\infty} (\|v_x\|_{L^\infty} + 1) \int_{\mathbb{R}} v_x^2 dx + (\int_{\mathbb{R}} v^2 dx)^{\frac{1}{2}} (\int_{\mathbb{R}} v_x^2 dx)^{\frac{1}{2}} + \|v\|_{L^\infty} \\
&+ (\|v_x\|_{L^\infty} + 1) \int_{\mathbb{R}} v_x^2 dx + (\int_{\mathbb{R}} v_x^2 dx)^{\frac{1}{2}} \Big) + O(e^{-\frac{L}{8}})
\end{aligned}
$$

$$\begin{aligned}&\le C\big(\|v\|_{H^1}^2+\|v\|_{H^1}+1\big)\|v\|_{H^1}+O(e^{-\frac{L}{8}})\\&\le O(\sqrt{\alpha})\big(O(\alpha)+O(\sqrt{\alpha})+O(1)\big)+O(e^{-\frac{L}{8}})\le O(\sqrt{\alpha})+O(e^{-\frac{L}{8}}).\end{aligned}$$

In a similar manner as above, noting that $(1-\partial_x^2)^{-1}f=\frac{1}{2}e^{-|x|}*f$, one can easily deduce that $I_3+I_4\le O(\sqrt{\alpha})+O(e^{-\frac{L}{8}})$. Therefore, combining the above estimations of I_1-I_4, with (4.2) and the exponential decay of R_j, we deduce from (3.10)-(3.11) that

$$\begin{aligned}a_i^2|\dot{\tilde{x}}_i-c_i|&\le\Big|\int_{\mathbb{R}}v_t\partial_xR_idx\Big|+\sum_{1\le j\le N,j\ne i}(|\dot{\tilde{x}}_j|+c_j)\Big|\int_{\mathbb{R}}(\partial_xR_j)(\partial_xR_i)dx\Big|\\&\quad+O(\sqrt{\alpha})+O(e^{-\frac{L}{8}})\\&\le O(\sqrt{\alpha})|\dot{\tilde{x}}_i-c_i|+O(\sqrt{\alpha})+O(e^{-\frac{L}{8}}),\end{aligned}$$

which yields (3.4).

To prove (3.5), taking $0<\alpha<\alpha_0$ and $L>L_0$, with $\alpha_0\ll 1$ and $L_0\gg 1$, and then combining (1.3)-(1.5), (3.4) with (4.2), we deduce for all $t\in[0,t_0]$ there exists $s\in[0,t]$ such that

$$\begin{aligned}\tilde{x}_i(t)-\tilde{x}_{i-1}(t)&=\tilde{x}_i(0)-\tilde{x}_{i-1}(0)+\big(\dot{\tilde{x}}_i(s)-\dot{\tilde{x}}_{i-1}(s)\big)t\\&=\tilde{x}_i(0)-\tilde{x}_{i-1}(0)+\big((\dot{\tilde{x}}_i(s)-c_i)-(c_{i-1}-\dot{\tilde{x}}_{i-1}(s))\big)t\\&\quad+(c_i-c_{i-1})t\\&\ge\frac{3}{4}L+\frac{c_i-c_{i-1}}{2}t.\end{aligned}$$

Finally, by the continuous embedding of $H^1(\mathbb{R})$ into $L^\infty(\mathbb{R})$ and (3.3), we have

$$u(x)=\sum_{i=1}^N\varphi_{c_i}(x-\tilde{x}_i(t))+O(\sqrt{\alpha}),\quad\forall x\in\mathbb{R}.$$

Applying the above formula with $x=x_i$ and $u(x_i)=\max_{x\in\mathcal{J}_i}u(x)$, and using (3.5), it holds

$$\begin{aligned}u(x_i)&=\max_{x\in\mathcal{J}_i}\left\{\sum_{i=1}^N\varphi_{c_i}(x-\tilde{x}_i(t))\right\}+O(\sqrt{\alpha})\\&=a_i+O(e^{-\frac{L}{4}})+O(\sqrt{\alpha})\ge\frac{2}{3}a_i.\end{aligned}$$

On the other hand, for $x\in\mathcal{J}_i\backslash[\tilde{x}_i(t)-\frac{L}{12},\tilde{x}_i(t)+\frac{L}{12}]$, we derive

$$u(x)\le a_ie^{-\frac{L}{12}}+O(e^{-\frac{L}{4}})+O(\sqrt{\alpha})\le\frac{a_i}{2},$$

which ensures that $x\in[\tilde{x}_i(t)-\frac{L}{12},\tilde{x}_i(t)+\frac{L}{12}]$. This completes the proof of Lemma 3.1. ☐

3.2. Monotonicity property

This subsection is devoted to proving the principal tool of our proof, which is the almost monotonicity of functionals that are very close to the energy at the right of the ith bump of the solution $u(t,x)$, $i = 1,\dots,N-1$. Firstly, we define Ψ to be a C^∞ function such that

$$\begin{cases} 0<\Psi(x)<1, \Psi'(x)>0, & x\in\mathbb{R},\\ |\Psi'''(x)|\le 10\Psi'(x), & x\in[-1,1],\end{cases} \quad \text{and } \Psi(x)=\begin{cases} e^{-|x|}, & x<-1,\\ 1-e^{-|x|}, & x>1.\end{cases}$$

Setting $\Psi_K=\Psi(\frac{\cdot}{K})$, $K>0$, then we introduce for $j=2,\dots,N$,

$$\mathcal{I}_{j,K}(t)=\int_{\mathbb{R}}\big(u^2(t,x)+u_x^2(t,x)\big)\Psi_{j,K}(t,x)dx, \tag{3.13}$$

where $\Psi_{j,K}(t,x)=\Psi_K(x-y_j(t))$ with y_j's defined in (3.6). Notice that $\mathcal{I}_{j,K}(t)$ is close to $\|u(t,x)\|^2_{H^1(x>y_j(t))}$, thus it measures the energy at the right of the $(j-1)$th bump of u. Finally, we set

$$\sigma_0=\frac{1}{4}\min\{c_1,c_2-c_1,\dots,c_N-c_{N-1}\}. \tag{3.14}$$

Lemma 3.2. *Let $u(t,x)$ be a strong solution of Eq.* (1.2) *satisfying* (3.3) *on* $[0,t_0]$ *with initial data $u(0,x)=u_0(x)$, which satisfies* (1.3)-(1.5). *There exist $\alpha_0>0$ and $L_0>0$ only depending on $(c_i)_{i=1}^N$, such that if $0<\alpha<\alpha_0$ and $L>L_0$ then for any $4\le K=O(\sqrt{L})$,*

$$\mathcal{I}_{j,K}(t)-\mathcal{I}_{j,K}(0)\le O(e^{-\frac{L}{8K}}),\quad \forall t\in[0,t_0],\quad i=2,\dots,N. \tag{3.15}$$

Proof. To prove this lemma, we first claim that for any smooth function $g(x):\mathbb{R}\mapsto\mathbb{R}$, it holds

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}}(u^2+u_x^2)g\,dx\\ =&-\frac{1}{2}\int_{\mathbb{R}}u(u^2-u_x^2)^2g'dx-2\int_{\mathbb{R}}u\big((1-\partial_x^2)^{-1}(uu_x^2y)\big)g'dx\\ &+\frac{1}{2}\int_{\mathbb{R}}u^5g'dx+\int_{\mathbb{R}}u\big((1-\partial_x^2)^{-1}(2u^4+5u^2u_x^2+\frac{1}{2}u_x^4)\big)g'dx,\end{aligned} \tag{3.16}$$

whose proof is given in Appendix A.2 in Section 4.

