

CHARACTERIZATIONS FOR THE FOCK-TYPE SPACES

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ABSTRACT. We obtain Lipschitz type characterization and double integral characterization for Fock-type spaces with the norm

$$\|f\|_{F_{m,\alpha,t}^p}^p = \int_{\mathbb{C}^n} |f(z)e^{-\alpha|z|^m}|^p \frac{dV(z)}{(1+|z|)^t},$$

where $\alpha > 0$, $t \in \mathbb{R}$, and $m \in \mathbb{N}$. The results of this paper are the extensions of the classical weighted Fock space $F_{2,\alpha,t}^p$.

1. Introduction

For a fixed positive integer n , let $H(\mathbb{C}^n)$ be the space of all entire functions on the complex n -space \mathbb{C}^n . For $z = (z_1, \dots, z_n)$ and $w = (w_1, \dots, w_n)$ in \mathbb{C}^n , we write

$$z \cdot w = z_1 \bar{w}_1 + \dots + z_n \bar{w}_n, \quad |z| = \sqrt{|z_1|^2 + \dots + |z_n|^2}.$$

For $\alpha > 0$, $t \in \mathbb{R}$, and $m \in \mathbb{N}$, we define $dG_{m,\alpha,t}$ the t -weighted (m, α) -Gaussian measure on \mathbb{C}^n by

$$dG_{m,\alpha,t}(z) = C_{m,\alpha,t} e^{-\alpha|z|^m} \frac{dV(z)}{(1+|z|)^t},$$

where dV is the volume measure on \mathbb{C}^n and $C_{m,\alpha,t}$ is the positive constant to be the normalized volume measure. For $0 < p < \infty$, we consider the Fock-type space $F_{m,\alpha,t}^p(\mathbb{C}^n)$ consisting of all $f \in H(\mathbb{C}^n)$, the class of all entire functions on \mathbb{C}^n , where the norm is defined by

$$\|f\|_{F_{m,\alpha,t}^p}^p = C_{m,\alpha,t} \int_{\mathbb{C}^n} |f(z)e^{-\alpha|z|^m}|^p \frac{dV(z)}{(1+|z|)^t}.$$

Then $F_{m,\alpha,t}^p(\mathbb{C}^n) = L^p(dG_{m,\alpha,t}) \cap H(\mathbb{C}^n)$. When $m = 2$, $F_{2,\alpha,t}^p(\mathbb{C}^n)$ is the classical weighted Fock space.

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In this paper, we characterize the Fock-type space $F_{m,\alpha,t}^p(\mathbb{C}^n)$. One of the main results is Lipschitz type characterization for $F_{m,\alpha,t}^p(\mathbb{C}^n)$ stated in Theorem 1.1 as follows.

Theorem 1.1. *Let $\alpha > 0$, $0 < p < \infty$, $s \geq 0$, $m \in \mathbb{N}$ and $t \in \mathbb{R}$. Then the following statements are equivalent for entire functions f on \mathbb{C}^n :*

- (a) $f \in F_{m,\alpha,t}^p(\mathbb{C}^n)$.
- (b) *There exists a nonnegative continuous function $g \in L^p(G_{m,\alpha p,t-sp(m-1)})$ such that*

$$\frac{|f(z) - f(w)|}{|z - w|} \leq (1 + |z|^{m-1} + |w|^{m-1})^{1+s} (g(z) + g(w))$$

for each $z, w \in \mathbb{C}^n$ with $z \neq w$.

For $f \in H(\mathbb{C}^n)$, we define

$$Lf(z, w) = f(z) - f(w).$$

Let m be an even positive integer and $s \in \mathbb{R}$. As a local type, we define

$$L_r^s f(z, w) = [f(z) - f(w)] e^{s(\bar{z} \cdot w) \frac{m}{2}} \chi_{E_r(z)}(w),$$

where $\chi_{E_r(z)}$ denotes the characteristic function in the Euclidean ball $E_r(z) = \{w \in \mathbb{C}^n : |w - z| < \frac{r}{1+|z|^{m-1}}\}$ for $r > 0$. Hence, the other main result is the double integral characterization for $F_{m,\alpha,t}^p$ as follows.

Theorem 1.2. *Let $\alpha > 0$, $0 < p < \infty$, and $t \in \mathbb{R}$. Let m be an even positive integer. For $s \geq 0$, let $\beta = \frac{s+\alpha}{2}$. Then the following statements are equivalent for entire functions f on \mathbb{C}^n :*

- (a) $f \in F_{m,\alpha,t}^p(\mathbb{C}^n)$.
- (b) $Lf \in L^p(G_{m,\alpha p,t} \times G_{m,\alpha p,t})$.
- (c) $L_r^s f \in L^p(G_{m,\beta p,\delta} \times G_{m,\beta p,\delta})$, where $\delta = \frac{t}{2} - n(m-1)$.

In Theorem 1.2, (a) and (b) are equivalent when $m \in \mathbb{N}$. On the other hand, (a) and (c) are equivalent when m is an even positive integer. An even positive integer m is necessary in order that the function $L_r^s f(z, w)$ is holomorphic with respect to w .

Lipschitz type characterization for weighted Bergman spaces with standard weights on the unit disc in the complex plane \mathbb{C} in terms of the Euclidean, hyperbolic, and pseudo-hyperbolic metrics was introduced by Hasi Wulan and Kehe Zhu [5]. Moreover, they generalized these results to the unit ball in \mathbb{C}^n . As an application, they proved the boundedness of the symmetric lifting operator in [4]. Moreover, double integral characterizations for weighted Bergman spaces in the unit ball in \mathbb{C}^n were proved in [3] and [4]. Our results were motivated by Lipschitz type characterization and double integral characterization for the weighted Fock spaces $F_{2,\alpha,t}^p(\mathbb{C}^n)$ in [1]. It's well known that $F_{2,\alpha,t}^p(\mathbb{C}^n)$ is closed

in $L^p(G_{2,\alpha p,t})$. In particular, $F_{2,\alpha,t}^2(\mathbb{C}^n)$ is a Hilbert space and the reproducing kernel K_z at $z \in \mathbb{C}^n$ for $F_{2,1/2,0}^2(\mathbb{C}^n)$ is given by

$$K_z(w) = e^{\bar{z} \cdot w}.$$

See [6]. In [1], the reproducing kernel K_z was used to get a norm estimate for $F_{2,\alpha,t}^p(\mathbb{C}^n)$. In this paper, we have the same norm estimate for the Fock-type space $F_{m,\alpha,t}^p(\mathbb{C}^n)$ without using K_z .

For nonnegative quantities X and Y , the notation $X \lesssim Y$ means that there exists a positive constant C such that $X \leq CY$. The constant C is independent of the relevant variables. Furthermore, the notation $X \approx Y$ means that $X \lesssim Y$ and $Y \lesssim X$. In this case, we say that X and Y are equivalent.

2. Preliminaries

In this section, we suppose $m \in \mathbb{N}$.

For $r > 0$, $E_r(z)$ is defined by

$$(2.1) \quad E_r(z) = \left\{ w \in \mathbb{C}^n : |w - z| < \frac{r}{1 + |z|^{m-1}} \right\}.$$

Lemma 2.1. *Let $m \in \mathbb{N}$ and $r > 0$. For any $w \in E_r(z)$, there exists a positive constant $C = C(m, r)$ such that*

$$(2.2) \quad C^{-1} \leq \frac{e^{|w|^m}}{e^{|z|^m}} \leq C.$$

Moreover, there exists a positive constant $C = C(m, r)$ such that

$$(2.3) \quad C^{-1} \leq \frac{1 + |w|^{m-1}}{1 + |z|^{m-1}} \leq C.$$

Proof. Note that the asserted inequalities are clear when $m = 1$. So, let $m \geq 2$. For $w \in E_r(z)$, we have

$$|w| < \frac{r}{1 + |z|^{m-1}} + |z|$$

so that

$$(2.4) \quad \begin{aligned} |w|^m &< |z|^m + \sum_{k=1}^m \binom{m}{k} r^k \frac{|z|^{m-k}}{(1 + |z|^{m-1})^k} \\ &< |z|^m + (1 + r)^m. \end{aligned}$$

Similarly, we get

$$(2.5) \quad 1 + |w|^{m-1} < (1 + |z|^{m-1})(1 + r)^{m-1}$$

which implies

$$\frac{r}{1 + |z|^{m-1}} < \frac{r(1 + r)^{m-1}}{1 + |w|^{m-1}}.$$

Thus, we have

$$(2.6) \quad E_r(z) \subset E_{2r(1+r)^{m-1}}(w), \quad w \in E_r(z).$$

Consequently, (2.4) and (2.6) give us (2.2). Moreover, (2.5) and (2.6) imply (2.3). The proof is complete. \square

We denote a multi-index $M = (m_1, \dots, m_n)$ which is an n -tuple of non-negative integers and use the following notation $|M| = m_1 + \dots + m_n$, $M! = m_1! \cdots m_n!$ and $\partial^M = \partial_1^{m_1} \cdots \partial_n^{m_n}$ where ∂_j denotes partial differentiation with respect to the j -th component.