Applying (3.16) with $g=\Psi_{j,K}$ and using $\frac{d}{dt}\Psi_{j,K}=-\dot{y}_j(t)\partial_x\Psi_{j,K}$, we get

$$\begin{aligned}\frac{d}{dt}\mathcal{I}_{j,K}(t)=&\frac{d}{dt}\int_{\mathbb{R}}(u^2+u_x^2)\Psi_{j,K}dx\\ =&-\dot{y}_j(t)\int_{\mathbb{R}}(u^2+u_x^2)\partial_x\Psi_{j,K}dx-\frac{1}{2}\int_{\mathbb{R}}u(u^2-u_x^2)^2\partial_x\Psi_{j,K}dx\\ &-2\int_{\mathbb{R}}u\big((1-\partial_x^2)^{-1}(uu_x^2y)\big)\partial_x\Psi_{j,K}dx+\frac{1}{2}\int_{\mathbb{R}}u^5\partial_x\Psi_{j,K}dx\\ &+\int_{\mathbb{R}}u\big((1-\partial_x^2)^{-1}(2u^4+5u^2u_x^2+\frac{1}{2}u_x^4)\big)\partial_x\Psi_{j,K}dx.\end{aligned} \tag{3.17}$$

Combining (3.4) with (3.6), it holds for $0 < \alpha < \alpha_0$ small enough and $L > L_0$ large enough,

$$\begin{aligned} \dot{y}_j(t) &= \frac{\dot{\tilde{x}}_j(t) - c_j}{2} + \frac{\dot{\tilde{x}}_{j-1}(t) - c_{j-1}}{2} + \frac{c_j + c_{j-1}}{2} \\ &\geq \frac{c_j + c_{j-1}}{2} + O(\sqrt{\alpha}) + O(L^{-1}) \geq \frac{c_1}{2}. \end{aligned} \tag{3.18}$$

Note that the assumptions on u_0 guarantee the positivity of u and y by Lemma 2.2. Hence, together with $\partial_x \Psi_{j,K} = \frac{1}{K}\Psi'(\frac{x-y_j(t)}{K}) > 0$ and (3.18), we deduce from (3.17) that

$$\begin{aligned} \frac{d}{dt}\mathcal{I}_{j,K}(t) \leq\ & -\frac{c_1}{2}\int_{\mathbb{R}} \big(u^2 + u_x^2\big)\partial_x \Psi_{j,K} dx + \frac{1}{2}\int_{\mathbb{R}} u^5 \partial_x \Psi_{j,K} dx \\ & + \int_{\mathbb{R}} u\big((1-\partial_x^2)^{-1}(2u^4 + 5u^2u_x^2 + \frac{1}{2}u_x^4)\big)\partial_x \Psi_{j,K} dx \\ :=\ & -\frac{c_1}{2}\int_{\mathbb{R}} \big(u^2 + u_x^2\big)\partial_x \Psi_{j,K} dx + J_1 + J_2. \end{aligned} \tag{3.19}$$

To estimate J_1-J_2, we firstly divide $\mathbb{R}$ into two regions $D_j := [\tilde{x}_{j-1}(t) + \frac{L}{4}, \tilde{x}_j(t) - \frac{L}{4}]$, $i = 2, \ldots, N$, and its complement D_j^c. Combining (3.5) with (3.6), we find that for $x \in D_j^c$,

$$|x - y_j(t)| \geq \frac{\tilde{x}_j(t) - \tilde{x}_{j-1}(t)}{2} - \frac{L}{4} \geq \frac{c_j - c_{j-1}}{4}t + \frac{L}{8} \geq \sigma_0 t + \frac{L}{8},$$

and then for $K = O(\sqrt{L})$ and L_0 large enough, we have $\frac{|x-y_j(t)|}{K} > 1$, which along with the definition of Ψ yields

$$\partial_x \Psi_{j,K} = \frac{1}{K}\Psi'(\frac{x-y_j(t)}{K}) \leq \frac{1}{K}e^{-\frac{1}{K}(\sigma_0 t + \frac{L}{8})}, \quad x \in D_j^c. \tag{3.20}$$

On the other hand, using the exponential decay of $\varphi_{c_i}(x - \tilde{x}_i(t))$ for any $x \in D_j$, and (3.3), it holds

$$\begin{aligned} \|u(t)\|_{L^\infty(D_j)} &\leq \|u - \sum_{i=1}^{N}\varphi_{c_i}(\cdot - \tilde{x}_i(t))\|_{L^\infty(D_j)} + \sum_{i=1}^{N}\|\varphi_{c_i}(\cdot - \tilde{x}_i(t))\|_{L^\infty(D_j)} \\ &\leq O(\sqrt{\alpha}) + O(e^{-\frac{L}{8}}). \end{aligned} \tag{3.21}$$

To estimate J_1, combining (3.20) with (3.21), for $0 < \alpha < \alpha_0$ and $L > L_0$, with $\alpha_0 \ll 1$ and $L_0 \gg 1$, gives rise to

$$\begin{aligned} J_1 &= \frac{1}{2}\int_{D_j} u^5 \partial_x \Psi_{j,K} dx + \frac{1}{2}\int_{D_j^c} u^5 \partial_x \Psi_{j,K} dx \\ &\leq \frac{1}{2}\|u\|_{L^\infty(D_j)}^3 \int_{D_j} u^2 \partial_x \Psi_{j,K} dx \\ &\quad + \frac{1}{2}\|\partial_x \Psi_{j,K}\|_{L^\infty(D_j^c)}\|u\|_{L^\infty(\mathbb{R})}^3 \|u\|_{L^2(\mathbb{R})}^2 \end{aligned}$$

$$\le \frac{1}{2}\|u\|^3_{L^\infty(D_j)}\int_{D_j} u^2\partial_x\Psi_{j,K}dx + \frac{C}{K}\|u_0\|^5_{H^1(\mathbb{R})}e^{-\frac{1}{K}(\sigma_0 t+\frac{L}{8})}$$

$$\le \frac{c_1}{8}\int_{\mathbb{R}}(u^2+u_x^2)\partial_x\Psi_{j,K}dx + \frac{C}{K}\|u_0\|^5_{H^1(\mathbb{R})}e^{-\frac{1}{K}(\sigma_0 t+\frac{L}{8})}. \tag{3.22}$$