Lemma 2.2. *Let $r > 0$, $m \in \mathbb{N}$ and $\alpha, t \in \mathbb{R}$. Given a multi-index M , there is a positive constant $C = C(\alpha, m, r, t, M)$ such that*

$$\frac{|\partial^M g(z)|^p e^{-\alpha|z|^m}}{(1 + |z|^{m-1})^{p|M|+2n+\frac{t}{m-1}}} \leq C \int_{E_r(z)} |g(w)|^p \frac{e^{-\alpha|w|^m}}{(1 + |w|)^t} dV(w), \quad z \in \mathbb{C}^n,$$

for $0 < p < \infty$ and $g \in H(\mathbb{C}^n)$.

Proof. Let $0 < p < \infty$ and $g \in H(\mathbb{C}^n)$. Let $z \in \mathbb{C}^n$. By subharmonicity we have

$$\begin{aligned} |g(z)|^p &\leq \frac{1}{V[E_r(z)]} \int_{E_r(z)} |g(w)|^p dV(w) \\ &= \frac{(1 + |z|^{m-1})^{2n}}{\omega_n r^{2n}} \int_{E_r(z)} |g(w)|^p dV(w), \end{aligned}$$

where ω_n is the volume of the unit ball of \mathbb{C}^n . And hence the Cauchy Estimates over the ball $E_{\frac{r}{2}}(z)$ implies that

$$|\partial^M g(z)|^p \leq C(1 + |z|^{m-1})^{p|M|+2n} \int_{E_r(z)} |g(w)|^p dV(w)$$

for some $C = C(m, r, M) > 0$. Moreover, for $z \in \mathbb{C}^n$, we know

$$1 + |z|^{m-1} \approx (1 + |z|)^{m-1}.$$

So, using (2.2) and (2.3), we complete the proof. \square

For a function $f \in H(\mathbb{C}^n)$, we define the radial derivative $\mathcal{R}f$ of f at z by

$$\mathcal{R}f(z) = \sum_{j=1}^n z_j \frac{\partial f}{\partial z_j}(z).$$

The complex gradient of f at z is defined by

$$|\nabla f(z)| = \left[\sum_{j=1}^n \left| \frac{\partial f}{\partial z_j}(z) \right|^2 \right]^{1/2}.$$

For a radial function $\phi(r)$, we deduce Proposition 2.5 from Definition 2.3.

Definition 2.3 ([2]). Assume $\phi : [0, \infty) \rightarrow \mathbb{R}^+$ is twice continuously differentiable and there exists $\rho > 0$ such that $\phi'(r) \neq 0$ for $r \geq \rho$. We say that ϕ is in the class \mathcal{W}_p if it satisfies the following conditions:

$$\begin{aligned} \lim_{r \rightarrow \infty} \frac{re^{-p\phi(r)}}{\phi'(r)} &= 0, \\ \limsup_{r \rightarrow \infty} \frac{1}{r} \left(\frac{r}{\phi'(r)} \right)' &< \frac{p}{n}, \\ \liminf_{r \rightarrow \infty} \frac{1}{r} \left(\frac{r}{\phi'(r)} \right)' &> -\infty. \end{aligned}$$

Lemma 2.4 ([2]). Let $0 < p < \infty$ and $\phi \in \mathcal{W}_p$. Then for $f \in H(\mathbb{C}^n)$,

$$(2.7) \quad \int_{\mathbb{C}^n} |f(z)|^p e^{-p\phi(|z|)} dV(z) \approx |f(0)|^p + \int_{\mathbb{C}^n} \frac{|\mathcal{R}f(z)|^p}{(1 + |z|\phi'(|z|))^p} e^{-p\phi(|z|)} dV(z).$$

Note that constant functions are contained in $F_{m,\alpha,t}^p(\mathbb{C}^n)$ so we have $f \in F_{m,\alpha,t}^p(\mathbb{C}^n)$ if and only if $f - f(0) \in F_{m,\alpha,t}^p(\mathbb{C}^n)$. Furthermore Lemma 2.4 implies that $F_{m,\alpha,t}^p(\mathbb{C}^n)$ -norm of $f - f(0)$ is equivalent to the norm of the product of its radial derivative and $\frac{1}{1+|z|^m}$ in the n -dimensional complex space. We know that this factor $\frac{1}{1+|z|^m}$ is the supplementary amount to control the growth of the radial derivative. See the following.