To bound J_2, since $(1-\partial_x^2)^{-1}f = \frac{1}{2}e^{-|x|}*f$, we deduce from (3.20) that

$$\begin{aligned}
&\int_{D_j^c} u\big((1-\partial_x^2)^{-1}(2u^4+5u^2u_x^2+\frac{1}{2}u_x^4)\big)\partial_x\Psi_{j,K}dx\\
&\le \frac{1}{4}\|u\|_{L^\infty(\mathbb{R})}\sup_{x\in D_j^c}|\partial_x\Psi_{j,K}(t,x)|\int_{\mathbb{R}} e^{-|x|}*(4u^4+10u^2u_x^2+u_x^4)dx\\
&\le \frac{1}{4}\|u\|_{L^\infty(\mathbb{R})}\sup_{x\in D_j^c}|\partial_x\Psi_{j,K}(t,x)|\int_{\mathbb{R}} e^{-|x|}dx\cdot\int_{\mathbb{R}}(5u^4+10u^2u_x^2)dx\\
&\le 5\|u\|^3_{L^\infty(\mathbb{R})}\sup_{x\in D_j^c}|\partial_x\Psi_{j,K}(t,x)|\int_{\mathbb{R}}(u^2+u_x^2)dx
\end{aligned}$$

$$\le \frac{C}{K}\|u_0\|^5_{H^1(\mathbb{R})}e^{-\frac{(\sigma_0 t+L/8)}{K}}, \tag{3.23}$$

where we used the fact that $|u_x|\le u$ in Lemma 2.2. Now for $x\in D_j$, noticing that u, $2u^4+5u^2u_x^2+\frac{1}{2}u_x^4$ and $\partial_x\Psi_{j,K}$ are non-negative, we have

$$\begin{aligned}
&\int_{D_j} u\big((1-\partial_x^2)^{-1}(2u^4+5u^2u_x^2+\frac{1}{2}u_x^4)\big)\partial_x\Psi_{j,K}dx\\
&\le \|u\|_{L^\infty(D_j)}\int_{D_j}\big((1-\partial_x^2)^{-1}(2u^4+5u^2u_x^2+\frac{1}{2}u_x^4)\big)\partial_x\Psi_{j,K}dx\\
&\le \frac{1}{2}\|u\|_{L^\infty(D_j)}\int_{\mathbb{R}}(4u^4+10u^2u_x^2+u_x^4)(1-\partial_x^2)^{-1}\partial_x\Psi_{j,K}dx
\end{aligned}$$

$$\le \frac{5}{2}\|u\|_{L^\infty(D_j)}\|u\|^2_{H^1(\mathbb{R})}\int_{\mathbb{R}}(u^2+u_x^2)(1-\partial_x^2)^{-1}\partial_x\Psi_{j,K}dx, \tag{3.24}$$

where we used the inequalities $|u_x|\le u$ and $\sup_{x\in\mathbb{R}}|u(x)|\le\frac{1}{\sqrt{2}}\|u\|_{H^1(\mathbb{R})}$. By the definition of Ψ and the property of $|\Psi'''(x)|\le 10\Psi'(x)$, we get

$$\begin{aligned}
&(1-\partial_x^2)\partial_x\Psi_{j,K}\ge(1-\frac{10}{K^2})\partial_x\Psi_{j,K}\\
&\Rightarrow(1-\partial_x^2)^{-1}\partial_x\Psi_{j,K}\le(1-\frac{10}{K^2})^{-1}\partial_x\Psi_{j,K}.
\end{aligned}$$

Taking $K\ge 4$, we infer from (3.21) and (3.24) for $0<\alpha<\alpha_0$ and $L>L_0$, with $\alpha_0\ll 1$ and $L_0\gg 1$ that

$$\begin{aligned}
&\int_{D_j} u\big((1-\partial_x^2)^{-1}(2u^4+5u^2u_x^2+\frac{1}{2}u_x^4)\big)\partial_x\Psi_{j,K}dx\\
&\le\big(O(\sqrt{\alpha})+O(e^{-\frac{L}{8}})\big)\|u_0\|^2_{H^1(\mathbb{R})}\int_{\mathbb{R}}(u^2+u_x^2)\partial_x\Psi_{j,K}dx
\end{aligned}$$

$$
\begin{aligned}
&\le \big(O(\sqrt{\alpha}) + O(e^{-\frac{L}{8}})\big)\big(\|u_0 - \sum_{i=1}^{N} \varphi_{c_i}(\cdot - \tilde{x}_i(0))\|_{H^1(\mathbb{R})} \\
&\quad + \|\sum_{i=1}^{N} \varphi_{c_i}(\cdot - \tilde{x}_i(0))\|_{H^1(\mathbb{R})}\big)^2 \int_{\mathbb{R}} \big(u^2 + u_x^2\big)\partial_x \Psi_{j,K} dx \\
&\le \big(O(\sqrt{\alpha}) + O(e^{-\frac{L}{8}})\big)\big(O(\sqrt{\alpha}) + \sum_{i=1}^{N} \sqrt{2} a_i\big)^2 \int_{\mathbb{R}} \big(u^2 + u_x^2\big)\partial_x \Psi_{j,K} dx \\
&\le \frac{c_1}{8} \int_{\mathbb{R}} \big(u^2 + u_x^2\big)\partial_x \Psi_{j,K} dx,
\end{aligned}
$$

which along with (3.23) yields

$$
J_2 \le \frac{c_1}{8} \int_{\mathbb{R}} \big(u^2 + u_x^2\big)\partial_x \Psi_{j,K} dx + \frac{C}{K}\|u_0\|_{H^1(\mathbb{R})}^5 e^{-\frac{1}{K}(\sigma_0 t + \frac{L}{8})}. \tag{3.25}
$$

Therefore, plugging (3.22) and (3.25) into (3.19), we find

$$
\frac{d}{dt}\mathcal{I}_{j,K}(t) \le -\frac{c_1}{4} \int_{\mathbb{R}} \big(u^2 + u_x^2\big)\partial_x \Psi_{j,K} dx + \frac{C}{K}\|u_0\|_{H^1(\mathbb{R})}^5 e^{-\frac{1}{K}(\sigma_0 t + \frac{L}{8})}.
$$

Thus the monotonicity property (3.15) can be obtained by integrating the above inequality from 0 to t, with $t \le t_0$. This completes the proof of Lemma 3.2. ☐

3.3. Localized estimate and global identity

Firstly, for $i = 1, \dots, N$, we define the following localized version of the conservation laws (2.1) as

$$
E_i(t) := E_i(u(t)) = \int_{\mathbb{R}} (u^2 + u_x^2)\Phi_i(t) dx, \tag{3.26}
$$

and

$$
F_i(t) := F_i(u(t)) = \int_{\mathbb{R}} \big(u^5 + 2u^3 u_x^2 - \frac{1}{3} u u_x^4\big)\Phi_i(t) dx. \tag{3.27}
$$

Here the weight functions $\Phi_i = \Phi_i(t,x)$ are given by

$$
\begin{cases}
\Phi_1 = 1 - \Psi_{2,K} = 1 - \Psi_K(\cdot - y_2(t)), \quad \Phi_N = \Psi_{N,K} = \Psi_K(\cdot - y_N(t)), \\
\Phi_i = \Psi_{i,K} - \Psi_{i+1,K} = \Psi_K(\cdot - y_i(t)) - \Psi_K(\cdot - y_{i+1}(t)), \quad i = 2, \dots, N-1,
\end{cases}
$$

where $\Psi_{i,K}$'s and $y_i(t)$'s are defined in Subsection 3.2 and (3.6), respectively. Then we find that the Φ_i's are positive functions and $\sum_{i=1}^{N} \Phi_i(t,x) \equiv 1$. Finally, taking $L/K > 0$ large enough, and using the exponentially asymptotic behavior of Φ_i, it is easy to check for $i = 1, \dots, N$ that

$$
\big|1 - \Phi_i\big| \le 4e^{-\frac{L}{4K}} \quad \text{for } x \in [\tilde{x}_i - \frac{L}{4}, \tilde{x}_i + \frac{L}{4}], \tag{3.28}
$$

and

$$
\big|\Phi_i\big| \le 4e^{-\frac{L}{4K}} \quad \text{for } x \in [\tilde{x}_j - \frac{L}{4}, \tilde{x}_j + \frac{L}{4}], \text{ whenever } j \ne i. \tag{3.29}
$$

We now derive a local version of an estimate, which establishes the connection between E_i and F_i by a polynomial inequality. Noticing that the functionals E_i and F_i are independent of time since we fix $\tilde{x}_1 < \cdots < \tilde{x}_N$.

Lemma 3.3. *Let be given $\tilde{x}_1 < \cdots < \tilde{x}_N$ with $\tilde{x}_i - \tilde{x}_{i-1} \geq \frac{3L}{4}$. Define the interval $\mathcal{J}_i$ as in Lemma 3.1 and assume that for $i = 1, \ldots N$, there exist $x_i \in \mathcal{J}_i$ such that $u(x_i) = \max_{x \in \mathcal{J}_i} u(x) := M_i$ and $|x_i - \tilde{x}_i| < \frac{L}{12}$. Then, for any fixed positive function $u \in H^s(\mathbb{R})$, $s > \frac{5}{2}$, it holds*

$$F_i(u) \leq \frac{4}{3} M_i^3 E_i(u) - \frac{8}{5} M_i^5 + \|u_0\|^5_{H^1(\mathbb{R})} O(L^{-\frac{1}{2}}), \quad i = 1, \ldots, N. \tag{3.30}$$

Proof. Let $i = 1, \ldots N$ be fixed. We first define the function g as in [14]

$$g(x) = \begin{cases} u(x) - u_x(x), & x < x_i, \\ u(x) + u_x(x), & x > x_i. \end{cases}$$

We thus get

$$\begin{aligned} \int_{\mathbb{R}} g^2(x) \Phi_i(x) dx &= \int_{-\infty}^{x_i} (u(x) - u_x(x))^2 \Phi_i dx + \int_{x_i}^{+\infty} (u(x) + u_x(x))^2 \Phi_i dx \\ &= E_i(u) - 2M_i^2 \Phi_i(x_i) + \int_{-\infty}^{x_i} u^2 \partial_x \Phi_i dx - \int_{x_i}^{+\infty} u^2 \partial_x \Phi_i dx. \end{aligned} \tag{3.31}$$

Next, following [27], we introduce the function $h(x)$ defined by

$$h(x) = \begin{cases} u^3(x) - \frac{2}{3} u^2(x) u_x(x) - \frac{1}{3} u u_x^2(x), & x < x_i, \\ u^3(x) + \frac{2}{3} u^2(x) u_x(x) - \frac{1}{3} u u_x^2(x), & x > x_i. \end{cases}$$

Integrating by parts, we compute

$$\begin{aligned} &\int_{\mathbb{R}} h(x) g^2(x) \Phi_i(x) dx \\ &= \int_{-\infty}^{x_i} (u^3 - \frac{2}{3} u^2 u_x - \frac{1}{3} u u_x^2)(u - u_x)^2 \Phi_i dx \\ &\quad + \int_{x_i}^{+\infty} (u^3 + \frac{2}{3} u^2 u_x - \frac{1}{3} u u_x^2)(u + u_x)^2 \Phi_i dx \\ &= \int_{-\infty}^{x_i} (u^5 + 2u^3 u_x^2 - \frac{1}{3} u u_x^4) \Phi_i dx - \frac{8}{3} \int_{-\infty}^{x_i} u^4 u_x \Phi_i dx \\ &\quad + \int_{x_i}^{+\infty} (u^5 + 2u^3 u_x^2 - \frac{1}{3} u u_x^4) \Phi_i dx + \frac{8}{3} \int_{x_i}^{+\infty} u^4 u_x \Phi_i dx \\ &= F_i(u) - \frac{8}{15} u^5 \Phi_i \Big|_{-\infty}^{x_i} + \frac{8}{15} \int_{-\infty}^{x_i} u^5 \partial_x \Phi_i dx + \frac{8}{15} u^5 \Phi_i \Big|_{x_i}^{+\infty} \\ &\quad - \frac{8}{15} \int_{x_i}^{+\infty} u^5 \partial_x \Phi_i dx \end{aligned}$$

$$= F_i(u) - \frac{16}{15}M_i^5\Phi_i(x_i) + \frac{8}{15}\int_{-\infty}^{x_i} u^5\partial_x\Phi_i dx - \frac{8}{15}\int_{x_i}^{+\infty} u^5\partial_x\Phi_i dx. \tag{3.32}$$

By the Cauchy-Schwarz inequality, we deduce for the positive solution $u(x)$ that

$$\begin{aligned} h(x) &= u^3(x) \mp \frac{2}{3}u^{\frac{3}{2}}(x)\cdot u^{\frac{1}{2}}(x)u_x(x) - \frac{1}{3}u(x)u_x^2(x) \\ &\leq u^3(x) + \frac{1}{3}u^3(x) = \frac{4}{3}u^3. \end{aligned} \tag{3.33}$$

Combining (3.31) with (3.33), we obtain

$$\begin{aligned} &\int_{\mathbb{R}} h(x)g^2(x)\Phi_i(x)dx \\ \leq\ & \frac{4}{3}\int_{\mathbb{R}} u^3(x)g^2(x)\Phi_i(x)dx \\ =\ & \frac{4}{3}\int_{\mathcal{J}_i} u^3(x)g^2(x)\Phi_i(x)dx + \frac{4}{3}\sum_{1\leq j\leq N, j\neq i}\int_{\mathcal{J}_j} u^3(x)g^2(x)\Phi_i(x)dx \\ \leq\ & \frac{4}{3}M_i^3\int_{\mathbb{R}} g^2(x)\Phi_i(x)dx + \frac{4}{3}\sum_{1\leq j\leq N, j\neq i}\int_{\mathcal{J}_j} u^3(x)g^2(x)\Phi_i(x)dx \\ =\ & \frac{4}{3}M_i^3E_i(u) - \frac{8}{3}M_i^5\Phi_i(x_i) + \frac{4}{3}M_i^3\int_{-\infty}^{x_i} u^2\partial_x\Phi_i dx - \frac{4}{3}M_i^3\int_{x_i}^{+\infty} u^2\partial_x\Phi_i dx \\ & + \frac{4}{3}\sum_{1\leq j\leq N, j\neq i}\int_{\mathcal{J}_j} u^3(x)g^2(x)\Phi_i(x)dx, \end{aligned}$$

which along with (3.32) gives rise to

$$\begin{aligned} F_i(u) \leq\ & \frac{4}{3}M_i^3E_i(u) + \frac{8}{5}M_i^5(1-\Phi_i(x_i)) - \frac{8}{5}M_i^5 \\ & + \frac{4}{3}\sum_{1\leq j\leq N, j\neq i}\int_{\mathcal{J}_j} u^3(x)g^2(x)\Phi_i(x)dx \\ & + \frac{4}{3}M_i^3\int_{-\infty}^{x_i} u^2\partial_x\Phi_i dx - \frac{4}{3}M_i^3\int_{x_i}^{+\infty} u^2\partial_x\Phi_i dx - \frac{8}{15}\int_{-\infty}^{x_i} u^5\partial_x\Phi_i dx \\ & + \frac{8}{15}\int_{x_i}^{+\infty} u^5\partial_x\Phi_i dx. \end{aligned} \tag{3.34}$$