Proposition 2.5. Let $\alpha > 0$, $0 < p < \infty$, $m \in \mathbb{N}$, and $t \in \mathbb{R}$. Then the following norms

$$\|f - f(0)\|_{F_{m,\alpha,t}}, \quad \left\| \frac{\mathcal{R}f(z)}{1 + |z|^m} \right\|_{L^p(G_{m,\alpha p,t})}, \quad \left\| \frac{\nabla f(z)}{1 + |z|^{m-1}} \right\|_{L^p(G_{m,\alpha p,t})}$$

are comparable to one another for $f \in H(\mathbb{C}^n)$.

Proof. Let $\phi(|z|) = \alpha|z|^m + \frac{t}{p} \ln(1 + |z|)$ for $z \in \mathbb{C}^n$. Then ϕ is in the class \mathcal{W}_p when $m \in \mathbb{N}$ and $t \in \mathbb{R}$. Also, by simple calculation, there exists a constant $C = C(\alpha, m, r, t) > 0$ such that

$$C^{-1} \leq \frac{1 + |z|^m}{1 + |z|\phi'(|z|)} \leq C.$$

Thus, applying Lemma 2.4 to $f - f(0)$, we obtain

$$\int_{\mathbb{C}^n} |f(z) - f(0)|^p e^{-\alpha p|z|^m} \frac{dV(z)}{(1 + |z|)^t} \approx \int_{\mathbb{C}^n} \frac{|\mathcal{R}f(z)|^p}{(1 + |z|^m)^p} e^{-\alpha p|z|^m} \frac{dV(z)}{(1 + |z|)^t}.$$

Also, due to $|\mathcal{R}f(z)| \leq |z||\nabla f(z)|$, this estimate implies

$$\|f - f(0)\|_{F_{m,\alpha,t}} \lesssim \left\| \frac{\nabla f(z)}{1 + |z|^{m-1}} \right\|_{L^p(G_{m,\alpha p,t})}.$$

Now, it remains to show

$$\left\| \frac{\nabla f(z)}{1 + |z|^{m-1}} \right\|_{L^p(G_{m,\alpha p,t})} \lesssim \|f - f(0)\|_{F_{m,\alpha,t}}.$$

It follows from Lemma 2.2 that

$$\frac{|\nabla f(z)|^p}{(1 + |z|^{m-1})^p} \lesssim (1 + |z|^{m-1})^{2n} \int_{E_r(z)} |f(w) - f(0)|^p dV(w).$$

Let $r > 0$ and $r_1 := 2r(1 + r)^{m-1}$. Integrating both sides of the above against $dG_{m,\alpha p,t}$, we have from (2.6), (2.2) and (2.3)

$$\begin{aligned} & \left\| \frac{\nabla f(z)}{1 + |z|^{m-1}} \right\|_{L^p(G_{m,\alpha p,t})}^p \\ & \lesssim \int_{\mathbb{C}^n} \int_{E_r(z)} |f(w) - f(0)|^p dV(w) dG_{m,\alpha p,t-2n(m-1)}(z) \\ & \lesssim \int_{\mathbb{C}^n} \int_{E_{r_1}(w)} |f(w) - f(0)|^p dG_{m,\alpha p,t-2n(m-1)}(z) dV(w) \\ & \approx \|f - f(0)\|_{F_{m,\alpha,t}}^p \end{aligned}$$

as desired. \square

3. Lipschitz type characterization

In this section, we prove our first result Theorem 1.1. For $r > 0$, we set

$$\Omega_r := \{(z, w) : |w - z|(1 + |z|^{m-1} + |w|^{m-1}) < r\}.$$

Theorem 3.1. *Let $\alpha > 0$, $0 < p < \infty$, $s \geq 0$, $m \in \mathbb{N}$, and $t \in \mathbb{R}$. Then the following statements are equivalent for entire functions f on \mathbb{C}^n :*

- (a) $f \in F_{m,\alpha,t}^p(\mathbb{C}^n)$.
- (b) *There exists a nonnegative continuous function $g \in L^p(G_{m,\alpha p,t-sp(m-1)})$ such that*

$$\frac{|f(z) - f(w)|}{|z - w|} \leq (1 + |z|^{m-1} + |w|^{m-1})^{1+s} (g(z) + g(w))$$

for each $z, w \in \mathbb{C}^n$ with $z \neq w$.