Taking $K = \sqrt{L}/8$, we deduce that with a constant $C > 0$, $|\partial_x\Phi_i| \leq C/K = O(\sqrt{L})$. Moreover, since $|x_i - \tilde{x}_i| < L/12$, it follows from (3.28) that $|1 - \Phi_i(x_i)| \leq 4e^{-L/4K} \leq O(\sqrt{L})$. Hence, with (3.29) and the Sobolev embedding $\|u\|_{L^\infty(\mathbb{R})} \leq \frac{\|u\|_{H^1(\mathbb{R})}}{\sqrt{2}}$ at hand, we infer from (3.34) that

$$F_i(u) \leq \frac{4}{3}M_i^3E_i(u) - \frac{8}{5}M_i^5 + \|u_0\|_{H^1(\mathbb{R})}^5 O(L^{-\frac{1}{2}}).$$

This completes the proof of Lemma 3.3. $\square$

Next, we present a global identity, which is the generalization of Lemma 3.1 in [27]. For $Z=(z_1,\dots,z_N)\in\mathbb{R}^N$, we set

$$R_Z(\cdot)=\sum_{i=1}^N R_{z_i}(\cdot)=\sum_{i=1}^N \varphi_{c_i}(\cdot-z_i)=\sum_{i=1}^N a_i\varphi(\cdot-z_i)=\sum_{i=1}^N a_ie^{-|\cdot-z_i|}, \tag{3.35}$$

where $a_i=\sqrt[3]{\frac{3c_i}{2}}$ by (2.4). Obviously, $R_{z_i}(x)$ has the peak at $x=z_i$, and hence $\max\limits_{x\in\mathbb{R}} R_{z_i}(x)=R_{z_i}(z_i)=a_i$. By a simple computation, we obtain

$$E(R_{z_i})=2a_i^2 \quad \text{and} \quad F(R_{z_i})=\frac{16}{15}a_i^5. \tag{3.36}$$

Lemma 3.4. *For any $(z_1,\dots,z_N)\in\mathbb{R}^N$ such that $|z_i-z_{i-1}|>\frac{L}{2}$ with $L>0$, $i=2,\dots,N$, and for any $u\in H^1(\mathbb{R})$, it holds*

$$\begin{aligned}E(u)-\sum_{i=1}^N E(\varphi_{c_i})=\|u-\sum_{i=1}^N R_{z_i}(x)\|^2_{H^1(\mathbb{R})}+4\sum_{i=1}^N a_i\big(u(z_i)-a_i\big)\\ +O(e^{-\frac{L}{4}}),\end{aligned} \tag{3.37}$$

where the constant involving in $O(e^{-\frac{L}{4}})$ depends only on $(c_i)_{i=1}^N$, since $a_i=\sqrt[3]{\frac{3c_i}{2}}$.

Proof. Integrating by parts, we have

$$\begin{aligned}
&\|u-\sum_{i=1}^N R_{z_i}(x)\|^2_{H^1(\mathbb{R})}\\
&=E(u)+E(\sum_{i=1}^N R_{z_i})-2\sum_{i=1}^N a_i\int_{\mathbb{R}}u(x)\varphi(\cdot-z_i)dx\\
&\quad-2\sum_{i=1}^N a_i\int_{\mathbb{R}}u_x(x)\varphi_x(\cdot-z_i)dx\\
&=E(u)+E(\sum_{i=1}^N R_{z_i})\\
&\quad+2\sum_{i=1}^N a_i\Big(\int_{z_i}^{+\infty}u_x(x)\varphi(\cdot-z_i)dx-\int_{-\infty}^{z_i}u_x(x)\varphi(\cdot-z_i)dx\Big)\\
&\quad-2\sum_{i=1}^N a_i\int_{\mathbb{R}}u(x)\varphi(\cdot-z_i)dx\\
&=E(u)-E(\sum_{i=1}^N R_{z_i})+4\Big(\frac{1}{2}E(\sum_{i=1}^N R_{z_i})-\sum_{i=1}^N a_iu(z_i)\Big). \qquad (3.38)
\end{aligned}$$

Since $|z_i - z_{i-1}| \geq \frac{L}{2}$, it follows from (3.36) that

$$E\Big(\sum_{i=1}^{N} R_{z_i}\Big) = \sum_{i=1}^{N} E(\varphi_{c_i}) + O(e^{-\frac{L}{4}}) = 2\sum_{i=1}^{N} a_i^2 + O(e^{-\frac{L}{4}}). \tag{3.39}$$

Combining (3.38) and (3.39), we obtain (3.37). This completes the proof of Lemma 3.4. □

We also need the following lemma, which enables us to control the distances between global and local energies at $t = 0$.

Lemma 3.5. *Let* $u_0 \in H^s(\mathbb{R})$, $s > \frac{5}{2}$ *satisfy* (1.3)-(1.5). *Then the following estimates hold*:

$$\Big|E(u_0) - \sum_{i=1}^{N} E(\varphi_{c_i})\Big| \leq O(\varepsilon^2) + O(e^{-\frac{L}{4}}), \tag{3.40}$$

$$\big|E_i(u_0) - E(\varphi_{c_i})\big| \leq O(\varepsilon^2) + O(e^{-\sqrt{L}}), \quad i = 1, \dots, N, \tag{3.41}$$

and

$$\big|F_i(u_0) - F(\varphi_{c_i})\big| \leq O(\varepsilon^2) + O(e^{-\sqrt{L}}), \quad i = 1, \dots, N, \tag{3.42}$$

where $O(\cdot)$ *depend only on* $(c_i)_{i=1}^N$, *since* $a_i = \sqrt[3]{\frac{3c_i}{2}}$.

Proof. For the first estimate, applying triangular inequality, and using (1.4), we get

$$\begin{aligned}\big|E(u_0) - E(R_{Z^0})\big| &= \big|\|u_0\|_{H^1(\mathbb{R})} - \|R_{Z^0}\|_{H^1(\mathbb{R})}\big| \cdot \big(\|u_0\|_{H^1(\mathbb{R})} + \|R_{Z^0}\|_{H^1(\mathbb{R})}\big)\\ &\leq \|u_0 - R_{Z^0}\|_{H^1(\mathbb{R})} \cdot \big(\|u_0 - R_{Z^0}\|_{H^1(\mathbb{R})} + 2\|R_{Z^0}\|_{H^1(\mathbb{R})}\big)\\ &\leq \varepsilon^2(\varepsilon^2 + 2\sqrt{2}\sum_{i=1}^{N} a_i),\end{aligned}$$

which together with (3.39) yields

$$\begin{aligned}\Big|E(u_0) - \sum_{i=1}^{N} E(\varphi_{c_i})\Big| &\leq \big|E(u_0) - E(R_{Z^0})\big| + \Big|E(R_{Z^0}) - \sum_{i=1}^{N} E(\varphi_{c_i})\Big|\\ &\leq \varepsilon^2(\varepsilon^2 + O(1)) + O(e^{-\frac{L}{4}}) \leq O(\varepsilon^2) + O(e^{-\frac{L}{4}}).\end{aligned}$$

For the second estimate, it follows from (1.4) and the exponential decay of φ_{c_i}'s and Φ_i's, and the definition of $E_i(\cdot)$ that

$$\begin{aligned}&\big|E_i(u_0) - E(\varphi_{c_i})\big|\\ &\leq \big|\|u_0\|^2_{H^1(\mathcal{J}_i(0))} - \|\varphi_{c_i}\|^2_{H^1(\mathcal{J}_i(0))}\big| + O(e^{-\sqrt{L}})\\ &= \big|\|u_0\|_{H^1(\mathcal{J}_i(0))} - \|\varphi_{c_i}\|_{H^1(\mathcal{J}_i(0))}\big|\big(\|u_0\|_{H^1(\mathcal{J}_i(0))} + \|\varphi_{c_i}\|_{H^1(\mathcal{J}_i(0))}\big) + O(e^{-\sqrt{L}})\\ &\leq \big(\|u_0 - R_{Z^0}\|_{H^1(\mathcal{J}_i(0))}\end{aligned}$$

$$+\sum_{1\le j\le N, j\ne i}\|\varphi_{c_j}\|_{H^1(\mathcal{J}_i(0))}\big)\big(\|u_0-R_{Z^0}\|_{H^1(\mathbb{R})}+2\sqrt{2}\sum_{i=1}^N a_i\big)+O(e^{-\sqrt{L}})$$
$$\le(\varepsilon^2+O(e^{-\frac{L}{8}})(\varepsilon^2+O(1))+O(e^{-\sqrt{L}})\le O(\varepsilon^2)+O(e^{-\sqrt{L}}).$$

For the third estimate, combining the above similar argument on the second estimate and the method developed for the estimate of $|F(u)-F(\varphi)|$ in Lemma 3.3 in [27], one can easily find that (3.42) holds. Thus we omit the details here. This completes the proof of Lemma 3.5. ☐

3.4. End of the proof of Theorem 1.1

Let $u(t,x)$ be a strong solution of Eq. (1.2) satisfying (3.1) on $[0,t_0]$ for some $0<t_0<T$, with initial data $u_0(x)\in H^s(\mathbb{R}), s>\frac{5}{2}$, which satisfies (1.3)-(1.5). Let us set $M_i=\max_{x\in\mathcal{J}_i}u(t_0,x)=u(t_0,x_i(t_0))$, with $\mathcal{J}_i$'s as in (3.6), and $\delta_i:=a_i-M_i$. Noticing that, by (3.5) and (3.7), we deduce for $i=2,\dots,N$ that

$$x_i(t_0)-x_{i-1}(t_0)\ge\tilde{x}_i(t_0)-\frac{L}{12}-(\tilde{x}_{i-1}(t_0)+\frac{L}{12})\ge\frac{3L}{4}-\frac{L}{6}>\frac{L}{2}. \tag{3.43}$$

Hence, applying (3.37) and (3.40) with $u(t_0)$ gives rise to

$$\|u(t_0,x)-\sum_{i=1}^N\varphi_{c_i}(x-x_i(t_0))\|^2_{H^1(\mathbb{R})}\le 4\sum_{i=1}^N a_i\delta_i+O(\varepsilon^2)+O(e^{-\frac{L}{4}}). \tag{3.44}$$

Therefore, to conclude the proof of Theorem 1.1, it is sufficient to prove that there exists $C>0$ only depending on $(c_i)_{i=1}^N$ such that

$$\delta_i\le C(\varepsilon+L^{-\frac{1}{4}}),\quad i=1,\dots,N, \tag{3.45}$$

and Theorem 1.1 follows by taking $A=2C$.

To prove (3.45), by (3.30), we have

$$F_i(u(t_0))\le\frac{4}{3}M_i^3E_i(u(t_0))-\frac{8}{5}M_i^5+O(L^{-\frac{1}{2}}),\quad i=1,\dots,N.$$

Taking the sum over i of the above inequality yields

$$F(u(t_0))=\sum_{i=1}^N F_i(u(t_0))\le\frac{4}{3}\sum_{i=1}^N M_i^3E_i(u(t_0))-\frac{8}{5}\sum_{i=1}^N M_i^5+O(L^{-\frac{1}{2}}). \tag{3.46}$$

Denoting $\Delta_0^{t_0}F(u):=F(u(t_0))-F(u_0)$ and $\Delta_0^{t_0}E(u):=E(u(t_0))-E(u_0)$, then it follows from (3.46) and the conservation laws (2.1) that

$$\begin{aligned}0=\Delta_0^{t_0}F(u)=\sum_{i=1}^N\Delta_0^{t_0}F_i(u)&\le\frac{4}{3}\sum_{i=1}^N M_i^3\Delta_0^{t_0}E_i(u)-\frac{8}{5}\sum_{i=1}^N M_i^5\\&+\sum_{i=1}^N\big(\frac{4}{3}M_i^3E_i(u_0)-F_i(u_0)\big)+O(L^{-\frac{1}{2}}).\end{aligned} \tag{3.47}$$

Using the conservation law $E(u)$ and (3.40), we obtain

$$\begin{aligned} M_i^2 &\le \|u(t,x)\|^2_{L^\infty(\mathbb{R})} \\ &\le \frac{\|u\|^2_{H^1(\mathbb{R})}}{2} = \frac{E(u_0)}{2} \\ &\le \frac{1}{2}\sum_{i=1}^N E(\varphi_{c_i}) + O(\varepsilon^2) + O(e^{-\frac{L}{4}}) \le 2\sum_{i=1}^N a_i^2 \end{aligned} \tag{3.48}$$

for $0 < \varepsilon < \varepsilon_0$ and $L > L_0 > 0$ with $\varepsilon_0 \ll 1$ and $L_0 \gg 1$ both depending only on $(c_i)_{i=1}^N$. Combining (3.41)-(3.42) with (3.48), and using (3.36), we thus obtain having substituted M_i by $a_i - \delta_i$ that

$$\begin{aligned} &\sum_{i=1}^N \big(-\frac{8}{5}M_i^5 + \frac{4}{3}M_i^3 E_i(u_0) - F_i(u_0)\big) \\ &= \sum_{i=1}^N \big(-\frac{8}{5}M_i^5 + \frac{4}{3}M_i^3(E_i(u_0) - E(\varphi_{c_i})) + \frac{4}{3}M_i^3 E(\varphi_{c_i}) \\ &\qquad - (F_i(u_0) - F(\varphi_{c_i})) - F(\varphi_{c_i})\big) \\ &\le 8\sum_{i=1}^N \delta_i^2\big(-a_i^3 + \frac{5}{3}a_i^2\delta_i - a_i\delta_i^2 + \frac{1}{5}\delta_i^3\big) + O(\varepsilon^2) + O(e^{-\sqrt{L}}) \\ &= -\frac{8}{15}\sum_{i=1}^N \delta_i^2\big(2a_i^3 + 4a_i^2 M_i + 6a_i M_i^2 + 3M_i^3\big) + O(\varepsilon^2) + O(e^{-\sqrt{L}}). \end{aligned} \tag{3.49}$$

Then, by (3.47) and (3.49), for $0 < \varepsilon < \varepsilon_0$ and $L > L_0 > 0$ with $\varepsilon_0 \ll 1$ and $L_0 \gg 1$, it holds

$$\begin{aligned} &\sum_{i=1}^N \delta_i^2\big(2a_i^3 + 4a_i^2 M_i + 6a_i M_i^2 + 3M_i^3\big) \\ &\le \frac{5}{2}\sum_{i=1}^N M_i^3 \Delta_0^{t_0} E_i(u) + O(\varepsilon^2) + O(L^{-\frac{1}{2}}). \end{aligned}$$