Proof. First, we assume that (b) holds. Fixing z and taking the limits $w \rightarrow z$ along the directions parallel to the coordinate axes,

$$|\partial_j f(z)| \lesssim (1 + |z|^{m-1})^{1+s} g(z)$$

for each j . Thus, we have

$$\frac{|\nabla f(z)|}{1 + |z|^{m-1}} \lesssim (1 + |z|^{m-1})^s g(z), \quad z \in \mathbb{C}^n$$

and thus

$$\int_{\mathbb{C}^n} \frac{|\nabla f(z)|^p}{(1 + |z|^{m-1})^p} e^{-\alpha p|z|^m} \frac{dV(z)}{(1 + |z|)^t} \lesssim \int_{\mathbb{C}^n} |g(z)|^p e^{-\alpha p|z|^m} \frac{dV(z)}{(1 + |z|)^{t-sp(m-1)}}.$$

Since $g(z) \in L^p(G_{m,\alpha p,t-sp(m-1)})$, by Proposition 2.5, we conclude

$$f \in F_{m,\alpha,t}^p(\mathbb{C}^n).$$

Second, we assume that (a) holds. Fix any $r > 0$. We consider $(z, w) \in \Omega_r$. Then $w \in E_r(z)$ and

$$1 + |z|^{m-1} + |w|^{m-1} \approx 1 + |z|^{m-1}.$$

By the fundamental theorem of calculus, we get

$$|f(z) - f(w)| \leq |z - w| \int_0^1 |\nabla f(\rho z + (1 - \rho)w)| d\rho.$$

Since $\rho z + (1 - \rho)w$ in $E_r(z)$, it follows

$$(3.1) \quad |f(z) - f(w)| \leq |z - w| \sup_{\zeta \in E_r(z)} |\nabla f(\zeta)|.$$

Furthermore, we note

$$(3.2) \quad \begin{aligned} |\nabla f(\zeta)| &\approx (1 + |z|^{m-1})^{1+s} \frac{|\nabla f(\zeta)|}{(1 + |\zeta|^{m-1})^{1+s}} \\ &\approx (1 + |z|^{m-1} + |w|^{m-1})^{1+s} \frac{|\nabla f(\zeta)|}{(1 + |\zeta|^{m-1})^{1+s}} \end{aligned}$$

for $\zeta \in E_r(z)$. Let

$$h_s(z) := \sup_{\zeta \in E_r(z)} \frac{|\nabla f(\zeta)|}{(1 + |\zeta|^{m-1})^{1+s}}.$$

Then we have by (3.1) and (3.2)

$$|f(z) - f(w)| \lesssim |z - w|(1 + |z|^{m-1} + |w|^{m-1})^{1+s}(h_s(z) + h_s(w))$$

for $(z, w) \in \Omega_r$.

Next, we consider $(z, w) \notin \Omega_r$. Then $|w - z|(1 + |z|^{m-1} + |w|^{m-1}) \geq r$. Therefore, for $s \geq 0$, we obtain

$$\begin{aligned} &|f(z) - f(w)| \\ &\leq \frac{|z - w|(1 + |z|^{m-1} + |w|^{m-1})}{r} (|f(z)| + |f(w)|) \\ &\leq \frac{|z - w|(1 + |z|^{m-1} + |w|^{m-1})^{1+s}}{r} \left(\frac{|f(z)|}{(1 + |z|^{m-1})^s} + \frac{|f(w)|}{(1 + |w|^{m-1})^s} \right). \end{aligned}$$

Hence, by setting $g(z) := h_s(z) + \frac{|f(z)|}{r(1 + |z|^{m-1})^s}$ for $z \in \mathbb{C}^n$, we have

$$|f(z) - f(w)| \lesssim |z - w|(1 + |z|^{m-1} + |w|^{m-1})^{1+s}(g(z) + g(w))$$

for each $z, w \in \mathbb{C}^n$ with $z \neq w$. Note that the constant suppressed above depends only on m, r and s . Also, the function $g(z)$ is continuous on \mathbb{C}^n . It remains for us to show the function $g(z)$ belongs to $L^p(G_{m,\alpha p,t-sp(m-1)})$. It is clear that $\frac{|f(z)|}{r(1 + |z|^{m-1})^s}$ is in $L^p(G_{m,\alpha p,t-sp(m-1)})$ for $f \in F_{m,\alpha,t}^p(\mathbb{C}^n)$.