Using the Abel transformation and the definition of the weight function Φ_i, we deduce from the above inequality that

$$\begin{aligned} &\sum_{i=1}^N \delta_i^2\big(2a_i^3 + 4a_i^2 M_i + 6a_i M_i^2 + 3M_i^3\big) \\ &\le \frac{5}{2}\sum_{i=2}^N (M_i^3 - M_{i-1}^3)\Delta_0^{t_0}\mathcal{I}_{j,K} + O(\varepsilon^2) + O(L^{-\frac{1}{2}}). \end{aligned} \tag{3.50}$$

where $\mathcal{I}_{j,K}(t)$ is given in (3.13) in Subsection 3.2. Recalling from (3.1) that if $u(t) \in U\big(\alpha, \frac{L}{2}\big), \forall t \in [0, t_0]$, in view of Lemma 3.1, then there exists $\tilde{X} =$

$(\tilde{x}_1, \dots, \tilde{x}_N)$ with $\tilde{x}_i \in \mathcal{J}_i$ such that $\|u(t_0) - R_{\tilde{X}}\|_{H^1(\mathbb{R})} \leq O(\sqrt{\alpha})$, where $R_{\tilde{X}}$ is defined in (3.35). Hence, for $X = (x_1, \dots, x_N)$, it follows from (3.7) that

$$\begin{aligned}\|u(t_0,\cdot) - R_X\|_{H^1(\mathbb{R})} &= \|u(t_0,\cdot) - \sum_{j=1}^{N} \varphi_{c_j}(\cdot - x_j(t_0))\|_{H^1(\mathbb{R})}\\ &\leq \|u(t_0,\cdot) - R_{\tilde{X}}\|_{H^1(\mathbb{R})} + \|R_{\tilde{X}} - R_X\|_{H^1(\mathbb{R})}\\ &\leq O(\sqrt{\alpha}) + \sum_{i=1}^{N} \|\varphi_{c_i}(\cdot - x_i(t_0)) - \varphi_{c_i}(\cdot - \tilde{x}_i(t_0))\|_{H^1(\mathbb{R})}\\ &\leq O(\sqrt{\alpha}) + O(e^{-\frac{L}{4}}),\end{aligned}$$

which along with the inequality (3.43) yields

$$\begin{aligned}u(t_0, x_i(t_0)) &= \sum_{j=1}^{N} \varphi_{c_j}(x_i(t_0) - x_j(t_0)) + O(\sqrt{\alpha}) + O(e^{-\frac{L}{4}})\\ &= a_i + \sum_{1\leq j\leq N, j\neq i} \varphi_{c_j}(x_i(t_0) - x_j(t_0)) + O(\sqrt{\alpha}) + O(e^{-\frac{L}{4}})\\ &= a_i + O(\sqrt{\alpha}) + O(e^{-\frac{L}{4}}).\end{aligned}$$

Taking $\alpha = A(\varepsilon + L^{-\frac{1}{4}})$, it follows from the above inequality that

$$M_i = a_i + O(\sqrt{\varepsilon}) + O(L^{-\frac{1}{8}}). \tag{3.51}$$

Owing to $0 < c_1 < \cdots < c_N$ and the relation $a_i = \sqrt[3]{\frac{3c_i}{2}}$, we deduce from (3.51), for $0 < \varepsilon < \varepsilon_0$ and $L > L_0 > 0$ with $\varepsilon_0 \ll 1$ and $L_0 \gg 1$, that

$$0 < M_1 < \cdots < M_N. \tag{3.52}$$

Thus, combining (3.48), (3.50), (3.52) with the monotonicity property (3.15), we have

$$3\sum_{i=1}^{N} \delta_i^2 M_i^3 \leq \sum_{i=1}^{N} \delta_i^2 \big(2a_i^3 + 4a_i^2 M_i + 6a_i M_i^2 + 3M_i^3\big) \leq O(\varepsilon^2) + O(L^{-\frac{1}{2}}).$$

Therefore, we find that there exists $C > 0$ only depending on $(c_i)_{i=1}^N$ and $\|u_0\|_{H^s(\mathbb{R})}$ such that

$$\delta_i \leq C(\varepsilon + L^{-\frac{1}{4}}), \quad i = 1, \dots, N,$$

which is the desired result (3.45). This completes the proof of Theorem 1.1.

4. Appendix

A.1. Construction of C^1-functions $(\tilde{x}_i(t))_{i=1}^N$ in Lemma 3.1. We firstly apply the implicit function theorem to prove the decomposition of the solution $u \in U(\alpha, \frac{L}{2})$ with no time dependency. For $Z = (z_1, \dots, z_N) \in \mathbb{R}^N$, such that $|z_i - z_{i-1}| > \frac{L}{2}$, we denote $R_Z = \sum_{i=1}^{N} \varphi_{c_i}(\cdot - z_i)$ and $B_{H^1}(R_Z, \alpha)$ as the ball in $H^1(\mathbb{R})$ of center R_Z with radius α. For $0<\alpha<\alpha_0$, we define the following mapping:

$$\begin{aligned} Y : B_{H^1}(R_Z,\alpha)\times\prod_{i=1}^{N}(-\alpha,\alpha) &\to \mathbb{R}^N,\\ (u,y_1,\dots,y_N) &\mapsto \big(Y^1(u,y_1,\dots,y_N),\dots,Y^N(u,y_1,\dots,y_N)\big), \end{aligned}$$

with

$$Y^i(u,y_1,\dots,y_N) := \int_{\mathbb{R}}\Big(u(x)-\sum_{j=1}^{N}\varphi_{c_j}(x-z_j-y_j)\Big)\partial_x\varphi_{c_i}(x-z_i-y_i)dx.$$

Next, we verify that the function Y satisfies the following three properties:

(i) $Y(R_Z,0,\dots,0)=(0,\dots,0)$.

(ii) By the dominated convergence theorem, we find that Y is a mapping of class C^1. Indeed, for $i=1,\dots,N$, we compute

$$\begin{aligned} &\frac{\partial Y^i}{\partial u}(u,y_1,\dots,y_N)=\int_{\mathbb{R}}\partial_x\varphi_{c_i}(x-z_i-y_i)dx,\\ &\frac{\partial Y^i}{\partial y_i}(u,y_1,\dots,y_N)=\int_{\mathbb{R}}\Big(u_x-\sum_{1\le j\le N, j\ne i}\partial_x\varphi_{c_j}(x-z_j-y_j)\Big)\partial_x\varphi_{c_i}(x-z_i-y_i)dx, \end{aligned}$$

and for $j\neq i$

$$\frac{\partial Y^i}{\partial y_j}(u,y_1,\dots,y_N)=\int_{\mathbb{R}}\partial_x\varphi_{c_j}(x-z_j-y_j)\partial_x\varphi_{c_i}(x-z_i-y_i)dx.$$

(iii) The determinant of the matrix $D_{(y_1,\dots,y_N)}Y(R_Z,0,\dots,0)\neq 0$. In fact, thanks to (ii), we have

$$\frac{\partial Y^i}{\partial y_i}(R_Z,0,\dots,0)=\int_{\mathbb{R}}(\partial_x\varphi_{c_i}(x-z_i))^2dx=a_i^2\ge a_1^2, \quad \text{where } a_i=\sqrt[3]{\frac{3c_i}{2}}.$$

and for $j\neq i$, using the exponential decay of peakons φ_{c_i} and $|z_i-z_{i-1}|>\frac{L}{2}$, for $L>L_0>0$ with $L_0\gg 1$, it holds

$$\frac{\partial Y^i}{\partial y_j}(R_Z,0,\dots,0)=\int_{\mathbb{R}}\partial_x\varphi_{c_j}(x-z_j)\partial_x\varphi_{c_i}(x-z_i)dx\le O(e^{-\frac{L}{4}}).$$