Now, we claim h_s is in $L^p(G_{m,\alpha p,t-sp(m-1)})$. Let $\zeta \in E_r(z)$. Then $E_{r_0}(\zeta) \subset E_{r_1}(z)$ by (2.6) where $r_0 = 2r(1+r)^{m-1}$ and $r_1 = 2r_0(1+r_0)^{m-1}$. By Lemma 2.2 and (2.3), we get

$$\begin{aligned} \frac{|\nabla f(\zeta)|^p}{(1+|\zeta|^{m-1})^{(1+s)p}} &\lesssim (1+|\zeta|^{m-1})^{2n-sp} \int_{E_{r_0}(\zeta)} |f(w)|^p dV(w) \\ &\lesssim (1+|z|^{m-1})^{2n-sp} \int_{E_{r_1}(z)} |f(w)|^p dV(w). \end{aligned}$$

Taking the supremum over $\zeta \in E_r(z)$, we have

$$|h_s(z)|^p \lesssim (1+|z|^{m-1})^{2n-sp} \int_{E_{r_1}(z)} |f(w)|^p dV(w)$$

for all $z \in \mathbb{C}^n$. Let $r_2 = 2r_1(1+r_1)^{m-1}$. By integrating both sides of the above against the measure $dG_{m,\alpha p,t-sp(m-1)}(z)$, it follows

$$\begin{aligned} &\|h_s\|_{L^p(G_{m,\alpha p,t-sp(m-1)})}^p \\ &\lesssim \int_{\mathbb{C}^n} (1+|z|^{m-1})^{2n} \int_{E_{r_1}(z)} |f(w)|^p dV(w) dG_{m,\alpha p,t}(z) \\ &= \int_{\mathbb{C}^n} \int_{\mathbb{C}^n} (1+|z|^{m-1})^{2n} |f(w)|^p \chi_{E_{r_1}(z)}(w) dG_{m,\alpha p,t}(z) dV(w) \\ &< \int_{\mathbb{C}^n} \int_{\mathbb{C}^n} (1+|z|^{m-1})^{2n} |f(w)|^p \chi_{E_{r_2}(w)}(z) dG_{m,\alpha p,t}(z) dV(w) \\ &= \int_{\mathbb{C}^n} |f(w)|^p \int_{E_{r_2}(w)} (1+|z|^{m-1})^{2n} dG_{m,\alpha p,t}(z) dV(w), \end{aligned}$$

where χ denotes the characteristic function in its subscripted set. For $z \in E_{r_2}(w)$, we know (2.2) and (2.3). Hence, it follows that

$$\begin{aligned} &\|h_s\|_{L^p(G_{m,\alpha p,t-sp(m-1)})}^p \\ &\lesssim \int_{\mathbb{C}^n} |f(w)|^p (1+|w|^{m-1})^{2n} \int_{E_{r_2}(w)} dV(z) dG_{m,\alpha p,t}(w) \\ &\lesssim \int_{\mathbb{C}^n} |f(w)|^p (1+|w|^{m-1})^{2n} \frac{\omega_n r_2^{2n}}{(1+|w|^{m-1})^{2n}} dG_{m,\alpha p,t}(w) \\ &\lesssim \|f\|_{F_{m,\alpha,t}^p}^p. \end{aligned}$$

This finishes the proof. \square

From the proof of Theorem 3.1, we have the following local version of Theorem 3.1 for arbitrary s real.

Theorem 3.2. *Let $\alpha > 0$, $0 < p < \infty$, $r > 0$, $m \in \mathbb{N}$, and $s, t \in \mathbb{R}$. Then the following statements are equivalent for entire functions f on \mathbb{C}^n :*

- (a) $f \in F_{m,\alpha,t}^p(\mathbb{C}^n)$.

- (b) *There exists a nonnegative continuous function $g \in L^p(G_{m,\alpha p,t-sp(m-1)})$ such that*

$$\frac{|f(z) - f(w)|}{|z - w|} \leq (1 + |z|^{m-1} + |w|^{m-1})^{1+s} (g(z) + g(w))$$

for $(z, w) \in \Omega_r$ with $z \neq w$.

4. Double integral characterization

In this section, we prove the main Theorem 1.2. By the same proof of Proposition 3.1 in [1], we have the following.

Theorem 4.1. *Let $0 < p < \infty$ and $\phi \in \mathcal{W}_p$. Then the estimate*

$$\int_{\mathbb{C}^n} |f(z) - f(0)|^p e^{-p\phi(|z|)} dV(z) \approx \int_{\mathbb{C}^n} \int_{\mathbb{C}^n} |Lf(z, w)|^p e^{-p\phi(|z|)} dV(z) e^{-p\phi(|w|)} dV(w)$$

holds for $f \in H(\mathbb{C}^n)$.

Letting $\phi(|z|) = \alpha|z|^m + \frac{t}{p} \ln(1 + |z|)$ for $z \in \mathbb{C}^n$, Theorem 4.1 gives us the following characterization.

Theorem 4.2. *Let $\alpha > 0$, $0 < p < \infty$, $m \in \mathbb{N}$, and $t \in \mathbb{R}$. Then the following statements are equivalent for entire functions f on \mathbb{C}^n :*

- (a) $f \in F_{m,\alpha,t}^p(\mathbb{C}^n)$.
- (b) $Lf \in L^p(G_{m,\alpha p,t} \times G_{m,\alpha p,t})$.

Moreover, the norms

$$\|f - f(0)\|_{F_{m,\alpha,t}^p} \quad \text{and} \quad \|Lf\|_{L^p(G_{m,\alpha p,t} \times G_{m,\alpha p,t})}$$

are comparable to each other.

Lemma 4.3. *Let $0 < p < \infty$ and $s \geq 0$. Let m be an even positive integer. For $w \in E_r(z)$, there exists a positive constant C such that $|e^{s(\bar{z}\cdot w)^{\frac{m}{2}}}|^p \lesssim e^{sp|z|^m}$.*

Proof. We prove that

$$\begin{aligned} |e^{s(\bar{z}\cdot w)^{\frac{m}{2}}}|^p &= e^{sp \operatorname{Re}[(\bar{z}\cdot w)^{\frac{m}{2}}]} \leq e^{sp|z|^{\frac{m}{2}}|w|^{\frac{m}{2}}} \\ &\leq e^{\frac{sp(|z|^m + |w|^m)}{2}} \\ &\lesssim e^{sp|z|^m} \end{aligned}$$

by the Cauchy-Schwarz inequality, arithmetic-geometric mean inequality and (2.2) in Lemma 2.1 for $w \in E_r(z)$ in turn. \square

Theorem 4.4. *Let $\alpha, r > 0$, $0 < p < \infty$, and $t \in \mathbb{R}$. Let m be an even positive integer. For $s \geq 0$, let $\beta := \frac{s+t}{2}$. Then the following statements are equivalent for entire functions f on \mathbb{C}^n :*

- (a) $f \in F_{m,\alpha,t}^p(\mathbb{C}^n)$.
- (b) $L_r^s f \in L^p(G_{m,\beta p,\delta} \times G_{m,\beta p,\delta})$, where $\delta = \frac{t}{2} - n(m-1)$.

Moreover, the norms

$$\|f - f(0)\|_{F_{m,\alpha,t}^p} \quad \text{and} \quad \|L_r^s f\|_{L^p(G_{m,\beta p,\delta} \times G_{m,\beta p,\delta})}$$

are comparable to each other.

Proof. We assume that (b) holds. Fix $f \in H(\mathbb{C}^n)$ and let $z \in \mathbb{C}^n$. Define a function

$$g_z(w) = [f(w) - f(z)]e^{s(\bar{z}\cdot w)\frac{m}{2}}.$$

Then $g_z \in H(\mathbb{C}^n)$ and $\nabla g_z(z) = \nabla f(z)e^{s|z|^m}$. By applying Lemma 2.2, (2.2) and (2.3), we get

$$\begin{aligned} & \left(\frac{|\nabla f(z)|e^{s|z|^m}}{1 + |z|^{m-1}} \right)^p \\ & \lesssim (1 + |z|^{m-1})^{2n} \int_{E_r(z)} |f(w) - f(z)|^p |e^{s(\bar{z}\cdot w)\frac{m}{2}}|^p dV(w) \\ & \approx e^{\frac{(s+\alpha)p}{2}|z|^m} (1 + |z|^{m-1})^n \\ & \quad \times \int_{E_r(z)} |f(w) - f(z)|^p |e^{s(\bar{z}\cdot w)\frac{m}{2}}|^p |e^{-\frac{(s+\alpha)p}{2}|w|^m} (1 + |w|^{m-1})^n dV(w). \end{aligned}$$