We deduce that for L_0 large enough, the Jacobi matrix

$$D_{(y_1,\dots,y_N)}Y(R_Z,0,\dots,0)=P+Q,$$

where P is an invertible diagonal matrix with the norms of $\|P^{-1}\|\le(a_1)^{-2}$ and $\|Q\|\le O(e^{-\frac{L}{4}})$. Hence, there exists $L_0>0$ such that for $L>L_0$, $D_{(y_1,\dots,y_N)}Y(R_Z,0,\dots,0)$ is invertible with an inverse matrix of norm smaller than $2(a_1)^{-2}$. Therefore, the implicit function theorem implies that there exists $0<\beta_0<\alpha$ and uniquely determined C^1 functions $(y_1(u),\dots,y_N(u))$ from $B_{H^1}(R_Z,\beta_0)$ to a neighborhood of $(0,\dots,0)$ such that $Y(u,y_1,\dots,y_N)=$

$(0,\dots,0)$, for all $u \in B_{H^1}(R_Z,\beta_0)$. Moreover, if $u \in B_{H^1}(R_Z,\beta)$ with $0<\beta\le\beta_0$, then there exists a constant $C_0>0$ such that

$$\sum_{i=1}^{N} \big|y_i(u)\big| \le C_0\beta. \tag{4.1}$$

Notice that β_0 and C_0 depend only on $a_1=\sqrt[3]{\frac{3c_1}{2}}$ and L_0, but not on $Z=(z_1,\dots,z_N)\in\mathbb{R}^N$. For $u\in B_{H^1}(R_Z,\beta_0)$, we set $\tilde{x}_i(u)=z_i+y_i(u)$. If we take $\beta_0\le\min\{\alpha,\frac{L_0}{8C_0}\}$, then $(\tilde{x}_1,\dots,\tilde{x}_N)$ are C^1-functions on $B_{H^1}(R_Z,\beta)$, satisfying

$$\tilde{x}_i(u)-\tilde{x}_{i-1}(u)=z_i-z_{i-1}+y_i(u)-y_{i-1}(u)>\frac{L}{2}-2C_0\beta\ge\frac{L}{4}. \tag{4.2}$$

For $L\ge L_0$ and $0<\alpha<\alpha_0<\frac{\beta_0}{2}$ to be chosen later, we define the modulation of $u\in U(\alpha,\frac{L}{2})$ as follows. Covering the trajectory of u by N_0 open balls in the following way:

$$\{u(t),t\in[0,t_0]\}\subset\bigcup_{k=1,\dots,N_0} B_{H^1}(R_{Z^k},2\alpha).$$

Owing to $0<\alpha<\alpha_0<\frac{\beta_0}{2}$, the functions $\tilde{x}_i(u)$ are uniquely determined for $u\in B(R_{Z^k},2\alpha)\cap B(R_{Z^{k'}},2\alpha)$. Hence, we define the functions $t\mapsto\tilde{x}_i(t)$ for all $t\in[0,t_0]$ by setting $\tilde{x}_i(t)=\tilde{x}_i(u(t))$. By construction, for $i=1,\dots,N$ and $t\in[0,t_0]$, the following orthogonality condition holds:

$$\int_{\mathbb{R}}\big(u(t,\cdot)-\sum_{j=1}^{N}\varphi_{c_j}(\cdot-\tilde{x}_j(t))\big)\partial_x\varphi_{c_i}(\cdot-\tilde{x}_i(t))dx=0. \tag{4.3}$$

A.2. Proof of the identity (3.16) in Lemma 3.2. To prove (3.16), let us first suppose that $u(t,x)$ is smooth since the case $u(t,x)\in C([0,T);H^s(\mathbb{R}))\cap C^1([0,T);H^{s-1}(\mathbb{R}))$, with $s>\frac{5}{2}$ follows by the density argument. Differentiating (2.3) with respect to x, we have

$$\begin{aligned}u_{tx}=&-(u^3u_{xx}+\frac{3}{2}u^2u_x^2-\frac{1}{4}u_x^4-uu_x^2u_{xx}-u^4)\\ &-(1-\partial_x^2)^{-1}(u^4+\frac{3}{2}u^2u_x^2-\frac{1}{12}u_x^4)-\frac{1}{3}(1-\partial_x^2)^{-1}\partial_x(uu_x^3).\end{aligned} \tag{4.4}$$

Using integration by parts, it follows from Eq. (1.2) and (4.4) that

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}}(u^2+u_x^2)g(x)dx=2\int_{\mathbb{R}}uy_tgdx-2\int_{\mathbb{R}}uu_{tx}g'dx\\ =&-2\int_{\mathbb{R}}u(\frac{1}{4}(u^2-u_x^2)^2+u(u^2-u_x^2)y)_xgdx\\ &+2\int_{\mathbb{R}}u(u^3u_{xx}+\frac{3}{2}u^2u_x^2-\frac{1}{4}u_x^4-uu_x^2u_{xx}-u^4)g'dx\end{aligned}$$

$$
\begin{aligned}
&+2\int_{\mathbb{R}} u(1-\partial_x^2)^{-1}\Big(u^4+\frac{3}{2}u^2u_x^2-\frac{1}{12}u_x^4\Big)g'dx\\
&+\frac{2}{3}\int_{\mathbb{R}} u(1-\partial_x^2)^{-1}\partial_x(uu_x^3)g'dx := K_1+K_2+K_3+K_4.
\end{aligned}
\tag{4.5}
$$

It is easy to check that

$$
\begin{aligned}
K_1+K_2 &= \frac{1}{2}\int_{\mathbb{R}} u_x(u^2-u_x^2)^2 g dx+\int_{\mathbb{R}} u(u^2-u_x^2)(u^2-u_x^2)_x g dx\\
&\quad +\frac{1}{2}\int_{\mathbb{R}} u^5 g'dx\\
&= \frac{1}{2}\int_{\mathbb{R}} (u(u^2-u_x^2)^2)_x g dx+\frac{1}{2}\int_{\mathbb{R}} u^5 g'dx\\
&= -\frac{1}{2}\int_{\mathbb{R}} u(u^2-u_x^2)^2 g'dx+\frac{1}{2}\int_{\mathbb{R}} u^5 g'dx.
\end{aligned}
$$

For the term K_4, we calculate

$$
\begin{aligned}
K_4 &= \frac{2}{3}\int_{\mathbb{R}} u(1-\partial_x^2)^{-1}(u_x^4+3uu_x^2u_{xx})g'dx\\
&= \frac{2}{3}\int_{\mathbb{R}} u((1-\partial_x^2)^{-1}u_x^4)g'dx-2\int_{\mathbb{R}} u((1-\partial_x^2)^{-1}uu_x^2y)g'dx\\
&\quad +2\int_{\mathbb{R}} u((1-\partial_x^2)^{-1}u^2u_x^2)g'dx.
\end{aligned}
$$

Thus, plugging the above identities of K_1+K_2 and K_4 into (4.5) yields the desired result (3.16).

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