By integrating both sides of the above against $dG_{m,(s+\alpha)p,t}(z)$ and applying (2.3) for $w \in E_r(z)$, we have

$$\begin{aligned} & \int_{\mathbb{C}^n} \left(\frac{|\nabla f(z)|}{1 + |z|^{m-1}} \right)^p dG_{m,\alpha p,t}(z) \\ & \lesssim \int_{\mathbb{C}^n} \int_{E_r(z)} |f(w) - f(z)|^p |e^{s(\bar{z}\cdot w)\frac{m}{2}}|^p dG_{m,\beta p,-n(m-1)}(w) dG_{m,\beta p,t-n(m-1)}(z) \\ & \approx \|L_r^s f\|_{L^p(G_{m,\beta p,\delta} \times G_{m,\beta p,\delta})}. \end{aligned}$$

Thus, by Proposition 2.5, we obtain

$$\|f - f(0)\|_{F_{m,\alpha,t}^p} \lesssim \|L_r^s f\|_{L^p(G_{m,\beta p,\delta} \times G_{m,\beta p,\delta})}.$$

The constant suppressed above is independent of f . We complete that (b) implies (a).

Now, we assume that (a) holds. Let $r_0 = 2r(1+r)^{m-1}$.

$$\begin{aligned} & \|L_r^s f\|_{L^p(G_{m,\beta p,\delta} \times G_{m,\beta p,\delta})} \\ & = \int_{\mathbb{C}^n} \int_{\mathbb{C}^n} |f(w) - f(z)|^p |e^{s(\bar{z}\cdot w)\frac{m}{2}}|^p \chi_{E_r(z)}(w) dG_{m,\beta p,\delta}(w) dG_{m,\beta p,\delta}(z) \\ & \lesssim \int_{\mathbb{C}^n} \int_{\mathbb{C}^n} |f(z) - f(0)|^p |e^{s(\bar{z}\cdot w)\frac{m}{2}}|^p \chi_{E_{r_0}(z)}(w) dG_{m,\beta p,\delta}(w) dG_{m,\beta p,\delta}(z) \\ & \quad + \int_{\mathbb{C}^n} \int_{\mathbb{C}^n} |f(w) - f(0)|^p |e^{s(\bar{z}\cdot w)\frac{m}{2}}|^p \chi_{E_{r_0}(w)}(z) dG_{m,\beta p,\delta}(z) dG_{m,\beta p,\delta}(w) \\ & \lesssim \int_{\mathbb{C}^n} |f(z) - f(0)|^p \int_{E_{r_0}(z)} |e^{s(\bar{z}\cdot w)\frac{m}{2}}|^p dG_{m,\beta p,\delta}(w) dG_{m,\beta p,\delta}(z). \end{aligned}$$

For the first inequality, we used $E_r(z) \subset E_{r_0}(z)$, $E_r(z) \subset E_{r_0}(w)$ for $w \in E_r(z)$ and Fubini's theorem. And for $w \in E_{r_0}(z)$, we have from (2.3)

$$\begin{aligned} & \|L_r^s f\|_{L^p(G_{m,\beta p,\delta} \times G_{m,\beta p,\delta})} \\ & \lesssim \int_{\mathbb{C}^n} |f(z) - f(0)|^p (1 + |z|^{m-1})^n \\ & \quad \times \int_{E_{r_0}(z)} |e^{s(\bar{z}\cdot w)^{\frac{m}{2}}}|^p dG_{m,\beta p,-n(m-1)}(w) dG_{m,\beta p,t}(z) \\ & = \int_{\mathbb{C}^n} |f(z) - f(0)|^p I_r(z) dG_{m,\alpha p,t}(z), \end{aligned}$$

where $I_r(z) := e^{\frac{(-s+\alpha)p}{2}|z|^m} (1 + |z|^{m-1})^n \int_{E_{r_0}(z)} |e^{s(\bar{z}\cdot w)^{\frac{m}{2}}}|^p dG_{m,\beta p,-n(m-1)}(w)$.

Now, we claim that $I_r(z) \lesssim 1$.

For $w \in E_{r_0}(z)$, we have Lemma 4.3, (2.2) and (2.3). It follows that

$$\begin{aligned} \int_{E_{r_0}(z)} |e^{s(\bar{z}\cdot w)^{\frac{m}{2}}}|^p dG_{m,\beta p,-n(m-1)}(w) & \lesssim \int_{E_{r_0}(z)} e^{sp|z|^m} dG_{m,\beta p,-n(m-1)}(w) \\ & \lesssim e^{\frac{(s-\alpha)p}{2}|z|^m} (1 + |z|^{m-1})^n V[E_{r_0}(z)]. \end{aligned}$$

Since $V[E_{r_0}(z)] \approx (1 + |z|^{m-1})^{-2n}$, we have $I_r(z) \lesssim 1$. Hence we complete that (a) implies (b). \square

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